RIGHT-HAND RULE (a proof) AND VECTOR PRODUCT

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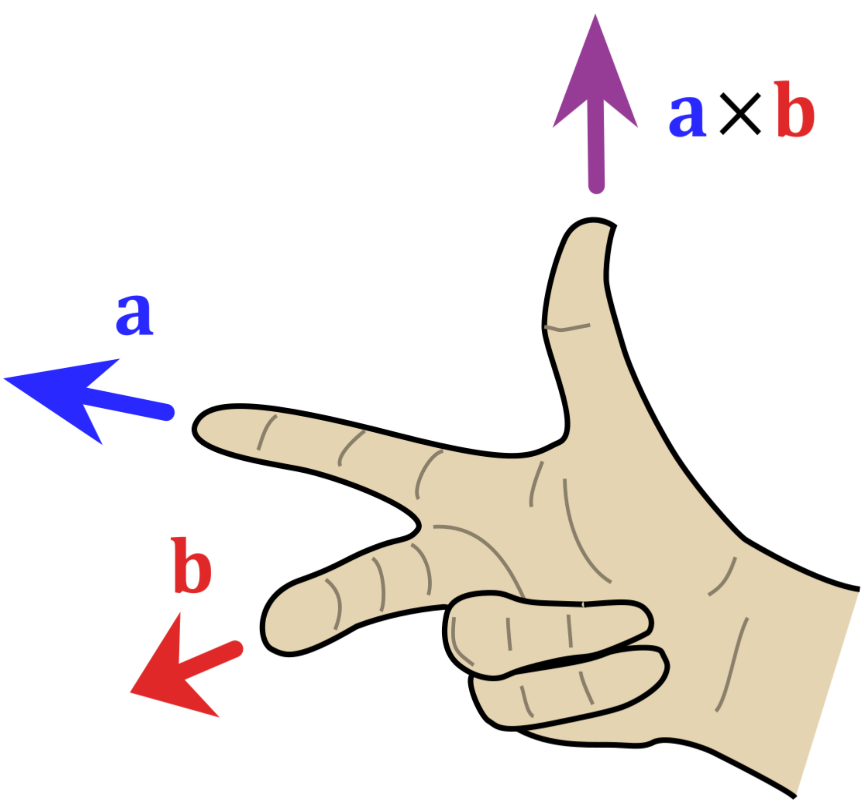

Figura 1: The practical right-hand rule.

1. VECTOR PRODUCT (CROSS PRODUCT)

Let us consider the vector space $\mathbb{R}^3$ and the standard set of vectors $\{\vec{i}, \vec{j}, \vec{k}\}$. Thus, the vectors $\vec{i}, \vec{j}$ and $\vec{k}$ are orthogonal to each other and each one has length 1. We say that $\{\vec{i}, \vec{j}, \vec{k}\}$ is an orthonormal basis.

Next, we consider two vectors

$$\begin{cases} \vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \\ \text{and} \\ \vec{b} = (b_1, b_2, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}, \end{cases}$$

where $a_1, a_2, a_3, b_1, b_2,$ and b_3 are real numbers.

Let us search for a vector $\vec{x} = (x_1, x_2, x_3)$ satisfying the conditions

$$\begin{cases} \vec{x} \text{ is orthogonal to } \vec{a} \\ \text{and} \\ \vec{x} \text{ is orthogonal to } \vec{b}. \end{cases}$$

By using the scalar product [also called inner product or dot product and indicated by the symbol "$\cdot$", a dot], we rewrite such conditions as

$$\begin{cases} \vec{x} \cdot \vec{a} = 0 \\ \vec{x} \cdot \vec{b} = 0. \end{cases}$$

Thus, the triplet (x_1, x_2, x_3) must satisfy the linear system

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = 0 \\ b_1x_1 + b_2x_2 + b_3x_3 = 0 \end{cases}$$

or, equivalently,

$$\begin{cases} a_1x_1 + a_2x_2 = -a_3x_3 \\ b_1x_1 + b_2x_2 = -b_3x_3. \end{cases}$$

