§6. Sheaves

In complex analysis one frequently has to deal with functions which have various domains of definition. The notion of a sheaf gives a suitable formal setting to handle this situation.

6.1. Definition. Suppose X is a topological space and $\mathfrak{T}$ is the system of open sets in X. A *presheaf* of abelian groups on X is a pair $(\mathscr{F}, \rho)$ consisting of

(i) a family $\mathscr{F} = (\mathscr{F}(U))_{U \in \mathfrak{T}}$ of abelian groups,
(ii) a family $\rho = (\rho_V^U)_{U, V \in \mathfrak{T}, V \subset U}$ of group homomorphisms

$$\rho_V^U: \mathscr{F}(U) \to \mathscr{F}(V), \quad \text{where } V \text{ is open in } U,$$

with the following properties:

$$\rho_U^U = \mathrm{id}_{\mathscr{F}(U)} \quad \text{for every } U \in \mathfrak{T},$$
$$\rho_W^V \circ \rho_V^U = \rho_W^U \quad \text{for } W \subset V \subset U.$$

Remark. Generally one just writes $\mathscr{F}$ instead of $(\mathscr{F}, \rho)$. The homomorphisms ρ_V^U are called *restriction homomorphisms*. Instead of $\rho_V^U(f)$ for $f \in \mathscr{F}(U)$ one writes just $f \mid V$. Analogous to presheaves of abelian groups one can also define presheaves of vector spaces, rings, sets, etc.

6.2. Example. Suppose X is an arbitrary topological space. For any open subset $U \subset X$ let $\mathscr{C}(U)$ be the vector space of all continuous functions $f: U \to \mathbb{C}$. For $V \subset U$ let $\rho_V^U: \mathscr{C}(U) \to \mathscr{C}(V)$ be the usual restriction mapping. Then $(\mathscr{C}, \rho)$ is a presheaf of vector spaces on X.

6.3. Definition. A presheaf $\mathscr{F}$ on a topological space X is called a *sheaf* if for every open set $U \subset X$ and every family of open subsets $U_i \subset U$, $i \in I$, such that $U = \bigcup_{i \in I} U_i$ the following conditions, which we will call the Sheaf Axioms, are satisfied:

(I) If $f, g \in \mathscr{F}(U)$ are elements such that $f \mid U_i = g \mid U_i$ for every $i \in I$, then $f = g$.

(II) Given elements $f_i \in \mathscr{F}(U_i)$, $i \in I$, such that

$$f_i \mid U_i \cap U_j = f_j \mid U_i \cap U_j \quad \text{for all } i, j \in I,$$

then there exists an $f \in \mathscr{F}(U)$ such that $f \mid U_i = f_i$ for every $i \in I$.

Remark. The element f, whose existence is assured by (II), is by (I) uniquely determined.

Applying (I) and (II) to the case $U = \varnothing = \bigcup_{i \in \varnothing} U_i$ implies $\mathscr{F}(\varnothing)$ consists of exactly one element.

6.4. Examples

(a) For every topological space X the presheaf $\mathscr{C}$ defined in (6.2) is a sheaf. Both Sheaf Axioms (I) and (II) are trivially fulfilled.

(b) Suppose X is a Riemann surface and $\mathcal{O}(U)$ is the ring of holomorphic functions defined on the open set $U \subset X$. Taking the usual restriction mapping $\mathcal{O}(U) \to \mathcal{O}(V)$ for $V \subset U$ one gets the sheaf $\mathcal{O}$ of holomorphic functions on X. The sheaf $\mathscr{M}$ of meromorphic functions on X is defined analogously.

(c) For an open subset U of a Riemann surface X let $\mathcal{O}^*(U)$ be the multiplicative group of all holomorphic maps $f\colon U \to \mathbb{C}^*$. With the usual restriction maps $\mathcal{O}^*$ is a sheaf of (multiplicative) abelian groups. The sheaf $\mathscr{M}^*$ is defined analogously: For any open set $U \subset X$, $\mathscr{M}^*(U)$ consists of all meromorphic functions $f \in \mathscr{M}(U)$ which do not vanish identically on any connected component of U.

(d) Suppose X is an arbitrary topological space and G is an abelian group. Define a presheaf $\mathscr{G}$ on X as follows: For any non-empty open subset $U \subset X$ let $\mathscr{G}(U) := G$ and let $\mathscr{G}(\emptyset) := 0$. As for the restriction mappings, let $\rho_V^U = \mathrm{id}_G$ if $V \neq \emptyset$ and let $\rho_\emptyset^U$ be the zero homomorphism. If G contains at least two distinct elements g_1, g_2 and if X has two disjoint non-empty open subsets U_1, U_2, then $\mathscr{G}$ is not a sheaf. This is because Sheaf Axion (II) does not hold. For, since $U_1 \cap U_2 = \emptyset$, one has $g_1 \mid U_1 \cap U_2 = 0 = g_2 \mid U_1 \cap U_2$ but there is no $f \in \mathscr{G}(U_1 \cup U_2) = G$ such that $f \mid U_1 = g_1$ and $f \mid U_2 = g_2$.

(e) One can easily modify the previous example to obtain a sheaf. For any open set U, let $\tilde{\mathscr{G}}(U)$ be the abelian group of all locally constant mappings $g\colon U \to G$. Then if U is a non-empty connected open set, one has $\tilde{\mathscr{G}}(U) = G$. For $V \subset U$ let $\tilde{\mathscr{G}}(U) \to \tilde{\mathscr{G}}(V)$ be the usual restriction. Then $\tilde{\mathscr{G}}$ is a sheaf on X which is called the sheaf of locally constant functions with values in G. Often it is just denoted by G.

6.5. The Stalk of a Presheaf. Suppose $\mathscr{F}$ is a presheaf of sets on a topological space X and $a \in X$ is a point. On the disjoint union

$$\dot{\bigcup_{U \ni a}} \mathscr{F}(U),$$

where the union is taken over all the open neighborhoods U of a, introduce an equivalence relation $\underset{a}{\sim}$ as follows: Two elements $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$ are related $f \underset{a}{\sim} g$ precisely if there exists an open set W with $a \in W \subset U \cap V$ such that $f \mid W = g \mid W$. One can easily check that this really is an equivalence relation. The set $\mathscr{F}_a$ of all equivalence classes, the so-called inductive limit of $\mathscr{F}(U)$, is given by

$$\mathscr{F}_a := \varinjlim_{U \ni a} \mathscr{F}(U) := \left(\dot{\bigcup_{U \ni a}} \mathscr{F}(U) \right) \Big/ \underset{a}{\sim},$$

and is called the *stalk* of $\mathscr{F}$ at the point a. If $\mathscr{F}$ is a presheaf of abelian groups (resp. vector spaces, rings), then the stalk $\mathscr{F}_a$ with the operation defined on the equivalence classes by means of the operation defined on representatives, is also an abelian group (resp. vector space, ring).

