Revisiting Hamiltonian Decomposition of the Hypercube

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Abstract

A hypercube or binary n-cube is an interconnection network very suitable for implementing computing elements. In this paper we study a useful namely the Hamiltonian decomposition, i.e. the partitioning of its edge set into Hamiltonian cycles. It is known that there are $\lfloor n/2 \rfloor$ disjoint Hamiltonian cycles on a binary n-cube. The proof of this result, however, does not give rise to any simple construction algorithm of such cycles. In a previous work Song presents ideas towards a simple method to this problem. First decompose the hypercube into cycles of length 16, C_{16}, and then apply a merge operator to join the C_{16} cycles into larger Hamiltonian cycles. The case of dimension $n = 6$ (a 64-node hypercube) is illustrated. He conjectures the method can be generalized for any even n. In this paper, we generalize the first phase of that method for any even n and prove its correctness. Also we show four possible merge operators for the case of $n = 8$ (a 256-node hypercube). This result can be viewed as a step toward the general merge operator, thus proving the conjecture.

1 Introduction

A hypercube is an interconnection network very suitable for connecting computing elements. In this paper we study an interesting property namely the Hamiltonian decomposition. Many results on the existence of Hamiltonian cycles in graphs are known [1, 3, 4, 5, 6]. When an application uses processing elements joined as a cycle, it is important to know alternative cycles in case of communication failure in one cycle [2]. It is desirable to have a simple algorithm to construct the alternative cycles.

It is known that there are $\lfloor n/2 \rfloor$ disjoint Hamiltonian cycles on a hypercube of dimension n [1]. The proof of this result, however, does not give rise to any simple construction algorithm of such cycles. In [7] Song presents ideas towards a simple and interesting method to this problem. Two phases are involved. (1) Decompose the hypercube into C_{16} (cycles of length 16) and then (2) apply a merge operator to join the obtained C_{16} cycles into larger cycles. The case of dimension $n = 6$ (a 64-node hypercube) was illustrated in [7] where he conjectured this method can be generalized for any even n.

In this paper, we generalize the first phase of that method for any even n and prove its correctness. Also we show a merge operator for the case of $n = 8$ (a 256-node hypercube).

This result can be viewed as a step toward the general merge operator, thus proving the conjecture.

2 Hamiltonian decomposition

The Hamiltonian decomposition of a graph is the partitioning of its edge set into Hamiltonian cycles. Consider a binary n-cube or hypercube of dimension n. For simplicity and without loss of generality, consider n to be even. (If n is odd, the edge set can be partitioned into $(n-1)/2$ Hamiltonian cycles and a perfect matching [1].) Observe first that the binary n-cube is equivalent to a 4-ary $n/2$-cube, that is the Cartesian product of $n/2$ cycles of length 4: $C_4 \times C_4 \times \cdots \times C_4$. We start with the following theorem (see [1] for details and proof).

Theorem 1 *The binary n-cube with even n, or equivalently the product of $n/2$ cycles, $C_4 \times C_4 \times \cdots \times C_4$, can be partitioned into $n/2$ Hamiltonian cycles.*

In a previous work [7] Song suggested ideas for a very simple method to construct the disjoint Hamiltonian cycle of a binary n-cube. He illustrated the method for the case of $n = 6$. In the following we summarize this method. It consists of two phases.

1. partition the edge set into cycles of length 16 or C_{16}.
2. merge the resulting cycles into larger cycles to get the desired Hamiltonian cycles. (This second phase is realized by using a merge operator to be seen later.)

