SCHWARZ THEOREM (mixed partial derivatives)

Oswaldo Rio Branco de Oliveira

http://www.ime.usp.br/~oliveira   oliveira@ime.usp.br

Year 2017 —- Universidade de São Paulo, SP - Brasil

Let us write $\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R} \text{ e } y \in \mathbb{R}\}$.

In this note we employ the following lemma.

Lemma (Limit X Iterated Limit). *Let us consider $(a,b) \in \mathbb{R}^2$ and a function $g : \mathbb{R}^2 \setminus \{(a,b)\} \to \mathbb{R}$. Let us suppose that the following limits exist,*

$$\begin{cases} \displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = L \in \mathbb{R} \\ \textit{and} \\ \displaystyle\lim_{x\to a} g(x,y) = G(y) \in \mathbb{R}, \ \textit{for all } y \textit{ in an open neighborhood of } b. \end{cases}$$

Then, the following iterated limit exists and satisfies

$$\lim_{y\to b}\lim_{x\to a} g(x,y) = L.$$

Proof. See https://www.ime.usp.br/~oliveira/ELE-IteratedLimits.pdf♣

Given a real function $F = F(x,y)$, we also write

$$F_x = \frac{\partial F}{\partial x}, \ F_y = \frac{\partial F}{\partial y}, \quad F_{xy} = \frac{\partial^2 F}{\partial y \partial x} \text{ and } F_{yx} = \frac{\partial^2 F}{\partial x \partial y}.$$

Theorem (Schwarz). *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be such that F_x, F_y and F_{xy} exist on a neighborhood of $(0,0)$, with F_{xy} continuous at $(0,0)$. Then, $F_{yx}(0,0)$ exists and*

$$F_{yx}(0,0) = F_{xy}(0,0).$$

Proof.

◇ Let us consider $h \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{R} \setminus \{0\}$, both small enough. We have

$$F_{xy}(0,0) \approx \frac{F_x(0,k) - F_x(0,0)}{k} \approx \frac{\frac{F(h,k)-F(0,k)}{h} - \frac{F(h,0)-F(0,0)}{h}}{k}.$$

Without logical rigor, this points out to

$$F_{xy}(0,0) \approx \frac{F(h,k) - F(0,k) - F(h,0) + F(0,0)}{hk}.$$

◇ From now on, let us be precise. We may write

$$F(h,k)-F(0,k)-F(h,0)+F(0,0) = [F(h,k)-F(h,0)]-[F(0,k)-F(0,0)].$$

The function $x \mapsto F(x,k) - F(x,0)$ is differentiable near $x = 0$. The mean-value theorem gives a point $\overline{h}$, between 0 and h, such that

$$[F(h,k) - F(h,0)] - [F(0,k) - F(0,0)] = [F_x(\overline{h},k) - F_x(\overline{h},0)]h.$$

The function $y \mapsto F_x(\overline{h},y)$ is differentiable near $y = 0$. The mean-value theorem gives a point $\overline{k}$, between 0 and k, such that

$$F_x(\overline{h},k) - F_x(\overline{h},0) = F_{xy}(\overline{h},\overline{k})k.$$

◇ The last two highlighted identities show that

$$F_{xy}(\overline{h},\overline{k}) = \frac{1}{h}\left[\frac{F(h,k) - F(h,0)}{k} - \frac{F(0,k) - F(0,0)}{k}\right].$$

◇ By the continuity of F_{xy} at the origin we know that

$$\lim_{(h,k)\to(0,0)} F_{xy}(\overline{h},\overline{k}) = F_{xy}(0,0).$$

However, fixing h, it also exists the limit

$$\lim_{k\to 0} \frac{1}{h}\left[\frac{F(h,k) - F(h,0)}{k} - \frac{F(0,k) - F(0,0)}{k}\right] = \frac{F_y(h,0) - F_y(0,0)}{h}.$$

By the lemma we conclude that

$$\begin{aligned} F_{xy}(0,0) &= \lim_{h\to 0}\lim_{k\to 0} \frac{1}{h}\left[\frac{F(h,k) - F(h,0)}{k} - \frac{F(0,k) - F(0,0)}{k}\right] \\ &= \lim_{h\to 0} \frac{F_y(h,0) - F_y(0,0)}{h} \\ &= F_{yx}(0,0) \clubsuit \end{aligned}$$

REFERENCE

1. Hairer, E. and Wanner, G., *Analysis by Its History*, Springer, 1996, pp. 317-318