By employing matrix notation we arrive at

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -a_3x_3 \\ -b_3x_3 \end{pmatrix}.$$

Now, let us freely develop some computations. To begin with, let us suppose that the 2×2 matrix right above is invertible. Hence we find that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}\begin{pmatrix} -a_3x_3 \\ -b_3x_3 \end{pmatrix}.$$

This shows that

$$\begin{cases} x_1 = \frac{a_2b_3x_3 - a_3b_2x_3}{a_1b_2-a_2b_1} = \frac{x_3}{a_1b_2-a_2b_1}(a_2b_3 - a_3b_2) \\ \text{and} \\ x_2 = \frac{a_3b_1x_3 - a_1b_3x_3}{a_1b_2-a_2b_1} = \frac{x_3}{a_1b_2-a_2b_1}(a_3b_1 - a_1b_3). \end{cases}$$

Choosing $x_3 = a_1b_2 - a_2b_1$ (we may pick any value for x_3) we find the vector

$$\overrightarrow{x} = (x_1, x_2, x_3) = (a_2b_3 - a_3b_2)\overrightarrow{i} - (a_1b_3 - a_3b_1)\overrightarrow{j} + (a_1b_2 - a_2b_1)\overrightarrow{k}.$$

Such a vector $\overrightarrow{x}$ may be written as the 3×3 "informal determinant"

$$\boxed{\overrightarrow{x} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\overrightarrow{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\overrightarrow{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\overrightarrow{k}}.$$

Now, let us investigate the properties of this highlighted vector $\overrightarrow{x}$.

Lemma 1. *Let $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbb{R}^3$. Then,*

$$\overrightarrow{x} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\overrightarrow{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\overrightarrow{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\overrightarrow{k}$$

is orthogonal to the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

Proof.

- ⋄ Let us show that $\overrightarrow{x} = (x_1, x_2, x_3)$ is orthogonal to $\overrightarrow{a}$. We have

$$\begin{aligned} \overrightarrow{a}\cdot\overrightarrow{x} &= a_1x_1 + a_2x_2 + a_3x_3 \\ &= a_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \end{aligned}$$

The determinant of a matrix with two equal lines is 0. Thus, $\overrightarrow{x}\cdot\overrightarrow{a} = 0$.

⋄ Analogouly it follows that $\overrightarrow{x}$ is orthogonal to $\overrightarrow{b}$ ♣

Lemma 2. *Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{x}$ be as in Lemma 1. Let θ, with $0 \leq \theta \leq \pi$, be the (smallest) angle between $\overrightarrow{a}$ and $\overrightarrow{b}$. Then, we have*

$$\|\overrightarrow{x}\| = \|\overrightarrow{a}\|\,\|\overrightarrow{b}\|\sin\theta.$$

Proof.

We have

$$\begin{aligned}
\|\overrightarrow{x}\|^2 &= x_1^2 + x_2^2 + x_3^2 \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\
&= (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_1^2b_3^2 - 2a_1a_3b_1b_3 + a_3^2b_1^2 \\
&\quad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 \\
&\quad - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 \\
&\quad - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - (a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1a_2b_1b_2 + 2a_1a_3b_1b_3 + 2a_2a_3b_2b_3) \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= \|\overrightarrow{a}\|^2\,\|\overrightarrow{b}\|^2 - (\overrightarrow{a} \cdot \overrightarrow{b})^2 \\
&= \|\overrightarrow{a}\|^2\,\|\overrightarrow{b}\|^2 - (\|\overrightarrow{a}\|\,\|\overrightarrow{b}\|\cos\theta)^2 \\
&= \|\overrightarrow{a}\|^2\,\|\overrightarrow{b}\|^2 - \|\overrightarrow{a}\|^2\|\overrightarrow{b}\|^2\cos^2\theta^2 \\
&= \|\overrightarrow{a}\|^2\,\|\overrightarrow{b}\|^2(1 - \cos^2\theta) \\
&= \|\overrightarrow{a}\|^2\,\|\overrightarrow{b}\|^2\sin^2\theta \\
&= (\|\overrightarrow{a}\|\,\|\overrightarrow{b}\|\sin\theta)^2.
\end{aligned}$$

Since $\theta \in [0, \pi]$, and thus $\sin\theta \geq 0$, we are allowed to conclude that

$$\|\overrightarrow{x}\| = \|\overrightarrow{a}\|\,\|\overrightarrow{b}\|\sin\theta ♣$$

Lemma 3. *Let* $\vec{a}$, $\vec{b}$ *and* $\vec{x}$ *be as in Lemma 1. Then we have*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = x_1^2 + x_2^2 + x_3^2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$

Proof.