2.1 Phase 1 – decomposition into C_{16}

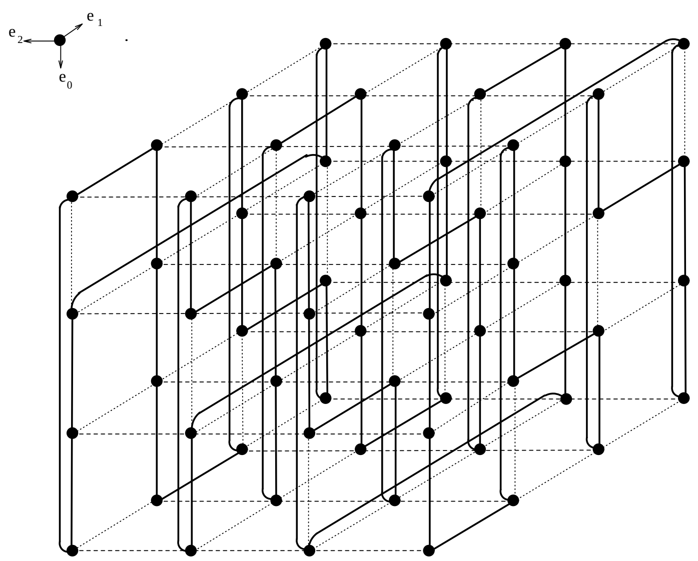

Figure 1: Cycles of color 0

Phase 1 decomposes the n-cube into cycles of length 16, or C_{16}. Foregger [5] gave a solution for the case of $n = 4$, i.e. $C_4 \times C_4$ is decomposed into two C_{16}. For the case $n = 6$, phase 1 consists of decomposing $C_4 \times C_4 \times C_4$ into 12 C_{16}. This decomposition is done as follows.

Divide the 12 cycles into three groups: 4 cycles of color 0, 4 cycles of color 1, and 4 cycles of color 2. In the next section the cycles C_{16} of same color will be merged to form a Hamiltonian cycle.

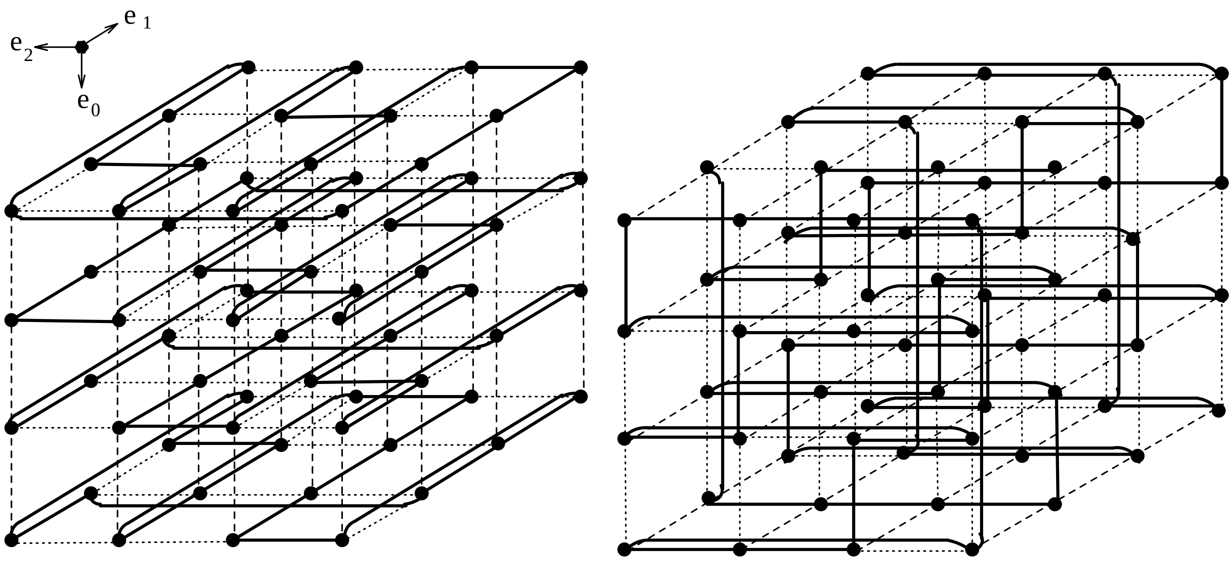

Figure 2: Cycles of color 1 (left) and of color 2 (right)

In Figure 1 we illustrate the 4 cycles of color 0. Notice that the cycles are situated on planes parallel to the plane defined by the axes e_0 and e_1. This observation will be formalized in the next section. In Figure 2 we have the cycles of color 1 and color 2. Notice the symmetries between cycles of this figure and those of Figure 1.

The edge set of the product $C_4 \times C_4 \times \cdots \times C_4$ ($n/2$ times) can be partitioned into $n2^n/32$ disjoint cycles of length 16, $2^n/16$ cycles of the same color.

2.2 Phase 2 – the merge operator

We show how the 12 cycles of length 16 of the previous section can be merged into 3 Hamiltonian cycles. Cycles of the same color will be merged together to form one Hamiltonian cycle.

Consider a vertex and the edges incident with it. An edge permutation operator is an operator that permutes the colors of the edges. We use a set of edge permutation operators to merge cycles of a given color to form a large cycle of the same color.

Definition 1 *A set of edge permutation operators is a (cycle) merge operator if it transforms a partition of the edge set of* r *cycles to a partition of the edge set of* s *cycles* ($s < r$).