For any open neighborhood U of a, let

$$\rho_a: \mathscr{F}(U) \to \mathscr{F}_a$$

be the mapping which assigns to each element $f \in \mathscr{F}(U)$ its equivalence class modulo $\underset{a}{\sim}$. One calls $\rho_a(f)$ the *germ* of f at a. As an example consider the sheaf $\mathcal{O}$ of holomorphic functions on a domain $X \subset \mathbb{C}$. Let $a \in X$. A *germ of a holomorphic function* $\varphi \in \mathcal{O}_a$ is represented by a holomorphic function in an open neighborhood of a and thus has a Taylor series expansion $\sum_{\nu=0}^{\infty} c_\nu(z-a)^\nu$ with a positive radius of convergence. Two holomorphic functions on neighborhoods of a determine the same germ at a precisely if they have the same Taylor series expansion about a. Thus there is an isomorphism between the stalk $\mathcal{O}_a$ and the ring $\mathbb{C}\{z-a\}$ of all convergent power series in $z-a$ with complex coefficients. In an analogous way, the ring $\mathcal{M}_a$ of *germs of meromorphic functions* at a is isomorphic to the ring of all convergent Laurent series

$$\sum_{\nu=k}^{\infty} c_\nu(z-a)^\nu, \qquad k \in \mathbb{Z}, \qquad c_\nu \in \mathbb{C},$$

which have finite principal parts.

For any germ of a function $\varphi \in \mathcal{O}_a$ the value of the function, $\varphi(a) \in \mathbb{C}$, is well-defined, i.e., is independent of the choice of representative.

6.6. Lemma. *Suppose $\mathscr{F}$ is a sheaf of abelian groups on the topological space X and $U \subset X$ is an open subset. Then an element $f \in \mathscr{F}(U)$ is zero precisely if all germs $\rho_x(f) \in \mathscr{F}_x$, $x \in U$, vanish.*

This follows directly from Sheaf Axiom (I).

6.7. The Topological Space Associated to a Presheaf. Suppose X is a topological space and $\mathscr{F}$ is a presheaf on X. Let

$$|\mathscr{F}| := \dot{\bigcup_{x \in X}} \mathscr{F}_x$$

be the disjoint union of all the stalks. Denote by

$$p: |\mathscr{F}| \to X$$

the mapping which assigns to each element $\varphi \in \mathscr{F}_x$ the point x. Now introduce a topology on $|\mathscr{F}|$ as follows: For any open subset $U \subset X$ and an element $f \in \mathscr{F}(U)$, let

$$[U, f] := \{\rho_x(f): x \in U\} \subset |\mathscr{F}|.$$

6.8. Theorem. *The system $\mathfrak{B}$ of all sets $[U, f]$, where U is open in X and $f \in \mathscr{F}(U)$, is a basis for a topology on $|\mathscr{F}|$. The projection $p\colon |\mathscr{F}| \to X$ is a local homeomorphism.*

PROOF

(a) To see that $\mathfrak{B}$ forms a basis for a topology on $|\mathscr{F}|$, one has to verify the following two conditions:

(i) Every element $\varphi \in |\mathscr{F}|$ is contained in at least one $[U, f]$. This is trivial.

(ii) If $\varphi \in [U, f] \cap [V, g]$, then there exists a $[W, h] \in \mathfrak{B}$ such that $\varphi \in [W, h] \subset [U, f] \cap [V, g]$. For suppose $p(\varphi) = x$. Then $x \in U \cap V$ and $\varphi = \rho_x(f) = \rho_x(g)$. Hence there exists an open neighborhood $W \subset U \cap V$ of x such that $f \,|\, W = g \,|\, W =: h$. This implies $\varphi \in [W, h] \subset [U, f] \cap [V, g]$.

(b) Now we will show that $p\colon |\mathscr{F}| \to X$ is a local homeomorphism. Suppose $\varphi \in |\mathscr{F}|$ and $p(\varphi) = x$. There exists a $[U, f] \in \mathfrak{B}$ with $\varphi \in [U, f]$. Then $[U, f]$ is an open neighborhood of φ and U is an open neighborhood of x. The mapping $p \,|\, [U, f] \to U$ is bijective and also continuous and open as one sees immediately from the definition. Thus $p\colon |\mathscr{F}| \to X$ is a local homeomorphism. □

6.9. Definition. A presheaf $\mathscr{F}$ on a topological space X is said to satisfy the *Identity Theorem* if the following holds. If $Y \subset X$ is a domain and f, $g \in \mathscr{F}(Y)$ are elements whose germs $\rho_a(f)$ and $\rho_a(g)$ coincide at a point $a \in Y$, then $f = g$.

For example, this condition is satisfied by the sheaf $\mathcal{O}$ (resp. $\mathscr{M}$) of holomorphic (resp. meromorphic) functions on a Riemann surface X.

6.10. Theorem. *Suppose X is a locally connected Hausdorff space and $\mathscr{F}$ is a presheaf on X which satisfies the Identity Theorem. Then the topological space $|\mathscr{F}|$ is Hausdorff.*

PROOF. Suppose $\varphi_1, \varphi_2 \in |\mathscr{F}|$ and $\varphi_1 \neq \varphi_2$. We have to find disjoint neighborhoods of φ_1 and φ_2.

Case 1. Suppose $p(\varphi_1) =: x \neq y := p(\varphi_2)$. Since X is Hausdorff, there exist disjoint neighborhoods U and V of x and y respectively. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint neighborhoods of φ_1 and φ_2, respectively.

Case 2. Suppose $p(\varphi_1) = p(\varphi_2) =: x$. Suppose the germs $\varphi_i \in \mathscr{F}_x$ are represented by elements $f_i \in \mathscr{F}(U_i)$, where the U_i are open neighborhoods of x, $i = 1, 2$. Let $U \subset U_1 \cap U_2$ be a connected open neighborhood of x. Then $[U, f_i | U]$ are open neighborhoods of φ_i. Now suppose there exists $\psi \in [U, f_1 | U] \cap [U, f_2 | U]$. Let $p(\psi) = y$. Then $\psi = \rho_y(f_1) = \rho_y(f_2)$. From the Identity Theorem it follows that $f_1 | U = f_2 | U$, thus $\varphi_1 = \varphi_2$. Contradiction! Hence $[U, f_1 \,|\, U]$ and $[U, f_2 \,|\, U]$ are disjoint. □

EXERCISES (§6)

6.1. Suppose X is a Riemann surface. For $U \subset X$ open, let $\mathscr{B}(U)$ be the vector space of all bounded holomorphic functions $f: U \to \mathbb{C}$. For $V \subset U$ let $\mathscr{B}(U) \to \mathscr{B}(V)$ be the usual restriction map. Show that $\mathscr{B}$ is a presheaf which satisfies sheaf axiom (I) but not sheaf axiom (II).