- From the formulas

$$\begin{aligned} \vec{x} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \\ &= x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} \end{aligned}$$

  we easily obtain the second claimed identity (the one that is not related to the 3×3 determinant).

- Moreover, we have

$$\begin{aligned} \begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= x_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - x_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + x_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\ &= x_1^2 + x_2^2 + x_3^2 \clubsuit \end{aligned}$$

In short, we have

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \|\vec{x}\|^2 \geq 0$$

for any two vectors $\vec{a}$ and $\vec{b}$, both in $\mathbb{R}^3$.

Definition (Parallelism).

- The null vector $(0,0,0)$ is parallel to every vector in the vector space $\mathbb{R}^3$.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where A, B, C, and D are points in the Cartesian space $\mathbb{R}^3$ such that $A \neq B$ and $C \neq D$, are parallel if the segments $\overline{AB}$ and $\overline{CD}$ are parallel.

Definition (Linear Combination). Given two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, both in the vector space $\mathbb{R}^3$, and two real numbers α and β, the vector

$$\overrightarrow{w} = \alpha\overrightarrow{u} + \beta\overrightarrow{v}$$

is a linear combination of $\overrightarrow{u}$ and $\overrightarrow{v}$, with coefficients α and β. We also say that $\overrightarrow{w}$ is generated by $\overrightarrow{u}$ and $\overrightarrow{v}$.

Definition (Linear Dependence or LD). Two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, in the vector space $\mathbb{R}^3$, are LD if there exist two real numbers α and β, not both zero, satisfying

$$\alpha\overrightarrow{u} + \beta\overrightarrow{v} = \overrightarrow{0} \qquad (\text{with } \alpha \neq 0 \text{ or } \beta \neq 0).$$

If $\overrightarrow{u}$ and $\overrightarrow{v}$ are LD, we also say that the set $\{\overrightarrow{u}, \overrightarrow{v}\}$ is LD.

Definition (Linear Independence or LI).

- Two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, in the vector space $\mathbb{R}^3$, are LI if they are not LD. That is, $\overrightarrow{u}$ and $\overrightarrow{v}$ are LI if given two real numbers α and β such that

  $$\alpha\overrightarrow{u} + \beta\overrightarrow{v} = \overrightarrow{0},$$

  then we have $\alpha = 0$ and $\beta = 0$.

  If $\overrightarrow{u}$ and $\overrightarrow{v}$ are LI, we also say that $\{\overrightarrow{u}, \overrightarrow{v}\}$ is LI.

  Summing up, $\overrightarrow{u}$ and $\overrightarrow{v}$ are LI if the following implication is true,

  $$\alpha\overrightarrow{u} + \beta\overrightarrow{v} = \overrightarrow{0} \Longrightarrow \begin{cases} \alpha = 0 \\ \beta = 0. \end{cases}$$

The following remarks are trivial.

$$\overrightarrow{u} \text{ and } \overrightarrow{v} \text{ are LD} \iff \overrightarrow{u} \text{ and } \overrightarrow{v} \text{ are parallel.}$$

$$\overrightarrow{u} \text{ and } \overrightarrow{v} \text{ are LI} \iff \overrightarrow{u} \text{ and } \overrightarrow{v} \text{ are not parallel.}$$

Lemma 4. *Let us consider* $\overrightarrow{a} = (a_1, a_2, a_3)$ *and* $\overrightarrow{b} = (b_1, b_2, b_3)$, *two arbitrary vectors in the vector space* $\mathbb{R}^3$, *and the* 2×3 *real matrix*

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Then,

$$\{\overrightarrow{a}, \overrightarrow{b}\} \text{ is LD} \iff \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0.$$

First Proof.