Consider the partition of the edges of $C_4 \times C_4 \times C_4$ into C_{16} as before. Figure 3 shows a merge operator that joins two cycles of each color into a large cycle of the same color. Point **A** is a reference point for the application of the merge operator. Figure 4 shows the effect of applying the merge operator. For each color the curves (i.e. the additional part not present in Figure 3) indicate the remaining of the C_{16} cycle. On the left of Figure 5 we have the positions (indicated by larger circles) of possible reference points **A** to apply the merge operator, according to Figure 3. On the right of the same figure we have the three points chosen in [7] resulting in 3 Hamiltonian cycles of length 64, one for each color 0, 1 and 2.

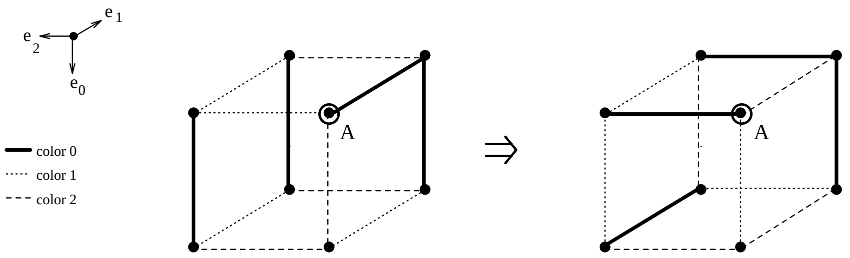

Figure 3: Merge operator

3 Generalization of phase 1

In the following we formalize phase 1, the decomposition of the edge set of a binary n-cube, for any even n, into cycles C_{16}. Let $n = 2m$ and H_m a binary n-cube.

$H_m = \underbrace{C_4 \times C_4 \times \cdots \times C_4}_{m\ times}$.

Consider a vertex x of H_m. Let $x = (x_0, x_1, \cdots, x_{m-1})$, where $x_i \in \{0, 1, 2, 3\}$ for $0 \leq i \leq m-1$. Denote by $x(i)$ the edge joining $x = (x_0, x_1, \cdots, x_i, \cdots, x_{m-1})$ and $x' = (x_0, x_1, \cdots, x_i + 1\ (mod\ 4), \cdots, x_{m-1})$.

Definition 2 *The initial coloring of the edge set of a binary n-cube is defined as follows. Color each edge $x(i)$ with color i, $\forall x \in H_m$ and $0 \leq i \leq m-1$.*

Thus each edge parallel to axis e_i is colored by color i with the initial coloring.

Definition 3 *The shift operator is an operator that, applied to a vertex $x \in H_m$, defines the colors of half of the edges incident with x in the following manner: each $x(i)$ is colored with color $(i-1)\ (mod\ m)$, for $0 \leq i \leq m-1$.*

The above operator is named *shift operator* because, given a vertex x and an initial coloring of the edges, the application of the shift operator to x has the effect of *shifting* the colors. See Figure 6 for $n = 6$. (Note however that the shift operator does not really shift the edge colors but always gives the same coloring as defined, independent of the initial colors before its application.)

Thus, given an initial coloring of the edges, by applying the shift operator at some points, say x, of H_m, edges $x(i+1\ (mod\ m))$ parallel to axis $e_{i+1\ (mod\ m)}$ will have color i but all the other edges also with color i are parallel to e_i. Furthermore, after applying shift, any path formed by edges of color i only lies within a $H_2 = C_4 \times C_4$ parallel to the plane defined by axes e_i and $e_{i+1\ (mod\ m)}$. This observation takes us to the following definition.

Definition 4 *For each $0 \leq i \leq m-1$, $H_2(i)$ is a H_2 parallel to the plane defined by axes e_i and $e_{i+1\ (mod\ m)}$.*