6.2. Suppose X is a Riemann surface. For $U \subset X$ open, let

$$\mathscr{F}(U) := \mathcal{O}^*(U)/\exp \mathcal{O}(U).$$

Show that $\mathscr{F}$ with the usual restriction maps is a presheaf which does not satisfy sheaf axiom (I).

6.3. Suppose $\mathscr{F}$ is a presheaf on the topological space X and $p: |\mathscr{F}| \to X$ is the associated covering space. For $U \subset X$ open, let $\tilde{\mathscr{F}}(U)$ be the space of all sections of p over U, i.e., the space of all continuous maps

$$f: U \to |\mathscr{F}|$$

with $p \circ f = \mathrm{id}_U$. Prove the following:

(a) $\tilde{\mathscr{F}}$ together with the natural restriction maps is a sheaf,
(b) There is a natural isomorphism of the stalks

$$\mathscr{F}_x \xrightarrow{\sim} \tilde{\mathscr{F}}_x, \quad \text{for every } x \in X.$$

§7. Analytic Continuation

Next we consider the construction of Riemann surfaces which arise from the analytic continuation of germs of functions.

7.1. Definition. Suppose X is a Riemann surface, $u: [0, 1] \to X$ is a curve and $a := u(0)$, $b := u(1)$. The holomorphic function germ $\psi \in \mathcal{O}_b$ is said to result from the *analytic continuation along the curve* u of the holomorphic function germ $\varphi \in \mathcal{O}_a$ if the following holds. There exists a family $\varphi_t \in \mathcal{O}_{u(t)}$, $t \in [0, 1]$ of function germs with $\varphi_0 = \varphi$ and $\varphi_1 = \psi$ with the property that for every $\tau \in [0, 1]$ there exists a neighborhood $T \subset [0, 1]$ of τ, an open set $U \subset X$ with $u(T) \subset U$ and a function $f \in \mathcal{O}(U)$ such that

$$\rho_{u(t)}(f) = \varphi_t \quad \text{for every } t \in T.$$

Here $\rho_{u(t)}(f)$ is the germ of f at the point $u(t)$. Because of the compactness of $[0, 1]$ this condition is equivalent to the following (see Fig. 5). There exist a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of the interval $[0, 1]$, domains $U_i \subset X$ with $u([t_{i-1}, t_i]) \subset U_i$ and holomorphic functions $f_i \in \mathcal{O}(U_i)$ for $i = 1, \ldots, n$ such that:

(i) φ is the germ of f_1 at the point a and ψ is the germ of f_n at the point b.
(ii) $f_i \mid V_i = f_{i+1} \mid V_i$ for $i = 1, \ldots, n-1$, where V_i denotes the connected component of $U_i \cap U_{i+1}$ containing the point $u(t_i)$.

CHAPTER 2

Compact Riemann Surfaces

Amongst all Riemann surfaces the compact ones are especially important. They arise, for example, as those covering surfaces of the Riemann sphere defined by algebraic functions. As well their function theory is subject to interesting restrictions, like the Riemann–Roch Theorem and Abel's Theorem. More recently the theory of Riemann surfaces has been generalized to an extensive theory for complex manifolds of higher dimension. And the methods developed for this are very well suited to proving the classical theorems. One such method is sheaf cohomology and we give a short introduction to this in the present chapter.

To a large extent Chapter 2 is independent of Chapter 1. Essentially only §1 (the definition of Riemann surfaces), the first half of §6 (the definition of sheaves) and §§9 and 10 (differential forms) will be needed.

§12. Cohomology Groups

The goal of this section is to define the cohomology groups $H^1(X, \mathscr{F})$, where $\mathscr{F}$ is a sheaf of abelian groups on a topological space X. In our further study of Riemann surfaces, these cohomology groups play a very decided role.

12.1. Cochains, Cocycles, Coboundaries. Suppose X is a topological space and $\mathscr{F}$ is a sheaf of abelian groups on X. Also suppose that an open covering of X is given, i.e., a family $\mathfrak{U} = (U_i)_{i \in I}$ of open subsets of X such that $\bigcup_{i \in I} U_i = X$. For $q = 0, 1, 2, \ldots$ define the *qth cochain group* of $\mathscr{F}$, with respect to $\mathfrak{U}$, as

$$C^q(\mathfrak{U}, \mathscr{F}) := \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

The elements of $C^q(\mathfrak{U}, \mathscr{F})$ are called q-cochains. Thus a q-cochain is a family

$$(f_{i_0, \ldots, i_q})_{i_0, \ldots, i_q} \in I^{q+1} \quad \text{such that } f_{i_0, \ldots, i_q} \in \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

for all $(i_0, \ldots, i_q) \in I^{q+1}$. The addition of two cochains is defined component-wise. Now define *coboundary operators*

$$\delta \colon C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$$

$$\delta \colon C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$$

as follows:

(i) For $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathscr{F})$ let $\delta((f_i)_{i \in I}) = (g_{ij})_{i, j \in I}$ where

$$g_{ij} := f_j - f_i \in \mathscr{F}(U_i \cap U_j).$$

Here it is understood that one restricts f_i and f_j to the intersection $U_i \cap U_j$ and then takes their difference.

(ii) For $(f_{ij})_{i, j \in I} \in C^1(\mathfrak{U}, \mathscr{F})$ let $\delta((f_{ij})) = (g_{ijk})$ where

$$g_{ijk} := f_{jk} - f_{ik} + f_{ij} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

Again the terms on the right are restricted to their common domain of definition $U_i \cap U_j \cap U_k$.

These coboundary operators are group homomorphisms. Let

$$Z^1(\mathfrak{U}, \mathscr{F}) := \operatorname{Ker}(C^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathscr{F})),$$

$$B^1(\mathfrak{U}, \mathscr{F}) := \operatorname{Im}(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F})).$$

The elements of $Z^1(\mathfrak{U}, \mathscr{F})$ are called *1-cocycles*. Thus by definition a 1-cochain $(f_{ij}) \in C^1(\mathfrak{U}, \mathscr{F})$ is a cocycle precisely if

$$f_{ik} = f_{ij} + f_{jk} \quad \text{on } U_i \cap U_j \cap U_k \tag{*}$$

for all $i, j, k \in I$. One calls (*) the *cocycle relation* and it implies

$$f_{ii} = 0, \qquad f_{ij} = -f_{ji}.$$

One obtains these from (*) by letting $i = j = k$ for the first and $i = k$ for the second.