⋄ A quite easy proof follows from the formula (see Lemma 2)

$$\|\overrightarrow{a} \times \overrightarrow{b}\| = \|\overrightarrow{a}\| \|\overrightarrow{b}\| \sin\theta,$$

and I leave this trivial task to the reader. However, it is also important (and instructive) to develop a proof that does not depend on Lemma 2.

Second Proof (independent of Lemma 2).

($\Rightarrow$) Let us suppose that $\overrightarrow{a}$ and $\overrightarrow{b}$ are LD. Hence, we have either $\overrightarrow{a} = \lambda \overrightarrow{b}$ or $\overrightarrow{b} = \lambda \overrightarrow{a}$ (for some real λ). We may suppose without loss of generality that

$$\overrightarrow{a} = \lambda \overrightarrow{b}.$$

Hence, we obtain the identity $(a_1, a_2, a_3) = (\lambda b_1, \lambda b_2, \lambda b_3)$ and thus

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} \lambda b_1 & \lambda b_2 \\ b_1 & b_2 \end{vmatrix} = \lambda b_1 b_2 - \lambda b_1 b_2 = 0,$$

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = \begin{vmatrix} \lambda b_1 & \lambda b_3 \\ b_1 & b_3 \end{vmatrix} = \lambda b_1 b_3 - \lambda b_1 b_3 = 0$$

and

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \lambda b_2 & \lambda b_3 \\ b_2 & b_3 \end{vmatrix} = \lambda b_2 b_3 - \lambda b_2 b_3 = 0.$$

($\Leftarrow$) The claim is obvious if $\vec{a} = \vec{b} = \vec{0}$. Hence, we may suppose that $\vec{b} \neq \vec{0}$. Furthermore, we may suppose without loss of generality $b_3 \neq 0$ (the cases $b_1 \neq 0$ and $b_2 \neq 0$ are analogous to the case $b_3 \neq 0$).

From the hypotheses

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0$$

we see that $\{(a_1, a_3), (b_1, b_3)\}$ is LD and $\{(a_2, a_3), (b_2, b_3)\}$ is also LD. Thus, since $b_3 \neq 0$, we see that there exist two real numbers α and β satisfying

$$(a_1, a_3) = \alpha(b_1, b_3) \text{ and } (a_2, a_3) = \beta(b_2, b_3).$$

Hence, we arrive at

$$a_3 = \alpha b_3, \quad a_3 = \beta b_3 \text{ and } b_3 \neq 0.$$

Then, we obviously have $\alpha b_3 = \beta b_3$, with $b_3 \neq 0$, and thus $\alpha = \beta$. Hence,

$$(a_1, a_2, a_3) = \alpha(b_1, b_2, b_3) \text{ and thus } \{\vec{a}, \vec{b}\} \text{ is LD} \clubsuit$$

Corollary 5. *Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be as in Lemma 4. Then, the following equivalences are true.*

$$\{\vec{a}, \vec{b}\} \text{ is LI} \iff \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \neq 0$$
$$\iff \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 \neq 0.$$

Proof. It is immediate from Lemma 4 $\clubsuit$

Corollary 6. *Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be as in Lemma 4. Then,*

$$\{\vec{a}, \vec{b}\} \text{ is LI} \iff \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \neq \vec{0}.$$

Proof. It is immediate from Corollary 5 $\clubsuit$

Corollary 7. *Let* $\overrightarrow{a} = (a_1, a_2, a_3)$ *and* $\overrightarrow{b} = (b_1, b_2, b_3)$ *be as in Lemma 4. Then,*

$$\{\overrightarrow{a}, \overrightarrow{b}\} \text{ is LD} \Longleftrightarrow \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \overrightarrow{0}.$$

Proof. It is immediate from Corollary 6 ♣

We already analized the direction of

$$\overrightarrow{x} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

[On the one hand, the vector $\overrightarrow{x}$ is null if the set $\{\overrightarrow{a}, \overrightarrow{b}\}$ is LD. On the other hand, the vector $\overrightarrow{x}$ is orthogonal to $\overrightarrow{a}$ and $\overrightarrow{b}$ if the set $\{\overrightarrow{a}, \overrightarrow{b}\}$ is LI.] Moreover, we already established the norm of $\overrightarrow{x}$ [we have seen that $\|\overrightarrow{x}\| = \|\overrightarrow{a}\|\|\overrightarrow{b}\|\sin\theta$].