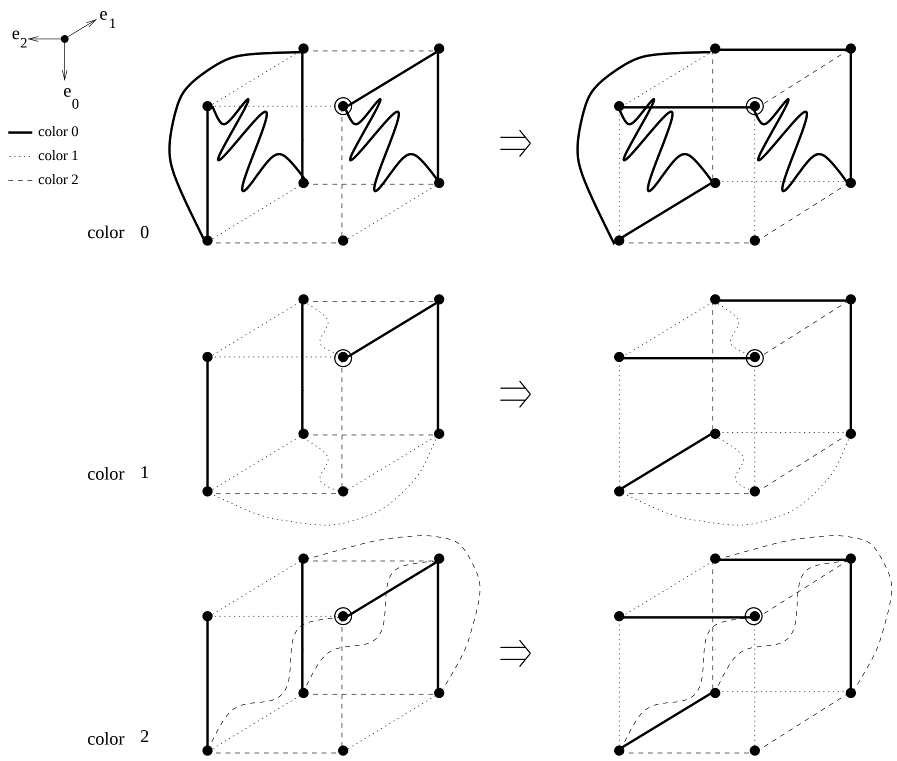

Figure 4: The effect of the merge operator on colors 0, 1 and 2

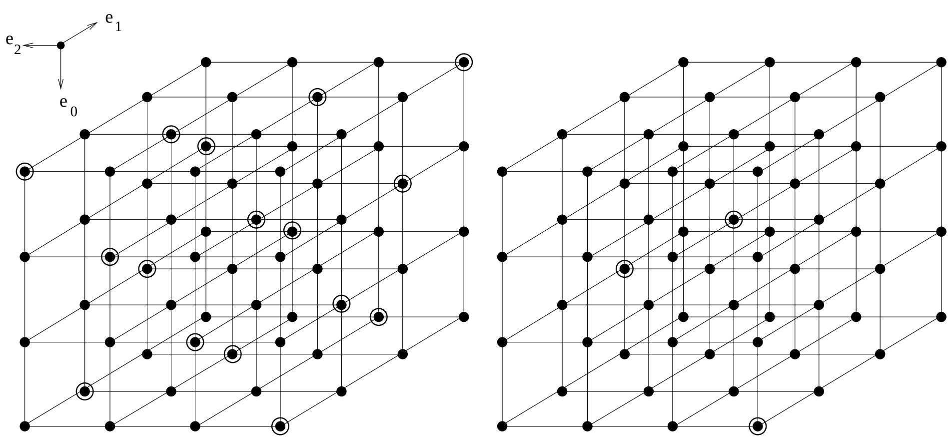

Figure 5: Candidate points to apply the merge operator (left) and the three chosen points (right)