The elements of $B^1(\mathfrak{U}, \mathscr{F})$ are called *1-coboundaries*. In particular every coboundary is a cocycle. A coboundary is also called a *splitting cocycle*. Thus a 1-cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ splits if and only if there is a 0-cochain $(g_i) \in C^0(\mathfrak{U}, \mathscr{F})$ such that

$$f_{ij} = g_i - g_j \quad \text{on } U_i \cap U_j \quad \text{for every } i, j \in I.$$

12.2. Definition. The quotient group

$$H^1(\mathfrak{U}, \mathscr{F}) := Z^1(\mathfrak{U}, \mathscr{F})/B^1(\mathfrak{U}, \mathscr{F})$$

is called the *1st cohomology group* with coefficients in $\mathscr{F}$ with respect to the covering $\mathfrak{U}$. Its elements are called cohomology classes and two cocycles which belong to the same cohomology class are called *cohomologous*. Thus two cocycles are cohomologous precisely if their difference is a coboundary.

The groups $H^1(\mathfrak{U}, \mathscr{F})$ depend on the covering U. In order to have cohomology groups which depend only on X and $\mathscr{F}$, one has to use finer and finer coverings and then take a limit. We shall now make this idea precise.

An open covering $\mathfrak{V} = (V_k)_{k \in K}$ is called *finer* than the covering $\mathfrak{U} = (U_i)_{i \in I}$, denoted $\mathfrak{V} < \mathfrak{U}$, if every V_k is contained in at least one U_i. Thus there is a mapping $\tau\colon K \to I$ such that

$$V_k \subset U_{\tau k} \quad \text{for every } k \in K.$$

By means of the mapping τ we can define a mapping

$$t_{\mathfrak{V}}^{\mathfrak{U}}\colon Z^1(\mathfrak{U}, \mathscr{F}) \to Z^1(\mathfrak{V}, \mathscr{F})$$

in the following way. For $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ let $t_{\mathfrak{V}}^{\mathfrak{U}}((f_{ij})) = (g_{kl})$ where

$$g_{kl} := f_{\tau k, \tau l} \,|\, V_k \cap V_l \quad \text{for every } k, l \in K.$$

This mapping takes coboundaries into coboundaries and thus induces a homomorphism of the cohomology groups $H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$, which we also denote by $t_{\mathfrak{V}}^{\mathfrak{U}}$.

12.3. Lemma. *The mapping*

$$t_{\mathfrak{V}}^{\mathfrak{U}}\colon H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is independent of the choice of the refining mapping $\tau\colon K \to I$.

PROOF. Suppose $\tilde{\tau}\colon K \to I$ is another mapping such that $V_k \subset U_{\tilde{\tau} k}$ for every $k \in K$. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and let

$$g_{kl} := f_{\tau k, \tau l} \,|\, V_k \cap V_l \quad \text{and} \quad \tilde{g}_{kl} := f_{\tilde{\tau} k, \tilde{\tau} l} \,|\, V_k \cap V_l.$$

We have to show that the cocycles (g_{kl}) and $(\tilde{g}_{kl})$ are cohomologous. Since $V_k \subset U_{\tau k} \cap U_{\tilde{\tau} k}$, one can define

$$h_k := f_{\tau k, \tilde{\tau} k} \,|\, V_k \in \mathscr{F}(V_k).$$

On $V_k \cap V_l$ one has

$$\begin{aligned} g_{kl} - \tilde{g}_{kl} &= f_{\tau k, \tau l} - f_{\tilde{\tau} k, \tilde{\tau} l} \\ &= f_{\tau k, \tau l} + f_{\tau l, \tilde{\tau} k} - f_{\tau l, \tilde{\tau} k} - f_{\tilde{\tau} k, \tilde{\tau} l} \\ &= f_{\tau k, \tilde{\tau} k} - f_{\tau l, \tilde{\tau} l} = h_k - h_l. \end{aligned}$$

Thus the cocycle $(g_{kl}) - (\tilde{g}_{kl})$ is a coboundary. □

12.4. Lemma. *The mapping*

$$t^{\mathfrak{U}}_{\mathfrak{V}}\colon H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$$

is injective.

PROOF. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ is a cocycle whose image in $Z^1(\mathfrak{V}, \mathscr{F})$ splits. One has to show that (f_{ij}) itself splits.

Now suppose $f_{\tau k, \tau l} = g_k - g_l$ on $V_k \cap V_l$, where $g_k \in \mathscr{F}(V_k)$. Then on $U_i \cap V_k \cap V_l$ one has

$$g_k - g_l = f_{\tau k, \tau l} = f_{\tau k, i} + f_{i, \tau l} = f_{i, \tau l} - f_{i, \tau k},$$

and thus $f_{i, \tau k} + g_k = f_{i, \tau l} + g_l$. Applying sheaf axiom II (see Definition (6.3)) to the family of open sets $(U_i \cap V_k)_{k \in K}$, one obtains $h_i \in \mathscr{F}(U_i)$ such that

$$h_i = f_{i, \tau k} + g_k \quad \text{on } U_i \cap V_k.$$

With the elements h_i found in this way, on $U_i \cap U_j \cap V_k$ one has

$$f_{ij} = f_{i, \tau k} + f_{\tau k, j} = f_{i, \tau k} + g_k - f_{j, \tau k} - g_k = h_i - h_j.$$

Since k is arbitrary, it follows from sheaf axiom I that this equation is valid over $U_i \cap U_j$, i.e., the cocycle (f_{ij}) splits with respect to the covering $\mathfrak{U}$. □

12.5. The definition of $H^1(X, \mathscr{F})$. If one has three open coverings such that $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$, then

$$t^{\mathfrak{V}}_{\mathfrak{W}} \circ t^{\mathfrak{U}}_{\mathfrak{V}} = t^{\mathfrak{U}}_{\mathfrak{W}}.$$

Thus one can define the following equivalence relation $\sim$ on the disjoint union of the $H^1(\mathfrak{U}, \mathscr{F})$, where $\mathfrak{U}$ runs through all open coverings of X. Two cohomology classes $\xi \in H^1(\mathfrak{U}, \mathscr{F})$ and $\eta \in H^1(\mathfrak{U}', \mathscr{F})$ are defined to be equivalent, denoted $\xi \sim \eta$, if there exists an open covering $\mathfrak{V}$ with $\mathfrak{V} < \mathfrak{U}$ and $\mathfrak{V} < \mathfrak{U}'$ such that $t^{\mathfrak{U}}_{\mathfrak{V}}(\xi) = t^{\mathfrak{U}'}_{\mathfrak{V}}(\eta)$. The set of equivalence classes is the so-called *inductive limit* of the cohomology groups $H^1(\mathfrak{U}, \mathscr{F})$ and is called the 1st cohomology group of X with coefficients in the sheaf F. In symbols

$$H^1(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}) = \left(\dot{\bigcup_{\mathfrak{U}}} H^1(\mathfrak{U}, \mathscr{F})\right)\Big/ \sim .$$

Addition in $H^1(X, \mathscr{F})$ is defined by means of representatives as follows. Suppose the elements $x, y \in H^1(X, \mathscr{F})$ are represented by $\xi \in H^1(\mathfrak{U}, \mathscr{F})$ resp. $\eta \in H^1(\mathfrak{U}', \mathscr{F})$. Let $\mathfrak{V}$ be a common refinement of $\mathfrak{U}$ and $\mathfrak{U}'$. Then $x + y \in H^1(X, \mathscr{F})$ is defined to be the equivalence class of $t^{\mathfrak{U}}_{\mathfrak{V}}(\xi) + t^{\mathfrak{U}'}_{\mathfrak{V}}(\eta) \in H^1(\mathfrak{V}, \mathscr{F})$. One can easily check that this definition is independent of the various choices made and makes $H^1(X, \mathscr{F})$ into an abelian group. If $\mathscr{F}$ is a sheaf of vector spaces, then in a natural way $H^1(\mathfrak{U}, \mathscr{F})$ and $H^1(X, \mathscr{F})$ are also vector spaces.

