ITERATED LIMITS FOR FUNCTIONS

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Definition. The limits of the type

$$\lim_{x\to a}\lim_{y\to b} f(x,y) \quad \text{and} \quad \lim_{y\to b}\lim_{x\to a} f(x,y)$$

are called **iterated limits**.

Let us see how to relate iterated limits with the usual limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

Example (The limit may exist but the iterated limit may not exist). Let us consider the function $f:\mathbb{R}^2\smallsetminus\{(0,0)\}\to\mathbb{R}$ defined by

$$f(x,y) = x\left(\sin\frac{1}{x}\right)\left(\sin\frac{1}{y}\right).$$

Then, we have

$$|f(x,y)|\leq|x| \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

However, given $x\neq 0$ such that $\sin(1/x)\neq 0$, we clearly see that the limit

$$\lim_{y\to 0} x\left(\sin\frac{1}{x}\right)\left(\sin\frac{1}{y}\right)$$

does not exist. Summing up, we have

$$\begin{cases} \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \ \text{ exists} \\ \text{but} \\ \text{the } \textbf{iterated limit } \ \displaystyle\lim_{x\to 0}\lim_{y\to 0} x\left(\sin\tfrac{1}{x}\right)\left(\sin\tfrac{1}{y}\right) \ \text{does not exist.} \end{cases}$$

Fortunately, we have the following positive result relating limits and iterated limits.

Proposition (Limit X Iterated Limit). *Let us consider* $(a,b) \in \mathbb{R}^2$ *and a function* $f : \mathbb{R}^2 \setminus \{(a,b)\} \to \mathbb{R}$*. Let us suppose that the following limits exist,*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \in \mathbb{R} \quad \textit{and} \quad \lim_{x\to a} f(x,y) = F(y) \in \mathbb{R},$$

for all y *inside an open interval containing* b*. Then, we have*

$$\lim_{y\to b}\lim_{x\to a} f(x,y) = \lim_{(x,y)\to(a,b)} f(x,y).$$

Proof.

- Given $\epsilon > 0$, there exists an open rectangle $I \times J$ centered at (a,b) such that

  $$f(x,y) \in (L-\epsilon, L+\epsilon) \text{ for all } (x,y) \in I \times J \setminus \{(a,b)\}.$$

  We may suppose, without loss of generality, J small enough so that

  $$\lim_{x\to a} f(x,y) = F(y) \text{ for all } y \in J.$$

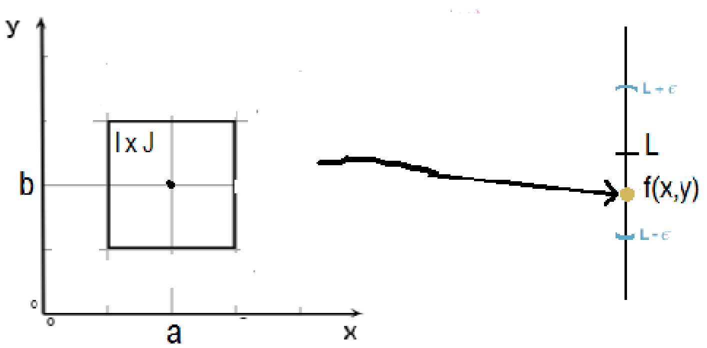

Figura 1: Ilustration to the lemma

- Fixing an arbitrary $y \in J \setminus \{b\}$, we have

  $$f(x,y) \in (L-\epsilon, L+\epsilon) \text{ for all } x \in I.$$

  Thus, for all $y \in J \setminus \{b\}$ we see that

  $$F(y) = \lim_{x\to a} f(x,y) \in [L-\epsilon, L+\epsilon].$$

  This shows that

  $$\lim_{y\to b} F(y) = L \clubsuit$$