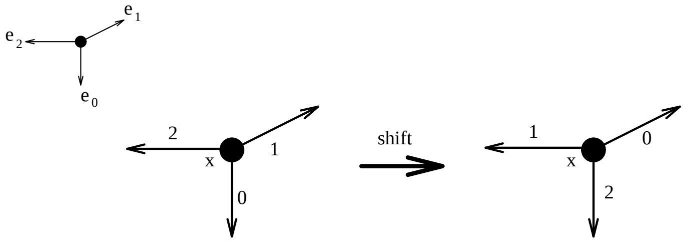

Figure 6: The shift operator for the case $n = 6$

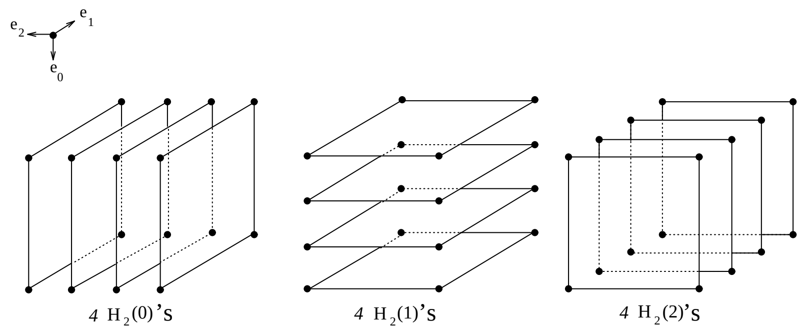

Figure 7: The 12 H_2's for H_3

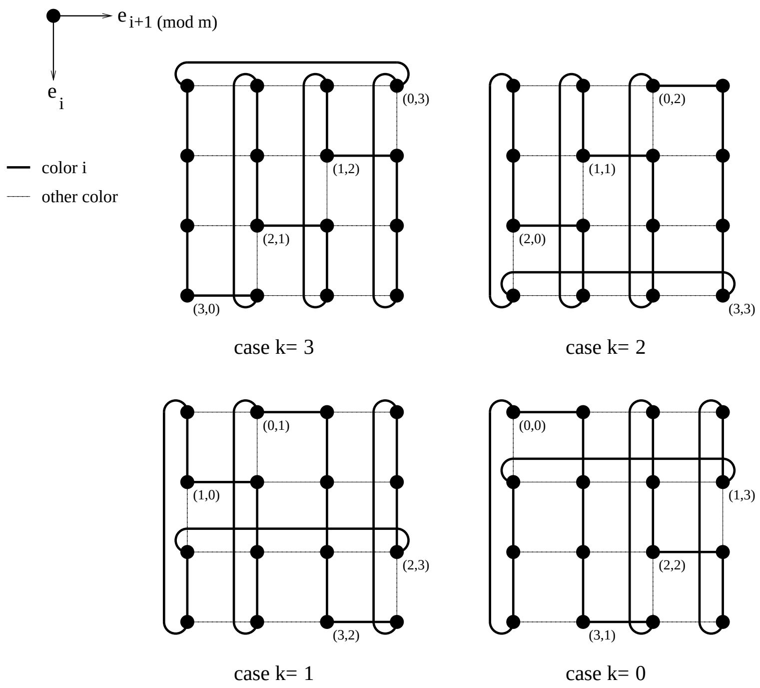

Figure 8: Illustration of Lemma 1

For example, for $n = 6$, $H_3 = C_4 \times C_4 \times C_4$ has 12 $H_2(i)$'s, with 4 $H_2(i)$'s for each $i = 0, 1, 2$ (see Figure 7). A vertex $x = (x_0, x_1, \cdots, x_i, x_{i+1\ (mod\ m)}, \cdots, x_{m-1})$ of $H_2(i)$ have constant coordinates except x_i and $x_{i+1\ (mod\ m)}$. It is therefore of the following type:

$x =$(constant, $\cdots$, constant, $x_i, x_{i+1\ (mod\ m)}$, constant, $\cdots$, constant)

To simplify the notation, we omit the $(m-2)$ constant coordinates of $H_2(i)$ and write $x =(x_i, x_{i+1\ (mod\ m)})$. The following lemma will be used in the proof of Theorem 2.

Lemma 1 *Consider a $H_2(i)$, for a fixed i, $0 \le i \le m-1$, a fixed $k \in \{0, 1, 2, 3\}$, and an initial coloring. Apply shift in this $H_2(i)$ to all $x = (x_i, x_{i+1\ (mod\ m)})$ such that $x_i + x_{i+1\ (mod\ m)} = k\ (mod\ 4)$. Then the edges of color i in this $H_2(i)$ form a C_{16}.*

Proof. The proof is straightforward. We consider all possible cases for k $(k = 0, 1, 2, 3)$ and obtain Figure 8. The lemma holds. □

The next theorem generalizes the decomposition in cycles C_{16} needed in phase 1 of the method of [7].

