DIVIDING A POWER SERIES BY A POLYNOMIAL

(Real and Complex Cases)

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This text proves the "Euclidean Division" of a power series by a polynomial, the complex case and the real case. It concludes with a quite easy and explicit formula for the remainder polynomial.

1. Introduction

Given two complex polynomials, or two real polynomials, in one variable (respectively, complex or real), we often compute the usual division of one by the other. This polynomial division is also called an Euclidean division for polynomials and it gives to us an unique quotient polynomial and an unique remainder polynomial (respectively, two complex polynomials or two real polynomials).

In this text we prove a similar division. We prove that given a power series A and a polynomial p, there exist an unique power series Q and an unique polynomial r satisfying

$$A = pQ + r, \text{ with degree}(r) < \text{degree}(p).$$

We also show how to compute the remainder polynomial r.

2. The Complex Case

Let z be the variable in the complex plane. We indicate the conjugate of a complex number z by $\overline{z}$. As a convention, the null polynomial has degree $-\infty$.

Let us consider an everywhere convergent complex power series, with complex coefficients,

$$A(z) = \sum_{j=0}^{+\infty} a_j z^j = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

and a complex polynomial $p = p(z)$, with complex coefficients and degree$(p) \geq 1$. Then, we have the following result.

Theorem. *There exist an unique everywhere convergent complex power series $Q(z)$, with complex coefficients, and an unique complex polynomial $r(z)$, with complex coefficients and degree$(r) < n =$degree(p), such that*

$$A(z) = p(z)Q(z) + r(z).$$

Proof. We may suppose without any loss that $p = p(z)$ is monic (i.e., dominant coefficient equal to $+1$). Now, we proceed the proof by induction on $n =$degree(p).

The Existence. We start by proving the existence of $Q = Q(z)$ and $r = r(z)$.

- The case $n = 1$. In such case we have $p(z) = z - \alpha$, for some $\alpha \in \mathbb{C}$.

  The sub-case $\alpha = 0$ is really trivial, since we have $p(z) = z$ and

  $$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = z(a_1 + a_2 z + \cdots) + a_0 = zQ(z) + r(z),$$

  with $Q(z) = (a_1 + a_2 z + a_3 z^2 + \cdots)$ and $r(z) = a_0$.

  For the sub-case $\alpha \neq 0$ we do the computations that are right below. In these, we advance that as a first step we rearrange $A(z)$ as an infinity power series $A(z) = B(z - \alpha)$, where $B(w) = \sum b_k w^k$ and $b_{k's}$ are complex coefficients, as we are allowed to do by the Rearrangement Theorem for Power Series (see Oliveira [6, p. 12], Apostol [1, p. 286], Lang [5]) which is also known as Taylor's Theorem. Then, as a second and final step, we conveniently apply the Rearrangement Theorem for Power Series one more time. Hence, as a first step, we write

$$
\begin{aligned}
A(z) &= \sum_{j=0}^{+\infty} a_j z^j \\
&= \sum_{j=0}^{+\infty} a_j [\alpha + (z-\alpha)]^j \\
&= \sum_{j=0}^{+\infty} a_j \sum_{0 \le k \le j} \alpha^{j-k} \binom{j}{k} (z-\alpha)^k \\
&= \sum_{j \ge 0} \sum_{0 \le k \le j} a_j \alpha^{j-k} \binom{j}{k} (z-\alpha)^k \\
&= \sum_{k \ge 0} \sum_{j \ge k} a_j \alpha^{j-k} \binom{j}{k} (z-\alpha)^k \\
&= \sum_{k=0}^{+\infty} \left[\sum_{j=k}^{+\infty} a_j \alpha^{j-k} \binom{j}{k} \right] (z-\alpha)^k \\
&= \sum_{k=0}^{+\infty} b_k (z-\alpha)^k \\
&= B(z-\alpha).
\end{aligned}
$$

As a second and final step we write

$$
\begin{aligned}
A(z) &= b_0 + (z-\alpha) \sum_{k=1}^{+\infty} b_k (z-\alpha)^{k-1} \\
&= b_0 + (z-\alpha) \sum_{l=0}^{+\infty} b_{l+1} (z-\alpha)^l \\
&= b_0 + (z-\alpha) \sum_{l=0}^{+\infty} b_{l+1} \sum_{0 \le m \le l} \binom{l}{m} (-\alpha)^{l-m} z^m \\
&= b_0 + (z-\alpha) \sum_{l \ge 0} \sum_{0 \le m \le l} b_{l+1} \binom{l}{m} (-\alpha)^{l-m} z^m \\
&= b_0 + (z-\alpha) \sum_{m \ge 0} \sum_{l \ge m} b_{l+1} \binom{l}{m} (-\alpha)^{l-m} z^m \\
&= b_0 + (z-\alpha) \sum_{m=0}^{+\infty} \left[\sum_{l=m}^{+\infty} b_{l+1} \binom{l}{m} (-\alpha)^{l-m} \right] z^m \\
&= b_0 + (z-\alpha) Q(z) \\
&= (z-\alpha) Q(z) + b_0.
\end{aligned}
$$

The case $n = 1$ is complete. [Its worth to notice that $b_0 = A(\alpha) = \sum_{j=0}^{+\infty} a_j \alpha^j$.]