Now we turn our attention to the orientation of the vector $\overrightarrow{x}$.

Lemma 8. *Keeping the notation, let us consider the vectors* $\overrightarrow{a}$, $\overrightarrow{b}$ *and* $\overrightarrow{x}$. *Let us suppose that* $\{\overrightarrow{a}, \overrightarrow{b}\}$ *is LI. Let us consider a vetor* $\overrightarrow{y}$ *satisfying the conditions*

$$\begin{cases} \overrightarrow{y} \text{ is orthogonal to } \overrightarrow{a} \text{ and } \overrightarrow{b}, \\ \|\overrightarrow{y}\| = \|\overrightarrow{x}\|. \end{cases}$$

Then we have

$$\overrightarrow{y} = \overrightarrow{x} \text{ or } \overrightarrow{y} = -\overrightarrow{x}.$$

Proof.

- Putting $\overrightarrow{a} = (a_1, a_2, a_3)$, $\overrightarrow{b} = (b_1, b_2, b_3)$, $\overrightarrow{x} = (x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_1, y_2, y_3)$, we consider the system in the real variables α, β and γ given by

$$S : \begin{cases} a_1\alpha + b_1\beta + x_1\gamma = y_1 \\ a_2\alpha + b_2\beta + x_2\gamma = y_2 \\ a_3\alpha + b_3\beta + x_3\gamma = y_3. \end{cases}$$

By determinants properties, Lemma 3, and Corollary 6, the determinant of this system is

$$\begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \|\overrightarrow{x}\|^2 \neq 0.$$

Therefore, by properties of linear systems, there exists a unique solution $\alpha = \alpha_0$, $\beta = \beta_0$ and $\gamma = \gamma_0$ of the linear system S under consideration.

In order to avoid heavy notation, let us write $(\alpha_0, \beta_0, \gamma_0)$ briefly by (α, β, γ).

From the system S we see that these three numbers α, β and γ satisfy

$$\alpha \overrightarrow{a} + \beta \overrightarrow{b} + \gamma \overrightarrow{x} = \overrightarrow{y}.$$

Thus, we arrive at

$$\overrightarrow{y} - \gamma \overrightarrow{x} = \alpha \overrightarrow{a} + \beta \overrightarrow{b}.$$

Now, since $\overrightarrow{y}$ is orthogonal to $\overrightarrow{a}$ and $\overrightarrow{b}$, it follows that $\overrightarrow{y}$ is orthogonal to the sum $\alpha \overrightarrow{a} + \beta \overrightarrow{b}$.

By the same reason, the vector $\overrightarrow{x}$ is also orthogonal to the sum $\alpha \overrightarrow{a} + \beta \overrightarrow{b}$.

Now, the same argument also shows that the vector $\alpha \overrightarrow{a} + \beta \overrightarrow{b}$ is orthogonal to the difference $\overrightarrow{y} - \gamma \overrightarrow{x}$.

Since we have the identity $\overrightarrow{y} - \gamma \overrightarrow{x} = \alpha \overrightarrow{a} + \beta \overrightarrow{b}$, we may conclude that the vector $\overrightarrow{y} - \gamma \overrightarrow{x}$ is orthogonal to itself. Hence, we arrive at the identity $(\overrightarrow{y} - \gamma \overrightarrow{x}) \cdot (\overrightarrow{y} - \gamma \overrightarrow{x}) = 0$ and thus

$$\| \overrightarrow{y} - \gamma \overrightarrow{x} \|^2 = 0.$$

This reveals that $\overrightarrow{y} - \gamma \overrightarrow{x} = \overrightarrow{0}$ and

$$\boxed{\overrightarrow{y} = \gamma \overrightarrow{x}.}$$

Therefore, by taking norms we obtain

$$\| \overrightarrow{y} \| = |\gamma| \, \| \overrightarrow{x} \|.$$

However, by hypothesis we also have $\| \overrightarrow{y} \| = \| \overrightarrow{x} \|$. Moreover, we already saw that $\| \overrightarrow{x} \| \neq 0$. Thus, we find that

$$\| \overrightarrow{x} \| = |\gamma| \, \| \overrightarrow{x} \|, \text{ with } \| \overrightarrow{x} \| \neq 0.$$