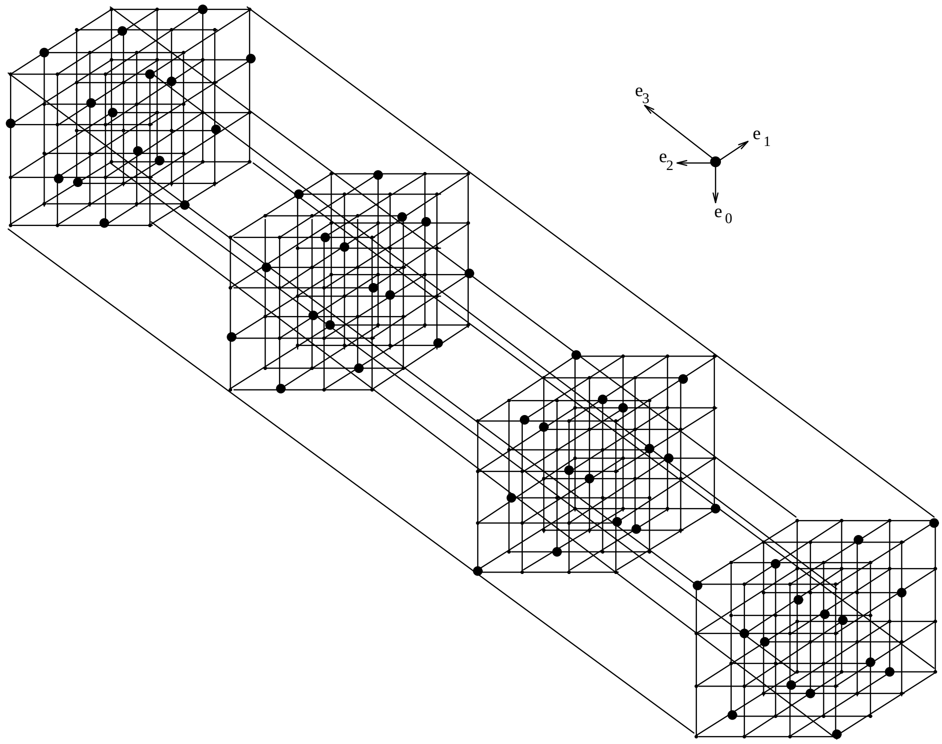

Figure 9: Points to apply the shift operator for H_4

Theorem 2 *Consider a binary n-cube H_m ($n = 2m$), with the initial coloring of its edges, and a fixed $K \in \{0, 1, 2, 3\}$. Apply the shift operator to all vertices $x = (x_0, x_1, \cdots, x_{m-1})$ such that $x_0 + x_1 + \cdots + x_{m-1} = K$ (mod 4). Then H_m is decomposed into cycles C_{16}, i.e., any edge is part of a cycle C_{16}.*

Proof. Consider any edge of color i, $0 \leq i \leq m-1$. It belongs to some $H_2(i)$. In this $H_2(i)$, consider the vertices at which the shift operator has been applied. As the $(m-2)$ vertex coordinates of this $H_2(i)$ are constant, the difference between K and the sum of these constants will also be constant. Call this difference d. Then the shift operator was applied at vertices x such that $x_i + x_{i+1 \ (mod\ m)} = d \ (mod\ 4)$ in this $H_2(i)$. This is exactly the situation described in Lemma 1. Therefore in this $H_2(i)$ this edge must be part of the cycle C_{16} of color i. □

Note that the decomposition in cycles C_{16} for $n = 6$ presented in [7] is a special case of this theorem with $K = 3$.

4 Hamiltonian decomposition for $n = 8$

We now present a Hamiltonian decomposition of the binary n-cube for $n = 8$.

4.1 Phase 1: decomposing H_4 into 64 cycles C_{16}

Apply Theorem 2 for H_4 with $K = 3$. (The choice of $K = 3$ is that applying the theorem for H_3 with this K would produce the same results and illustrations as in [7] for H_3.) In Figure 9 the dark points are those to apply the shift operator. For clarity of illustration,