From Lemma (12.4) it follows that for any open covering of X the canonical mapping

$$H^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is injective. In particular this implies that $H^1(X, \mathscr{F}) = 0$ precisely if $H^1(\mathfrak{U}, \mathscr{F}) = 0$ for every open covering $\mathfrak{U}$ of X.

12.6. Theorem. *Suppose X is a Riemann surface and $\mathscr{E}$ is the sheaf of differentiable functions on X. Then $H^1(X, \mathscr{E}) = 0$.*

PROOF. We give the proof under the assumption that X has a countable topology. However this assumption is always valid, see §23.

Suppose $\mathfrak{U} = (U_i)_{i \in I}$ is an arbitrary open covering of X. Then there is a partition of unity subordinate to U, i.e. a family $(\psi_i)_{i \in I}$ of functions $\psi_i \in \mathscr{E}(X)$ with the following properties (cf. the Appendix):

(i) $\operatorname{Supp}(\psi_i) \subset U_i$.

(ii) Every point of X has a neighborhood meeting only finitely many of the sets $\operatorname{Supp}(\psi_i)$.

(iii) $\sum_{i \in I} \psi_i = 1$.

We will show that $H^1(\mathfrak{U}, \mathscr{E}) = 0$, i.e., every cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{E})$ splits.

The function $\psi_j f_{ij}$, which is defined on $U_i \cap U_j$, may be differentiably extended to all of U_i by assigning it the value zero outside its support. Thus it may be considered as an element of $\mathscr{E}(U_i)$. Set

$$g_i := \sum_{j \in I} \psi_j f_{ij}.$$

Because of (ii), in a neighborhood of any point in U_i, this sum has only finitely many terms which are not zero and thus defines an element $g_i \in \mathscr{E}(U_i)$. For $i, j \in I$

$$\begin{aligned} g_i - g_j &= \sum_{k \in I} \psi_k f_{ik} - \sum_{k \in I} \psi_k f_{jk} = \sum_k \psi_k (f_{ik} - f_{jk}) \\ &= \sum_k \psi_k (f_{ik} + f_{kj}) = \sum_k \psi_k f_{ij} = f_{ij} \end{aligned}$$

on $U_i \cap U_j$ and thus (f_{ij}) is a coboundary. □

Remark. In exactly the same way one can show that on a Riemann surface X the 1st cohomology groups with coefficients in the sheaves $\mathscr{E}^{(1)}, \mathscr{E}^{1,0}, \mathscr{E}^{0,1}$ and $\mathscr{E}^{(2)}$ also vanish.

12.7. Theorem. *Suppose X is a simply connected Riemann surface. Then*

(a) $H^1(X, \mathbb{C}) = 0$,

(b) $H^1(X, \mathbb{Z}) = 0$.

Here $\mathbb{C}$ (resp. $\mathbb{Z}$) denotes the sheaf of locally constant functions with values in the complex numbers (resp. integers), cf. (6.4.e).

PROOF

(a) Suppose $\mathfrak{U}$ is an open covering of X and $(c_{ij}) \in Z^1(\mathfrak{U}, \mathbb{C})$. Since $Z^1(\mathfrak{U}, \mathbb{C}) \subset Z^1(\mathfrak{U}, \mathscr{E})$ and $H^1(\mathfrak{U}, \mathscr{E}) = 0$, there exists a cochain $(f_i) \in C^0(\mathfrak{U}, \mathscr{E})$ such that

$$c_{ij} = f_i - f_j \quad \text{on } U_i \cap U_j.$$

But $dc_{ij} = 0$ implies $df_i = df_j$ on $U_i \cap U_j$, and thus there exists a global differential form $\omega \in \mathscr{E}^{(1)}(X)$ such that $\omega \mid U_i = df_i$. Since $ddf_i = 0$, it follows that ω is closed. Because X is simply connected, by (10.7) there exists $f \in \mathscr{E}(X)$ such that $df = \omega$. Set

$$c_i := f_i - f \mid U_i.$$

Since $dc_i = df_i - df = \omega - \omega = 0$ on U_i, c_i is locally constant, i.e., $(c_i) \in C^0(\mathfrak{U}, \mathbb{C})$. On $U_i \cap U_j$ one has

$$c_{ij} = f_i - f_j = (f_i - f) - (f_j - f) = c_i - c_j,$$

and thus the cocycle (c_{ij}) splits.

(b) Suppose $(a_{jk}) \in Z^1(\mathfrak{U}, \mathbb{Z})$. By (a) there exists a cochain $(c_j) \in C^0(\mathfrak{U}, \mathbb{C})$ such that

$$a_{jk} = c_j - c_k \quad \text{on } U_j \cap U_k.$$

Since $\exp(2\pi i a_{jk}) = 1$, one has $\exp(2\pi i c_j) = \exp(2\pi i c_k)$ on the intersection $U_j \cap U_k$. Since X is connected, there exists a constant $b \in \mathbb{C}^*$ such that

$$b = \exp(2\pi i c_j) \quad \text{for every } j \in I.$$

Choose $c \in \mathbb{C}$ such that $\exp(2\pi i c) = b$ and let

$$a_j := c_j - c.$$

Since $\exp(2\pi i a_j) = \exp(2\pi i c_j)\exp(-2\pi i c) = 1$, it follows that a_j is an integer, i.e., $(a_j) \in C^0(\mathfrak{U}, \mathbb{Z})$. Moreover

$$a_{jk} = c_j - c_k = (c_j - c) - (c_k - c) = a_j - a_k,$$

i.e., the cocycle (a_{jk}) lies in $B^1(\mathfrak{U}, \mathbb{Z})$. □

The next theorem shows that in certain cases one can calculate $H^1(X, \mathscr{F})$ using only a single covering of X.

