# A Note on the Equality of the Column and Row Rank of a Matrix

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A basic result in linear algebra is that the row and column spaces of a matrix have dimensions that are equal. In this note we derive this result using an approach that is elementary yet different from that appearing in most current texts. Moreover, we do not rely on the echelon form in our arguments.

The column and row ranks of a matrix $A$ are the dimensions of the column and row spaces, respectively, of $A$. As is often noted, it is enough to establish that

$$\textit{row rank } A \leq \textit{column rank } A \tag{1}$$

for any matrix $A$. For, applying the result in (1) to $A^t$, the transpose of $A$, would produce the desired equality, since the row space of $A^t$ is precisely the column space of $A$. We present the argument for real matrices, indicating at the end the modifications necessary in the complex case and the case of matrices over arbitrary fields.

We take full advantage of the following two elementary observations:

1) for any vector $x$ in $R^n$, $Ax$ is a linear combination of the columns of $A$, and
2) vectors in the null space of $A$ are orthogonal to vectors in the row space of $A$, relative to the usual Euclidean product.

Both of these remarks follow easily from the very definition of matrix multiplication. For example, if the vector $x$ is in the null space of $A$ then $Ax = 0$, and thus the inner product of $x$ with the rows of $A$ must be zero. Since these rows span the row space of $A$, remark 2) follows. An immediate consequence is that only the zero vector can be common to both the null space and row space of $A$, since 2) requires such a vector to be orthogonal to itself.

Given an $m$ by $n$ matrix $A$, let the vectors $x_1, x_2, \ldots, x_r$ in $R^n$ form a basis for the row space of $A$. Then the $r$ vectors $Ax_1, Ax_2, \ldots, Ax_r$ are in the column space of $A$ and, further, we claim they are linearly independent. For, if $c_1 Ax_1 + c_2 Ax_2 + \cdots + c_r Ax_r = 0$ for some real scalars $c_1, c_2, \ldots, c_r$, then $A(c_1 x_1 + c_2 x_2 + \cdots + c_r x_r) = 0$ and the vector $v = c_1 x_1 + c_2 x_2 + \cdots + c_r x_r$ would be in the null space of $A$. But, $v$ is also in the row space of $A$, since it is a linear combination of basis elements. So, $v$ is the zero vector and the linear independence of $x_1, x_2, \ldots, x_r$ guarantees that $c_1 = c_2 = \cdots = c_r = 0$. The existence of $r$ linearly independent vectors in the column space requires that $r \leq \textit{column rank } A$. Since $r$ is the row rank of $A$, we have arrived at the desired inequality (1).

This approach also yields additional information. Let $y_1, y_2 \ldots y_k$ form a basis for the null space of $A$. Since the row space of $A$ and the null space of $A$ intersect trivially, it follows that the set $\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_k\}$ is linearly independent. Further, if $z$ is a vector in $R^n$ then $Az$ is in the column space of $A$ and hence expressible as a linear combination of basis elements. Thus, we can write $Az = \sum_{j=1}^{r} d_j Ax_j$, for scalars $d_j$. But then the vector $z - \sum_{j=1}^{r} d_j x_j$ is in the null space of $A$ and can be written as a linear combination of $y_1, y_2, \ldots, y_k$. Thus $z$ can be expressed as a linear combination of the vectors in the set $\{x_1, x_2, \ldots, x_r,$

$y_1, y_2, \ldots, y_k\}$ and this set then forms a basis for $R^n$. A dimension count yields $r + k = n$, giving the rank and nullity theorem without use of the echelon form.

Note that the reliance on orthogonality in the arguments above is elementary and does not require the Gram-Schmidt process. The ideas used here can be readily applied to complex matrices with minor modifications. The hermitian inner product is used instead in the complex vector space $C^n$, as is the hermitian transpose. Observation 2) would note then that vectors in the null space of $A$ are orthogonal to those in the row space $\bar{A}$.

The row and column rank theorem is a well-known result that is valid for matrices over arbitrary fields. The notion of orthogonal complement can be generalized using linear functionals and dual spaces, and the general structure of the arguments here can then be carried over to arbitrary fields. The text [**1**, pp. 97 ff.] contains such an approach.

Many authors base their discussion of rank on the echelon form. The fact that the non-zero rows of the echelon form are a basis for the row space, or that columns in the echelon form containing lead ones can be used to identify a basis for the column space in the original matrix $A$, are central to such a development of rank. Results such as these follow easily if it is established independently that row rank and column rank must be equal.

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## REFERENCE

1. Kenneth Hoffman and Ray Kunze, *Linear Algebra*, 2nd edition, Prentice-Hall, Englewood Cliffs, NJ, 1971.

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# A One-Sentence Proof That $\sqrt{2}$ Is Irrational

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If $\sqrt{2}$ were rational, say $\sqrt{2} = m/n$ in lowest terms, then also $\sqrt{2} = (2n - m)/(m - n)$ in *lower* terms, giving a contradiction.

(The three verifications needed—that the second denominator is less than the first and still positive, and that the two fractions are equal—are straightforward.)

The argument is not original; it's the algebraic version of a geometric argument given in [**1**, p. 84], and it was presented (in slightly different form) by Ivan Niven at a lecture in 1985. Note, though, that the algebraic argument, unlike the geometric, easily adapts to an arbitrary $\sqrt{k}$ where $k$ is any positive integer that is not a perfect square. Indeed, let $j$ be the integer such that $j < \sqrt{k} < j + 1$. If we had $\sqrt{k} = m/n$ in lowest terms $(m, n \in \mathbf{Z}^+)$, then also $\sqrt{k} = (kn - jm)/(m - jn)$ in lower terms, a contradiction.

## REFERENCE

1. H. Eves, *An Introduction to the History of Mathematics*, 6th edition, W. B. Saunders Co., Philadelphia, 1990.