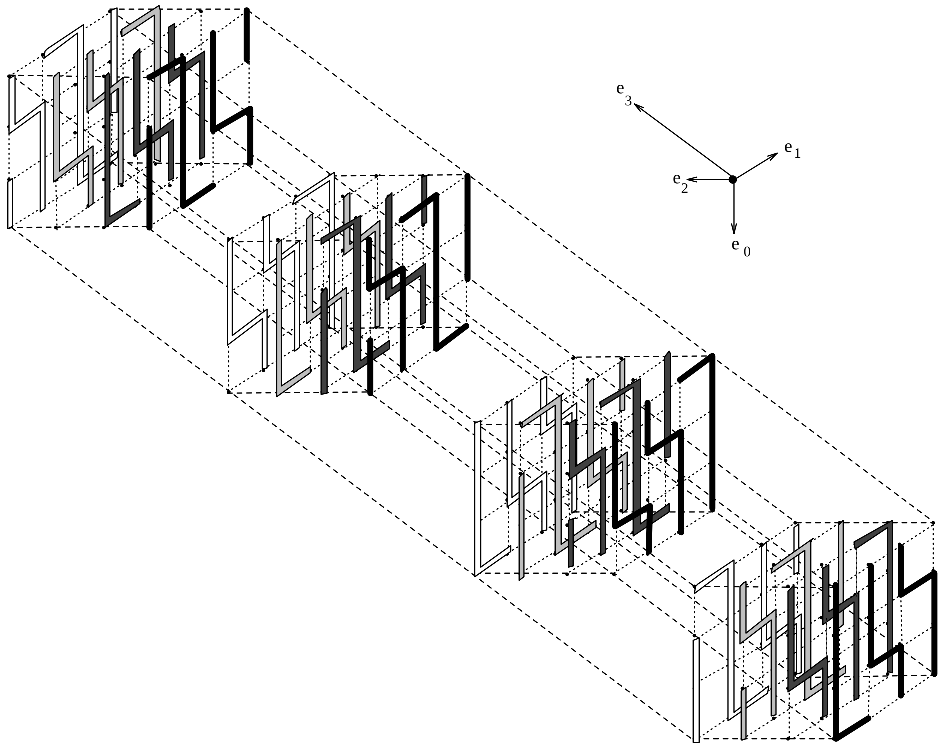

Figure 10: 16 cycles C_{16} of color 0

some of the edges are omitted in the figure (most of the edges of type $x(3)$ and all edges of type $x(i)$, $x = (x_0, x_1, x_2, x_3)$, with $x_i = 3$). H_4 is decomposed into 64 cycles C_{16}, with 16 C_{16} of color i for each $i = 0, 1, 2, 3$. In Figure 10 the 16 cycles C_{16} of color 0 are illustrated. In this figure we omit more edges than we did in the previous figure, for the sake of clarity. Compare this figure with Figure 8 to verify the omitted edges of color 0 that are part of the cycle.