12.8. Theorem (Leray). *Suppose F is a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ is an open covering of X such that $H^1(U_i, \mathscr{F}) = 0$ for every $i \in I$. Then*

$$H^1(X, \mathscr{F}) \cong H^1(\mathfrak{U}, \mathscr{F}).$$

Such a $\mathfrak{U}$ is called a *Leray covering* (of 1st order) for the sheaf $\mathscr{F}$.

PROOF. It suffices to show that, for every open covering $\mathfrak{V} = (V_\alpha)_{\alpha \in A}$, with $\mathfrak{V} < \mathfrak{U}$, the mapping $t^{\mathfrak{U}}_{\mathfrak{V}}: H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is an isomorphism. From (12.4) this mapping is injective.

Suppose $\tau: A \to I$ is a refining mapping with $V_\alpha \subset U_{\tau\alpha}$ for every $\alpha \in A$. To prove the surjectivity of $t^{\mathfrak{U}}_{\mathfrak{V}}$, we must show that given any cocycle $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathscr{F})$, there exists a cocycle $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ such that the cocycle

$$(F_{\tau\alpha, \tau\beta}) - (f_{\alpha\beta})$$

is cohomologous to zero relative to the covering $\mathfrak{V}$. Now the family $(U_i \cap V_\alpha)_{\alpha \in A}$ is an open covering of U_i which we denote by $U_i \cap \mathfrak{V}$. By assumption $H^1(U_i \cap \mathfrak{V}, \mathscr{F}) = 0$, i.e., there exist $g_{i\alpha} \in \mathscr{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{on } U_i \cap V_\alpha \cap V_\beta.$$

Now on the intersection $U_i \cap U_j \cap V_\alpha \cap V_\beta$ one has

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}$$

and thus by sheaf axiom II there exist elements $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

Clearly, (F_{ij}) satisfies the cocycle relation and thus lies in $Z^1(\mathfrak{U}, \mathscr{F})$. Let $h_\alpha := g_{\tau\alpha, \alpha} | V_\alpha \in \mathscr{F}(V_\alpha)$. Then on $V_\alpha \cap V_\beta$ one has

$$\begin{aligned} F_{\tau\alpha, \tau\beta} - f_{\alpha\beta} &= (g_{\tau\beta, \alpha} - g_{\tau\alpha, \alpha}) - (g_{\tau\beta, \alpha} - g_{\tau\beta, \beta}) \\ &= g_{\tau\beta, \beta} - g_{\tau\alpha, \alpha} = h_\beta - h_\alpha, \end{aligned}$$

and thus $(F_{\tau\alpha, \tau\beta}) - (f_{\alpha\beta})$ splits. □

12.9. Example. As an application of Leray's Theorem, we will show

$$H^1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}.$$

Let $U_1 := \mathbb{C}^* \backslash \mathbb{R}_-$ and $U_2 := \mathbb{C}^* \backslash \mathbb{R}_+$, where $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the positive and negative real axes respectively. Then $\mathfrak{U} = (U_1, U_2)$ is an open covering of $\mathbb{C}^*$. By (12.7) $H^1(U_i, \mathbb{Z}) = 0$ since U_i is star-shaped and thus simply connected. Thus $H^1(\mathbb{C}^*, \mathbb{Z}) = H^1(\mathfrak{U}, \mathbb{Z})$.

Since any cocycle $(a_{ij}) \in Z^1(\mathfrak{U}, \mathbb{Z})$ is alternating, i.e., $a_{ii} = 0$ and $a_{ij} = -a_{ji}$, it is completely determined by a_{12} and thus $Z^1(\mathfrak{U}, \mathbb{Z}) \cong \mathbb{Z}(U_1 \cap U_2)$. But the intersection $U_1 \cap U_2$ has two connected components, namely the upper and lower half planes, and thus $\mathbb{Z}(U_1 \cap U_2) \cong \mathbb{Z} \times \mathbb{Z}$. Since U_i is connected, $\mathbb{Z}(U_i) \cong \mathbb{Z}$ and hence $C^0(\mathfrak{U}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. The coboundary operator $\delta: C^0(\mathfrak{U}, \mathbb{Z}) \to Z^1(\mathfrak{U}, \mathbb{Z})$ is given with respect to these isomorphisms by

$$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, \qquad (b_1, b_2) \mapsto (b_2 - b_1, b_2 - b_1).$$

Thus the coboundaries are exactly the subgroup $B \subset \mathbb{Z} \times \mathbb{Z}$ of those elements (a_1, a_2) with $a_1 = a_2$. Hence $H^1(\mathfrak{U}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}/B \cong \mathbb{Z}$.

Similarly one can show $H^1(\mathbb{C}^*, \mathbb{C}) \cong \mathbb{C}$.

12.10. The Zeroth Cohomology Group. Suppose $\mathscr{F}$ is a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ is an open covering of X. Set

$$Z^0(\mathfrak{U}, \mathscr{F}) := \operatorname{Ker}(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F})),$$
$$B^0(\mathfrak{U}, \mathscr{F}) := 0,$$
$$H^0(\mathfrak{U}, \mathscr{F}) := Z^0(\mathfrak{U}, \mathscr{F})/B^0(\mathfrak{U}, \mathscr{F}) = Z^0(\mathfrak{U}, \mathscr{F}).$$

From the definition of δ it follows that a 0-cochain $(f_i) \in C^0(\mathfrak{U}, \mathscr{F})$ belongs to $Z^0(\mathfrak{U}, \mathscr{F})$ precisely if $f_i | U_i \cap U_j = f_j | U_i \cap U_j$ for every $i, j \in I$. By sheaf axiom II the elements f_i piece together to give a global element $f \in \mathscr{F}(X)$ and there is a natural isomorphism

$$H^0(\mathfrak{U}, \mathscr{F}) = Z^0(\mathfrak{U}, \mathscr{F}) \cong \mathscr{F}(X).$$

Thus the groups $H^0(\mathfrak{U}, \mathscr{F})$ are entirely independent of the covering $\mathfrak{U}$ and one can define

$$H^0(X, \mathscr{F}) := \mathscr{F}(X).$$

Exercises (§12)

12.1. Suppose $p_1, \ldots, p_n$ are distinct points of $\mathbb{C}$ and let

$$X := \mathbb{C} \setminus \{p_1, \ldots, p_n\}.$$

Prove

$$H^1(X, \mathbb{Z}) \cong \mathbb{Z}^n.$$

[*Hint*: Construct a covering $\mathfrak{U} = (U_1, U_2)$ of X such that U_1 and U_2 are connected and simply connected and $U_1 \cap U_2$ has $n + 1$ connected components.]

12.2. (a) Let X be a manifold, $U \subset X$ open and $V \Subset U$. Show that V meets only a finite number of connected components of U.

