$6^{\underline{a}}$ Lista de MAT221 - Cálculo Diferencial e Integral IV - IME
$2^{\underline{o}}$ semestre de 2008
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11. a) Determine uma expressão em série de cossenos, em $(0,\pi)$, para $\quad f(x) = \operatorname{sen} x$.

b) Compute a soma $\quad \frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{c^2-1} + \frac{1}{8^2-1} + ...$

c) Compute o valor da série $\quad \sum_{n\geq 1} \frac{(-1)^{n+1}}{(2n)^2-1} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} + ...$

Resolução

(a) Para a função par $f(x) = |sen\,x|$, $-\pi \leq x \leq \pi$, os coeficientes de Fourier são:

$$b_n = 0, \qquad , \quad a_n = \frac{2}{\pi}\int_0^{\pi} sen\,x\,cos\,nx\,dx.$$

Logo,

$$\frac{\pi a_n}{2} = \frac{1}{2}\int_0^{\pi}[sen(1+n)x + sen(1-n)x]dx = -\frac{1}{2}\left[\frac{cos\,(n+1)x}{n+1}\Big|_0^{\pi} + \frac{cos(1-n)x}{1-n}\Big|_0^{\pi}\right],$$

e assim,

$$-\pi a_n = \frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1} = [(-1)^{n+1}-1]\left(\frac{1}{n+1} - \frac{1}{n-1}\right) =$$

$$= [(-1)^{n+1}-1]\frac{-2}{n^2-1}.$$

Portanto, $a_n = 0$ se n é ímpar, $a_n = -\frac{4}{\pi(n^2-1)}$, se n é par, e a série de Fourier de f é

$$|sen\,x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{p=1}^{+\infty}\frac{cos\,2p\,x}{(2p)^2-1} \ .$$

(b) Para $x = 0$ temos

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1}......\right),$$

e então, $\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1}...... = \frac{1}{2}$.

(c) Para $x = \frac{\pi}{2}$ temos,

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{p=1}^{+\infty}\frac{(-1)^p}{(2p)^2-1},$$

e portanto, $\sum_{p=1}^{+\infty}\frac{(-1)^{p+1}}{(2p)^2-1} = \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} + ... = \frac{\pi}{4} - \frac{1}{2}$.

12. a) Determine a série de Fourier da função $f(x) = \dfrac{x^3 - \pi^2 x}{3}, \quad -\pi \leq x \leq \pi.$

b) Compute o valor da série $\sum\limits_{n\geq 0} \frac{(-1)^n}{(2n+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} + ... + \frac{(-1)^n}{(2n+1)^3} + ...$

Resolução

(a) A função f é ímpar e assim, $a_n = 0, \forall n$. Ainda, para $n \geq 1$,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^3 - \pi^2 x}{3} sen\, nx\, dx \;=\; \frac{2}{3\pi}\int_0^{\pi} (x^3 - \pi^2 x) sen\, nx\, dx =$$

$$= \frac{2}{3\pi}\left[-(x^3 - \pi^2 x)\frac{cos\, nx}{n}\Big|_0^{\pi} \;+\; \int_0^{\pi} (3x^2 - \pi^2)\frac{cos\, nx}{n} dx \right] =$$

$$= \frac{2}{3n\pi}\int_0^{\pi} (3x^2 - \pi^2) cos\, nx\, dx \;=\; \frac{2}{3n\pi}\int_0^{\pi} 3x^2 cos\, nx\, dx =$$

$$= \frac{2}{n\pi}\int_0^{\pi} x^2 cos\, nx\, dx \;=\; \frac{2}{n\pi}\left[x^2 \frac{sen\, nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{sen\, nx}{n}\, dx \right] =$$

$$= -\frac{4}{n^2\pi}\int_0^{\pi} x sen\, nx\, dx \;=\; -\frac{4}{n^2\pi}\left[-\frac{x cos\, nx}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{cos\, nx}{n} dx \right] =$$

$$= \frac{4}{n^2\pi}\frac{\pi(-1)^n}{n} = \frac{4(-1)^n}{n^3}.$$

Logo, a série de Fourier de f é,

$$\frac{x^3 - \pi^2 x}{3} \;=\; 4\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3} sen\, nx\ .$$

(b) Se $x = \frac{\pi}{2}$ então $sen\, n\frac{\pi}{2} = 0$, se n é par, $sen(2p+1)\frac{\pi}{2} = (-1)^p$ e,

$$f(\frac{\pi}{2}) = -\frac{\pi^3}{8} = 4\left(-1 + \frac{1}{3^3} - \frac{1}{5^3} + ... \right)\ .$$

Logo, $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ ...... = \frac{\pi^3}{32}$.

13. Seja $\alpha \in \mathbb{R} - \mathbb{Z}$ e $f(x) = e^{idx}$, $-\pi < x < \pi$ e $f(x+2\pi) = f(x)$.

a) Determine a série de Fourier de f.

b) Mostre que $\dfrac{\pi}{\operatorname{sen}\alpha\,\pi} = \dfrac{1}{\alpha} + 2\alpha \sum_{n\geq 1} \dfrac{(-1)^n}{\alpha^2 - n^2}$.

c) Mostre que $\left(\dfrac{\pi}{\operatorname{sen}\alpha\,\pi}\right)^2 = \sum_{n=-\infty}^{+\infty} \dfrac{1}{(\alpha - n)^2}$.