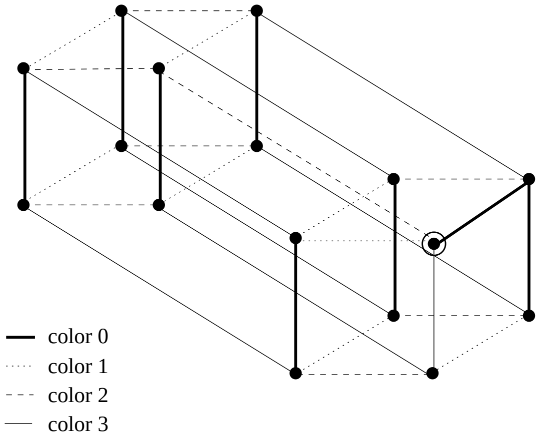

Figure 11: Before applying the merge operator

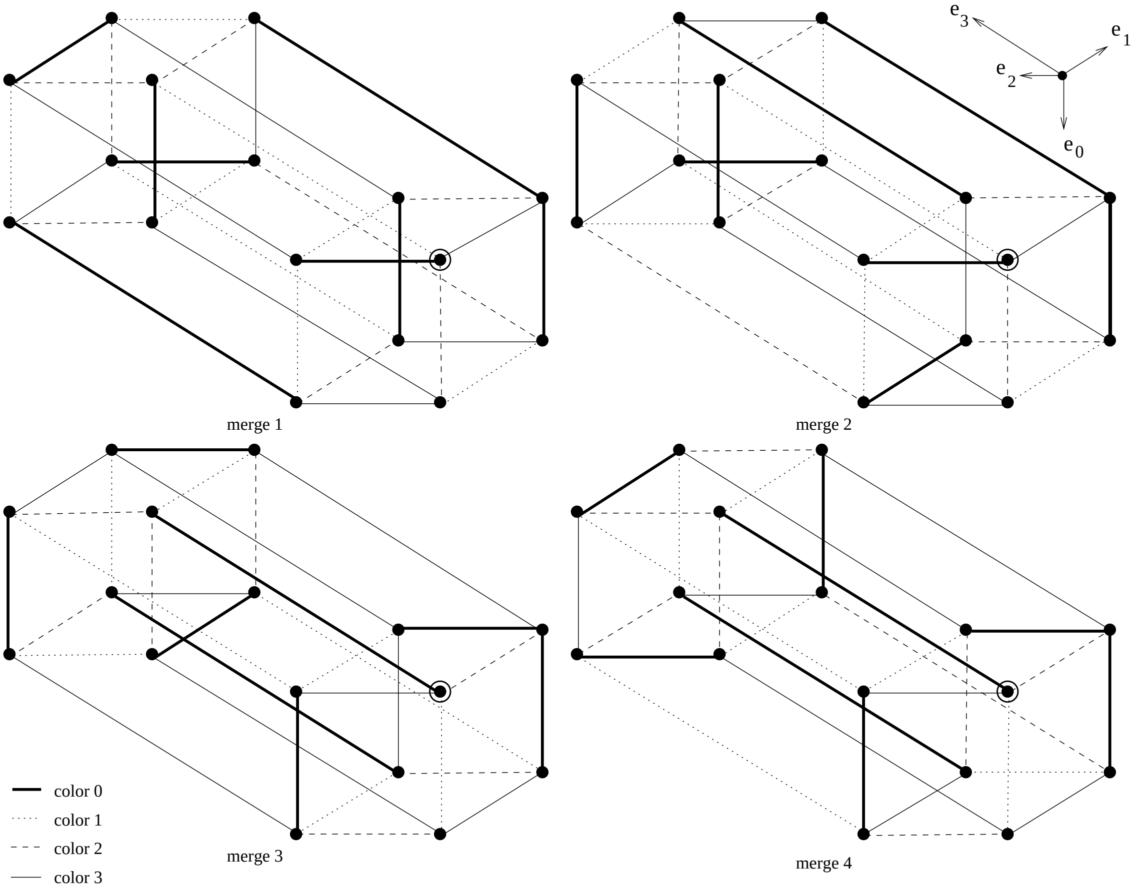

Figure 12: After applying each of the 4 merge operators for $n = 8$

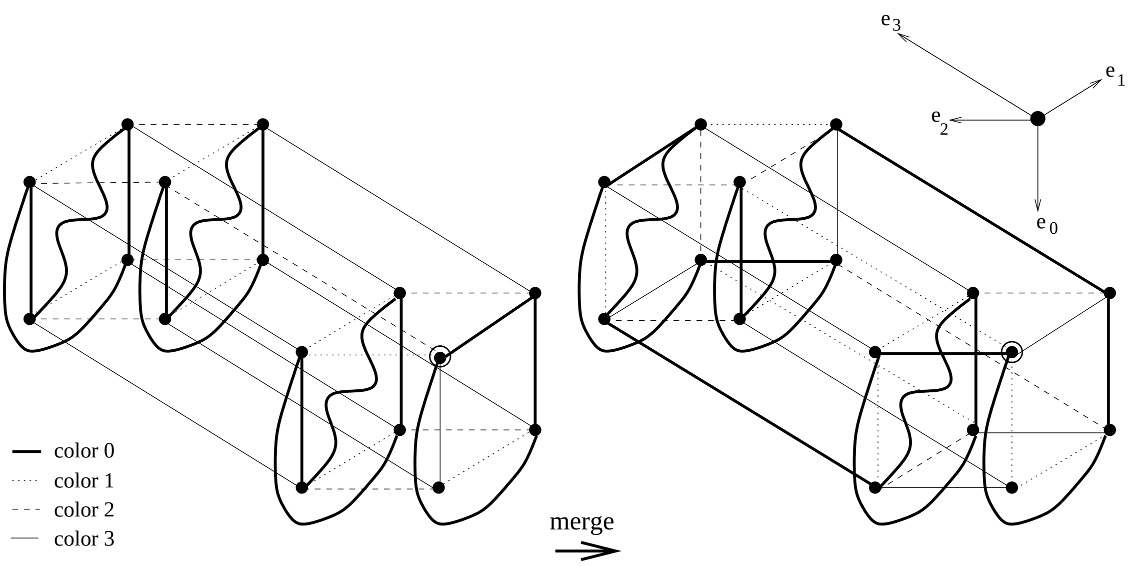

Figure 13: Effect of the merge operator 1 on color 0

4.2 Phase 2: obtaining a merge operator

We found 4 merge operators (see Figure 11 and Figure 12). A small example is given in Figure 13 to illustrate the effect of applying the merge operator. We choose the merge operator 1 and use color 0. On the left of this figure, the large black point is the point where the shift operator has been applied and the bold curves indicate the remaining of the cycle C_{16} of color 0 in $H_2(0)$. We can observe four cycles of color 0. On the right we notice that the merge operator (merge 1) joins the four cycles of color 0 into one sole cycle. The same effect can be observed for colors 1, 2, and 3, that is, four cycles of each color are joined by applying merge 1. The same is valid for the other merge operators (merge 2, merge 3 and merge 4). We apply the merge operator at the following five reference points: $(3,0,0,0), (1,0,2,0), (0,1,1,1), (3,2,0,2), (1,2,2,2)$ and we join the 16 cycles of each color $i = 0,1,2,3$ into one cycle only obtaining four cycles C_{256}.

5 Conclusion

We continue the work initiated by Song[7] of finding a simple constructive algorithm to obtain the disjoint Hamiltonian cycles of a binary n-cube. In this paper we generalize the first phase of that method for any even n and prove its correctness. Also we show four possible merge operators for the case of $n = 8$ (a 256-node hypercube). This result can be viewed as a step toward the general merge operator, thus proving the conjecture.

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