Hence, it follows that

$$|\gamma| = 1 \text{ and } \gamma = \pm 1.$$

Therefore, there are only two possibilities. We have

$$\overrightarrow{y} = \overrightarrow{x} \quad \text{or} \quad \overrightarrow{y} = -\overrightarrow{x} \; \clubsuit$$

The following theorem is a trivial consequence of the previous lemmas.

Theorem. *Let* $\overrightarrow{a} = (a_1, a_2, a_3)$ *and* $\overrightarrow{b} = (b_1, b_2, b_3)$ *be two any vectors in* $\mathbb{R}^3$. *Then, the vector*

$$\overrightarrow{x} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

has the following properties.

- *If* $\overrightarrow{a}$ and $\overrightarrow{b}$ *are LD, then* $\overrightarrow{x} = \overrightarrow{0}$.
- *If* $\overrightarrow{a}$ and $\overrightarrow{b}$ *are LI, then* $\overrightarrow{x}$ *is the only vector* $\overrightarrow{y} = (y_1, y_2, y_3)$ *satisfying the following three conditions*

$$\begin{cases} \overrightarrow{y} \text{ is orthogonal to } \overrightarrow{a} \text{ and orthogonal to } \overrightarrow{b}, \\ \text{the norm of } \overrightarrow{y} \text{ is the area of the parallelogram determined by } \overrightarrow{a} \text{ and } \overrightarrow{b}, \\ \text{the determinant } \begin{vmatrix} y_1 & y_2 & y_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{ is (strictly) positive.} \end{cases}$$

Proof. It follows from the previous lemmas ♣

Definition (vector product, or cross product). Given $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$, two arbitrary vectors in $\mathbb{R}^3$, the vector product of $\overrightarrow{a}$ by $\overrightarrow{b}$, in this order, is the vector denoted by $\overrightarrow{a} \times \overrightarrow{b}$ and given by (an informal determinant)

$$\boxed{\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.}$$

2. THE RIGHT-HAND RULE

A positive number is a real number x such that $x > 0$ (i.e, x is bigger than 0).

We indicate the determinant of a square real matrix M by $\det M$.

Let $M_{3\times3}(\mathbb{R})$ be the set of the 3×3 real matrices.

We denote by I the 3×3 identity matrix.

In this text, the symbol $\mathcal{D}^+$ denotes the set of the 3×3 real matrices with a positive determinant. That is,

$$\mathcal{D}^+ = \{M \in M_{3\times3}(\mathbb{R}) : \det(M) > 0\}.$$

Let us consider a 3×3 real matrix M with a positive determinant (that is, $\det M > 0$). Let us write

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Objective. Our objective is to prove that we can continuously deform the matrix M into the identity matrix I by using only matrices with a positive determinant along the deformation process.

Thus, we want to prove the following theorem.

Theorem. *Given a* 3×3 *real matrix*

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \text{ with } \det M > 0,$$

there exists a continuous curve inside $\mathcal{D}^+$ *connecting* M *to the identity matrix* I.

Proof. We split the proof into six (6) numbered steps.

(1) We may suppose that $a \neq 0$. Let us show this claim. In what follows, we describe a sequence of short steps. These are taken so that the determinants of all the appearing matrices do not change and are equal to $\det M$.

The first column of M is not null (otherwise, we have $\det M = 0$↯).

The case $a = 0$ and $d \neq 0$. Then

$$\begin{pmatrix} 0+td & b+te & c+tf \\ d & e & f \\ g & h & i \end{pmatrix}, \text{ where } t \text{ runs over } [0,1],$$

continuously connects (from the instant $t = 0$ up to the instant $t = 1$)

$$M = \begin{pmatrix} 0 & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix}.$$

This case is proven (since $d \neq 0$).