Resolução

(a) Computemos os coeficientes c_n, $n \in \mathbb{Z}$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\alpha x}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\alpha-n)x}dx = \frac{1}{2\pi}\frac{e^{i(\alpha-n)x}}{i(\alpha-n)}\Big|_{-\pi}^{\pi} =$$

$$= \frac{1}{2(\alpha-n)\pi i}[\, e^{i(\alpha-n)\pi} - e^{-i(n-\alpha)\pi}\,] = \frac{2sen(\alpha-n)\pi\, i}{2(\alpha-n)\pi\, i} =$$

$$= \frac{sen(\alpha-n)\pi}{(\alpha-n)\pi} = \frac{(-1)^n sen\,\alpha\pi}{(\alpha-n)\pi} \;.$$

Logo, a série de Fourier é,

$$e^{i\alpha x} \sim S(f;x) \;=\; \frac{sen\alpha\pi}{\pi} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{\alpha - n} e^{inx}.$$

(b) Em termos de $(a_n)'s$ e $(b_n)'s$, $e^{i\alpha x} \sim \frac{a_0}{2} + \sum_{n\geq 1}(a_n cos\,nx + b_n sen\,nx)$ e, em $x = 0$,

$$1 = \frac{a_0}{2} \;+\; \sum_{n\geq 1} a_n \;,$$

$$\frac{a_0}{2} = \frac{sen\alpha\pi}{\alpha\pi} \;,\quad \frac{a_n}{\frac{sen\alpha\pi}{\pi}} = (-1)^n\Big(\frac{1}{\alpha-n} + \frac{1}{\alpha+n}\Big) = \frac{2\alpha(-1)^n}{\alpha^2-n^2}.$$

Logo,

$$1 = \frac{sen\,\alpha\pi}{\alpha\pi} \;+\; \frac{2\alpha sen\alpha\pi}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{\alpha^2-n^2},$$

e multiplicando a equação acima por $\frac{\pi}{sen\,\alpha\pi}$,

$$\frac{\pi}{sen\,\alpha\pi} = \frac{1}{\alpha} \;+\; 2\alpha \sum_{n\geq 1} \frac{(-1)^n}{\alpha^2-n^2}.$$

(c) De $\frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{i\alpha x}|^2 dx = 1$, $|c_n|^2 = |\frac{(-1)^n sen\,\alpha\pi}{(\alpha-n)\pi}|^2$ e da fórmula de Parsevall:

$$1 = \Big(\frac{sen\,\alpha\pi}{\pi}\Big)^2 \sum_{n=-\infty}^{+\infty} \frac{1}{(\alpha-n)^2} \;.$$

14. Seja $f(x) = \cos x,\ 0 < x < \pi$.

a) Determine a série de senos de f.

b) Mostre que $\dfrac{\pi\sqrt{2}}{16} = \dfrac{1}{2^2-1} - \dfrac{3}{6^2-1} + \dfrac{5}{10^2-1} - \dfrac{7}{14^2-1} + ...$

Resolução

(a) Seja $F(x) = -cosx\,, x \in (-\pi, 0),\ F(x) = cosx\,, x \in (0, \pi),\ F(-\pi) = F(0) = F(\pi)$. Então, F é ímpar e os coeficientes $(a_n)'s$ são nulos. Ainda,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) sen\, nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} cos\, x sen\, nx\, dx = \frac{1}{\pi}\int_{0}^{\pi} [\, sen(n{+}1)x + sen(n{-}1)x\,] dx =$$

$$= -\frac{1}{\pi}\left[\frac{cos(n+1)x}{n+1}\Big|_0^{\pi} + \frac{cos(n-1)x}{n-1}\Big|_0^{\pi}\right] =$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right] = \frac{(-1)^n+1}{\pi}\frac{2n}{n^2-1}.$$

Logo, $b_n = 0$ se n é ímpar e $b_n = \frac{4n}{\pi(n^2-1)}$ se n é par e assim, $b_{2k} = \frac{8k}{\pi[(2k)^2-1]}$, $k \geq 1$.

Portanto,

$$F(x) \sim \frac{8}{\pi}\sum_{k\geq 1}\frac{k}{(2k)^2-1} sen(2kx)\ .$$

(b) Em $x = \frac{\pi}{4}$, $sen(2kx) = sen\,\frac{k\pi}{2} = 0$, se k é par e, para $k = 2p+1$, $p \geq 0$, $sen(2kx) = sen\frac{(2p+1)\pi}{2} = (-1)^p$, $\forall p \geq 1$. Portanto,

$$\frac{\sqrt{2}}{2} = \frac{8}{\pi}\sum_{p\geq 0}\frac{(2p+1)(-1)^p}{[\,2(2p+1)\,]^2-1} = \frac{8}{\pi}\left(\frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{(10)^2-1} + ...+\right).$$

Respostas:

11. a) $\dfrac{1}{2}$ b) $\dfrac{\pi}{4} - \dfrac{1}{2}$; 12. a) $\dfrac{\pi^3}{12}$