(b) Let X be a compact manifold and $\mathfrak{U} = (U_i)_{i \in I}$, $\mathfrak{V} = (V_i)_{i \in I}$ be two finite open coverings of X such that $V_i \Subset U_i$ for every $i \in I$. Prove that

$$\operatorname{Im}(Z^1(\mathfrak{U}, \mathbb{C}) \to Z^1(\mathfrak{V}, \mathbb{C}))$$

is a finite-dimensional vector space.

(c) Let X be a compact Riemann surface. Prove that $H^1(X, \mathbb{C})$ is a finite-dimensional vector space.

[*Hint*: Use finite coverings $\mathfrak{U} = (U_i)$, $\mathfrak{V} = (V_i)$ of X with $V_i \Subset U_i$, such that all the U_i and V_i are isomorphic to disks.]

12.3. (a) Let X be a compact Riemann surface. Prove that the map

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C}),$$

induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$, is injective.

(b) Let X be a compact Riemann surface. Show that $H^1(X, \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module.

[*Hint*: Show first, as in Ex. 12.2.c), that $H^1(X, \mathbb{Z})$ is finitely generated and then use 12.3.a) to prove that $H^1(X, \mathbb{Z})$ is free.]

§13. Dolbeault's Lemma

In this section we solve the inhomogeneous Cauchy–Riemann differential equation $(\partial f/\partial \bar{z}) = g$, where g is a given differentiable function on the disk X. This is then used to show that the cohomology group $H^1(X, \mathcal{O})$ vanishes.

13.1. Lemma. *Suppose $g \in \mathcal{E}(\mathbb{C})$ has compact support. Then there exists a function $f \in \mathcal{E}(\mathbb{C})$ such that*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

PROOF. Define the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(\zeta) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{z - \zeta} \, dz \wedge d\bar{z}.$$

Since the integrand has a singularity when $z = \zeta$, one has to show that the integral exists and depends differentiably on ζ. The simplest way to do this is to change variables by translation and then introduce polar coordinates r, θ, namely let

$$z = \zeta + re^{i\theta}.$$

With regard to the integration ζ is a constant and one has

$$dz \wedge d\bar{z} = -2i \, dx \wedge dy = -2ir \, dr \wedge d\theta.$$

Thus

$$f(\zeta) = -\frac{1}{\pi} \iint \frac{g(\zeta + re^{i\theta})}{re^{i\theta}} r \, dr \, d\theta$$

$$= -\frac{1}{\pi} \iint g(\zeta + re^{i\theta}) e^{-i\theta} \, dr \, d\theta.$$

Since g has compact support, one has only to integrate over a rectangle $0 \le r \le R$, $0 \le \theta \le 2\pi$, provided R is chosen sufficiently large. One may differentiate under the integral sign, i.e., $f \in \mathcal{E}(\mathbb{C})$ and

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = -\frac{1}{\pi} \iint \frac{\partial g(\zeta + re^{i\theta})}{\partial \bar{\zeta}} e^{-i\theta} \, dr \, d\theta.$$

Changing back to the original coordinates, one has

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \iint\limits_{B_\varepsilon} \frac{\partial g(\zeta + z)}{\partial \bar{\zeta}} \frac{1}{z} \, dz \wedge d\bar{z},$$

where $B_\varepsilon := \{z \in \mathbb{C} : \varepsilon \leq |z| \leq R\}$. Since

$$\frac{\partial g(\zeta + z)}{\partial \bar{\zeta}} \frac{1}{z} = \frac{\partial g(\zeta + z)}{\partial \bar{z}} \frac{1}{z} = \frac{\partial}{\partial \bar{z}} \left(\frac{g(\zeta + z)}{z} \right)$$

for $z \neq 0$, one has

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \iint\limits_{B_\varepsilon} \frac{\partial}{\partial \bar{z}} \left(\frac{g(\zeta + z)}{z} \right) dz \wedge d\bar{z} = -\lim_{\varepsilon \to 0} \iint\limits_{B_\varepsilon} d\omega,$$

where the differential form ω is given by

$$\omega(z) = \frac{1}{2\pi i} \frac{g(\zeta + z)}{z} \, dz$$

(here one considers z as a variable and ζ as a constant). By Stokes' Theorem

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = -\lim_{\varepsilon \to 0} \iint\limits_{B_\varepsilon} d\omega = -\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \omega = \lim_{\varepsilon \to 0} \int_{|z| = \varepsilon} \omega.$$

Parametrizing the circle $|z| = \varepsilon$ by $z = \varepsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$, one gets

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} g(\zeta + \varepsilon e^{i\theta}) \, d\theta.$$

Now the integral gives the average value of the function g over the circle $\zeta + \varepsilon e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Since g is continuous, this converges to $g(\zeta)$ as $\varepsilon \to 0$, i.e.,

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = g(\zeta).$$ □

The next theorem shows that one may drop the assumption that g has compact support.

13.2. Theorem. *Suppose $X := \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$, and $g \in \mathscr{E}(X)$. Then there exists $f \in \mathscr{E}(X)$ such that*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

This theorem is a special case of the so-called Dolbeault Lemma in several complex variables, see [32].

PROOF. In this case a solution cannot simply be given as an integral as in (13.1), for the integral will not converge in general. For this reason we use an exhaustion process which allows (13.1) to be applied in the present setting.

Suppose $0 < R_0 < R_1 < \cdots < R_n$ is a sequence of radii such that $\lim_{n\to\infty} R_n = R$ and set

$$X_n := \{z \in \mathbb{C}\colon |z| < R_n\}.$$

There exist functions $\psi_n \in \mathscr{E}(X)$ with compact supports $\operatorname{Supp}(\psi_n) \subset X_{n+1}$ and $\psi_n | X_n = 1$. The functions $\psi_n g$ vanish outside X_{n+1} and thus if one extends them by zero, they become functions on $\mathbb{C}$. By (13.1) there exist functions $f_n \in \mathscr{E}(X)$ such that

$$\bar\partial f_n = \psi_n g \quad \text{on } X.$$

Here and in the following we use the abbreviation $\bar\partial := (\partial/\partial\bar z)$.

By induction we alter the sequence (f_n) to another sequence $(\tilde f_n)$, which for all $n \geq 1$ satisfies

(i) $\bar\partial \tilde f_n = g \quad \text{on } X_n$,
(ii) $\|\tilde f_{n+1} - \tilde f_n\|_{X_{n-1}} \leq 2^{-n}$.