The case $a = d = 0$ and $g \neq 0$. In this case, we employ to the first and the third rows (horizontal lines) of

$$M = \begin{pmatrix} 0 & b & c \\ 0 & e & f \\ g & h & i \end{pmatrix}$$

the same argument that we employed to the first and second rows of M.

Thus, we see that we may continuously connect

$$M = \begin{pmatrix} 0 & b & c \\ 0 & e & f \\ g & h & i \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} g & b+h & c+i \\ 0 & e & f \\ g & h & i \end{pmatrix}.$$

This case is proven (since $g \neq 0$). The proof of step (1) is complete.

(2) We may suppose $d = g = 0$ [thanks to (1), we are already supposing $a \neq 0$]. Let us verify this claim. Once more, all the arguments are taken so that the determinants of all the appearing matrices are equal to $\det M$.

Clearly,

$$\begin{pmatrix} a & b & c \\ d - ta & e - tb & f - tc \\ g & h & i \end{pmatrix}$$

continuously connects (with the variable t running from $t = 0$ to $t = d/a$)

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ to } \begin{pmatrix} a & b & c \\ 0 & e - \frac{bd}{a} & f - \frac{cd}{a} \\ g & h & i \end{pmatrix}.$$

Analogously,

$$\begin{pmatrix} a & b & c \\ 0 & e - \frac{bd}{a} & f - \frac{cd}{a} \\ g - ta & h - tb & i - tc \end{pmatrix}$$

continuously connects (with the variable t running from $t = 0$ to $t = g/a$)

$$\begin{pmatrix} a & b & c \\ 0 & e - \frac{bd}{a} & f - \frac{cd}{a} \\ g & h & i \end{pmatrix} \text{ to } \begin{pmatrix} a & b & c \\ 0 & e - \frac{bd}{a} & f - \frac{cd}{a} \\ 0 & h - \frac{bg}{a} & i - \frac{cg}{a} \end{pmatrix}.$$

The proof of step (2) is complete.

(3) Up to here, we have shown that we may suppose that

$$M = \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix}, \text{ with } a \neq 0.$$

As before, all the arguments in this step are made so that the determinants of all the appearing matrices are equal to $\det M$.

We claim that we may suppose $e \neq 0$. In fact, if $e = 0$ then it follows that $h \neq 0$ (otherwise, we obtain $\det M = 0$↯) and thus

$$\begin{pmatrix} a & b & c \\ 0 & 0 - th & f - ti \\ 0 & h & i \end{pmatrix}$$

continuously connects (with t running from $t = 0$ to $t = 1$)

$$\begin{pmatrix} a & b & c \\ 0 & 0 & f \\ 0 & h & i \end{pmatrix} \text{ to } \begin{pmatrix} a & b & c \\ 0 & -h & f - i \\ 0 & h & i \end{pmatrix}, \text{ with } -h \neq 0.$$

Hence, as we claimed, we may suppose that $e \neq 0$.

The proof of step (3) is complete.

(4) We may suppose that $b = h = 0$. [We already saw that we may suppose $a \neq 0$, $d = g = 0$, and $e \neq 0$.]

All the arguments in this step are made so that the determinants of all the appearing matrices are equal to $\det M$.

Clearly,

$$\begin{pmatrix} a & b - te & c - tf \\ 0 & e & f \\ 0 & h & i \end{pmatrix}$$

continuously connects (with t running from $t = 0$ to $t = b/e$)

$$M = \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & 0 & c - \frac{bf}{e} \\ 0 & e & f \\ 0 & h & i \end{pmatrix}.$$

It is also clear that

$$\begin{pmatrix} a & 0 & c - \frac{bf}{e} \\ 0 & e & f \\ 0 & h - te & i - tf \end{pmatrix}$$

continuously connects (with t running from $t = 0$ to $t = h/e$)

$$\begin{pmatrix} a & 0 & c - \frac{bf}{e} \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \text{ to } \begin{pmatrix} a & 0 & c - \frac{bf}{e} \\ 0 & e & f \\ 0 & 0 & i - \frac{fh}{e} \end{pmatrix}.$$

The proof of step (4) is complete.

(5) From the four previous steps it follows that we may suppose that

$$M = \begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}, \text{ with } a \neq 0, \mathrm{e} \neq 0, \text{and } \det M > 0.$$

Now, let us show that we can suppose $c = f = 0$.

It is clear that $i \neq 0$ (otherwise, we have $\det M = 0$↯). Clearly,

$$\begin{pmatrix} a & 0 & c - ti \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}$$

continuously connects

$$M = \begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}.$$

It is also clear that

$$\begin{pmatrix} a & 0 & 0 \\ 0 & e & f - ti \\ 0 & 0 & i \end{pmatrix}$$

continuously connects

$$\begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}.$$

The proof of step (5) is complete.

(6) From the previous steps it follows that we may suppose that

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}, \text{ with } \det M = aei > 0.$$

Then we have two possibilities about the entries a, e, and i.

$$\begin{cases} \text{The entries } a, e, \text{ and } i \text{ are positive } (>0) \\ \text{or} \\ \text{one of them is positive and the other two are negative } (<0). \end{cases}$$

In this step, the determinants of all the appearing matrices are positive.

The case where a, e, and i are positive. Then

$$\begin{pmatrix} ta & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}$$

continuously connects (with t positive and running from $t = 1$ to $1/a$)

$$\begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Next,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & te & 0 \\ 0 & 0 & i \end{pmatrix}$$

continuously connects (with t positive and running from $t = 1$ to $t = 1/e$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & te & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Finally,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ti \end{pmatrix}$$

continuously connects (with t positive and running from $t = 1$ to $t = 1/i$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The proof of this case is complete.

The case where one element of $\{a, e, i\}$ is positive and the others are negative.

The subcase $a > 0$ (and thus $e < 0$ and $i < 0$).

Then, as we already saw, we may continuously connect

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Now, we notice that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & te & 0 \\ 0 & 0 & i \end{pmatrix}$$

continuously connects, with t positive and running from $t = 1$ to $t = -1/e > 0$ (the positive sign of the determinant is kept along the deformation),

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & ti \end{pmatrix}$$

continuously connects (with t positive and running from $t = 1$ to $t = -1/i > 0$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now, we notice that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

continuously connects, with θ running from $\theta = \pi$ to $\theta = 2\pi$ (the positive sign of the determinant is kept along the described deformation),

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The proof of the subcase $a > 0$ is complete.

The subcase $e > 0$ (and thus $a < 0$ and $i < 0$).

Analogously to the subcase above we may continuously connect

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now, we notice that

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

continuously connects, with θ running from $\theta = \pi$ to $\theta = 2\pi$ (again, the positive sign of the determinant is kept along the deformation),

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The proof of the subcase $e > 0$ is complete.

The subcase $i > 0$ (and thus $a < 0$ and $e < 0$).

Analogously to the two subcases above, we may continuously connect

$$\begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix} \text{ to } \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, we notice that

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

continuously connects, with θ running from $\theta = \pi$ to $\theta = 2\pi$ (once more, the positive sign of the determinant is kept along the deformation),

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The proof of the subcase $i > 0$ is complete.

Thus, the proof of step (6) is complete.

The proof of the theorem is complete ♣

Corollary. *Given a* 3×3 *real matrix* N *with* $\det N < 0$*, then there exists a continuous curve connecting* N *to the* 3×3 *matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

with all the 3×3 *real matrices along the deformation process having negative determinant.*

Proof.

- From the above theorem we conclude that there exists a continuous curve connecting $-N$ to the identity matrix I, with all the 3×3 matrices along the deformation process having positive determinant.

Thus, it is trivial to see that we also have a continuous curve connecting the given matrix N to the matrix $-I$, with all the matrices along the deformation process having negative determinant.

To complete this proof, we notice that

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

continuously connects, with θ running from π to 2π (we remark that the determinant of all the matrices along this last deformation are equal to -1),

$$-I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \clubsuit$$

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