(As usual let $\|f\|_K := \sup_{x\in K} |f(x)|$ denote the supremum norm.) Set $\tilde f_1 := f_1$. Suppose $\tilde f_1, \ldots, \tilde f_n$ are already constructed. Then

$$\bar\partial(f_{n+1} - \tilde f_n) = 0 \quad \text{on } X_n,$$

and thus $f_{n+1} - \tilde f_n$ is holomorphic on X_n. Hence there exists a polynomial P (e.g., a finite number of terms of the Taylor series of $f_{n+1} - \tilde f_n$) such that

$$\|f_{n+1} - \tilde f_n - P\|_{X_{n-1}} \leq 2^{-n}.$$

If we set $\tilde f_{n+1} := f_{n+1} - P$, then (ii) is satisfied. Moreover, on X_{n+1} one has

$$\bar\partial \tilde f_{n+1} = \bar\partial f_{n+1} - \bar\partial P = \bar\partial f_{n+1} = \psi_{n+1} g = g,$$

i.e., (i) also holds. Since every point $z \in X$ is contained in almost all X_n, the limit

$$f(z) := \lim_{n\to\infty} \tilde f_n(z)$$

exists. On X_n one may write

$$f = \tilde f_n + \sum_{k=n}^{\infty} (\tilde f_{k+1} - \tilde f_k).$$

For $k \geq n$, the functions $\tilde f_{k+1} - \tilde f_k$ are holomorphic on X_n, since $\bar\partial(\tilde f_{k+1} - \tilde f_k) = 0$.

Because of (ii), the series

$$F_n := \sum_{k=n}^{\infty} (\tilde f_{k+1} - \tilde f_k)$$

converges uniformly on X_n and is thus holomorphic there. Hence $f = \tilde{f}_n + F_n$ is infinitely differentiably on X_n for every n and thus $f \in \mathscr{E}(X)$. As well

$$\bar{\partial} f = \bar{\partial} \tilde{f}_n = g \quad \text{on } X_n$$

for every n and thus $\bar{\partial} f = g$ on all of X. □

Remark. Naturally the solution of the equation $\bar{\partial} f = g$ is not uniquely determined, only up to the addition of an arbitrary holomorphic function.

13.3. Corollary. *Suppose $X := \{z \in \mathbb{C}: |z| < R\}$, $0 < R \leq \infty$. Then given any $g \in \mathscr{E}(X)$, there exists $f \in \mathscr{E}(X)$ such that $\Delta f = g$.*

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \, \partial \bar{z}}$$

is the Laplace operator.

PROOF. Choose $f_1 \in \mathscr{E}(X)$ such that $\bar{\partial} f_1 = g$ and $f_2 \in \mathscr{E}(X)$ such that $\bar{\partial} f_2 = \bar{f}_1$. Then $f := \frac{1}{4}\bar{f}_2$ satisfies $\Delta f = g$, for

$$\Delta f = \frac{\partial^2 \bar{f}_2}{\partial z \, \partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial \bar{f}_2}{\partial z}\right) = \frac{\partial}{\partial \bar{z}}\left(\overline{\frac{\partial f_2}{\partial \bar{z}}}\right) = \frac{\partial f_1}{\partial \bar{z}} = g. \qquad \square$$

13.4. Theorem. *Suppose $X := \{z \in \mathbb{C}: |z| < R\}$, $0 < R \leq \infty$. Then $H^1(X, \mathcal{O}) = 0$.*

PROOF. Suppose $\mathfrak{U} = (U_i)$ is an open covering of X and $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ is a cocycle. Since $Z^1(\mathfrak{U}, \mathcal{O}) \subset Z^1(\mathfrak{U}, \mathscr{E})$ and $H^1(X, \mathscr{E}) = 0$, there exists a cochain $(g_i) \in C^0(\mathfrak{U}, \mathscr{E})$ such that

$$f_{ij} = g_i - g_j \quad \text{on } U_i \cap U_j.$$

Since $\bar{\partial} f_{ij} = 0$, one has $\bar{\partial} g_i = \bar{\partial} g_j$ on $U_i \cap U_j$ and thus there exists a global function $h \in \mathscr{E}(X)$ with $h | U_i = \bar{\partial} g_i$. By (13.2) we can find a function $g \in \mathscr{E}(X)$ such that $\bar{\partial} g = h$. Define

$$f_i := g_i - g.$$

Now f_i is holomorphic, since $\bar{\partial} f_i = \bar{\partial} g_i - \bar{\partial} g = 0$, and thus $(f_i) \in C^0(\mathfrak{U}, \mathcal{O})$. As well on $U_i \cap U_j$ one has

$$f_i - f_j = g_i - g_j = f_{ij},$$

i.e., the cocycle (f_{ij}) splits. □

13.5. Theorem. *For the Riemann sphere $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.*

PROOF. Set $U_1 := \mathbb{P}^1\backslash\infty$ and $U_2 := \mathbb{P}^1\backslash 0$. Since $U_1 = \mathbb{C}$ and U_2 is biholomorphic to $\mathbb{C}$, it follows from (13.4) that $H^1(U_i, \mathcal{O}) = 0$. Thus $\mathfrak{U} = (U_1, U_2)$ is a Leray covering of $\mathbb{P}^1$ and $H^1(\mathbb{P}^1, \mathcal{O}) = H^1(\mathfrak{U}, \mathcal{O})$ by (12.8). Thus the proof is complete once one shows that every cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ splits. In order to do this, it is clearly enough to find functions $f_i \in \mathcal{O}(U_i)$ such that

$$f_{12} = f_1 - f_2 \quad \text{on } U_1 \cap U_2 = \mathbb{C}^*.$$

Let

$$f_{12}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent expansion of f_{12} on $\mathbb{C}^*$. Set

$$f_1(z) := \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad f_2(z) := -\sum_{n=-\infty}^{-1} c_n z^n.$$

Then $f_i \in \mathcal{O}(U_i)$ and $f_1 - f_2 = f_{12}$. □

EXERCISES (§13)

13.1. Let $X = \{z \in \mathbb{C}: |z| < R\}$, where $0 < R \leq \infty$. Denote by $\mathscr{H}$ the sheaf of harmonic functions on X, i.e.

$$\mathscr{H}(U) = \{f: U \to \mathbb{C} : f \text{ is harmonic}\}$$

for $U \subset X$ open. Prove

$$H^1(X, \mathscr{H}) = 0.$$

13.2. (a) Show that $\mathfrak{U} = (\mathbb{P}^1\backslash\infty, \mathbb{P}^1\backslash 0)$ is a Leray covering for the sheaf Ω of holomorphic 1-forms on $\mathbb{P}^1$.

(b) Prove that

$$H^1(\mathbb{P}^1, \Omega) \cong H^1(\mathfrak{U}, \Omega) \cong \mathbb{C}$$

and that the cohomology class of

$$\frac{dz}{z} \in \Omega(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \Omega)$$

is a basis of $H^1(\mathbb{P}^1, \Omega)$.

13.3. Suppose $g \in \mathscr{E}(\mathbb{C})$ is a function with compact support. Prove that there is a solution $f \in \mathscr{E}(\mathbb{C})$ of the equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

having compact support if and only if

$$\iint_{\mathbb{C}} z^n g(z)\, dz \wedge d\bar{z} = 0 \quad \text{for every } n \in \mathbb{N}.$$