- The induction step. Let us suppose that the claim of the theorem is true for all complex polynomials with degree less or equal to n. Then, let us consider a polynomial $p = p(z)$ with $\text{degree}(p) = n+1$. Hence, we may write

  $$p(z) = (z-\beta)q(z),$$

  where $\beta \in \mathbb{C}$ and $q = q(z)$ is a polynomial with $\text{degree}(q) = n$.

  Thus, by induction hypothesis we may write

  $$A(z) = q(z)C(z) + r(z),$$

  where the quotient $C(z)$ is an everywhere convergent power series and the remainder $r = r(z)$ is a polynomial with $\text{degree}(r) < n$.

  By the case $n = 1$ we obtain

  $$C(z) = (z-\beta)D(z) + c,$$

  with $D(z)$ an everywhere convergent power series and c a complex constant. Substituting this equation into the previous one on display, we find

  $$\begin{aligned} A(z) &= q(z)\big[(z-\beta)D(z) + c\big] + r(z) \\ &= p(z)D(z) + \big[cq(z) + r(z)\big]. \end{aligned}$$

  It is clear that

  $$\text{degree}(cq + r) \leq \text{degree}(q) = n < n + 1 = \text{degree}(p).$$

  Thus, the induction step is proven.

The existence of $Q = Q(z)$ and $r = r(z)$ is proven.

The Uniqueness. Let us suppose that $K = K(z)$ and $\rho = \rho(z)$ form another pair of a complex power series and a complex polynomial, respectively, satisfying

$$\begin{cases} A(z) = p(z)K(z) + \rho(z), \text{ for all } z, \\ \text{with degree}(\rho) < \text{degree}(p). \end{cases}$$

We then have

$$p(z)[Q(z) - K(z)] = \rho(z) - r(z), \text{ for all } z \in \mathbb{C}.$$

Let us prove, by contradiction, that the polynomial $\rho - r$ is the null polynomial.

Given an arbitrary zero λ_k of algebraic multiplicity n_k of $p(z) = 0$, it is not hard to see that there exists

$$\lim_{z \to +\lambda_k} \frac{\rho(z) - r(z)}{(z - \lambda_k)^{n_k}}.$$

Therefore, such λ_k is a zero of $\rho(z) - r(z) = 0$, with algebraic multiplicity greater or equal to n_k (at this point we are admitting that $\rho - r$ is not null). Since this is true for all zeros of $\rho(z) - r(z) = 0$, it follows that degree$(\rho - r) \geq$ degree(p) $\lightning$

This contradiction shows that $\rho - r$ is indeed the null polynomial. We then have

$$\begin{cases} \rho(z) = r(z), \text{ for all } z \in \mathbb{C}, \\ \text{and} \\ K(z) = Q(z), \text{ for all } z \in \mathbb{C}. \end{cases}$$

The uniqueness of the remainder polynomial $r = r(z)$ and of the quotient power series $Q = Q(z)$ are proven.

The proof of the theorem is complete$\clubsuit$

In section 4 “Explicit Computation of the Remainder (and its uniqueness)” we show how to compute explicitly the remainder $r = r(z)$. Such computation gives another proof of the uniqueness of $r(z)$ and, as a consequence, another proof of the uniqueness of $Q(z)$.

3. The Real Case

Let x be the variable on the real line.

We consider an everywhere convergent real power series

$$A(x) = \sum_{j=0}^{+\infty} a_j x^j = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

and a real polynomial $p = p(x)$ with degree$(p) \geq 1$. Accordingly, the coefficients of the power series $A = A(x)$ and of the polynomial $p = p(x)$ are all real numbers.

Theorem. *There exist an unique everywhere convergent real power series $Q(x)$ and an unique real polynomial $r(x)$ satisfying*

$$\begin{cases} A(x) = p(x)Q(x) + r(x), \ \textit{for all } x \in \mathbb{R}, \\ \textit{degree}(r) < \textit{degree}(p). \end{cases}$$

Proof.

- **The Existence.** From the complex case, proven in the previous section, it follows that there exist a complex power series $Q = Q(z)$ and a complex polynomial $r = r(z)$ satisfying

$$\begin{cases} A(z) = p(z)Q(z) + r(z), \ \text{for all } z \in \mathbb{C}, \\ \text{degree}(r) < \text{degree}(p). \end{cases}$$

  Let us show that the coefficients of the power series $Q = Q(z)$ and of the polynomial $r = r(z)$ are all real numbers.

  In the following, we indicate by $\overline{Q} = \overline{Q}(z)$ the power series whose coefficients are, one by one, the conjugates of the respective coefficients of the complex power series $Q = Q(z)$. Similarly, we indicate by $\overline{r} = \overline{r}(z)$ the polynomial whose coefficients are the conjugates of the respective ones of the complex polynomial $r = r(z)$.

  Hence, taking conjugates, we have the equation

$$\overline{A(\overline{z})} = \overline{p(\overline{z})}\,\overline{Q(\overline{z})} + \overline{r(\overline{z})}, \ \text{for all } z.$$

  Since the power series $A = A(z)$ and also the polynomial $p = p(z)$ have real coefficients, we have the identities

$$\overline{A(\overline{z})} = A(z) \ \text{and} \ \overline{p(\overline{z})} = p(z).$$

Thus, substituting these identities into the previous equation, we arrive at

$$\begin{cases} A(z) = p(z)\overline{Q}(z) + \overline{r}(z), \text{for all } z \in \mathbb{C}, \\ \\ \text{degree}(\overline{r}) < \text{degree}(p). \end{cases}$$

By the uniqueness proven in the complex case, it follows that

$$\begin{cases} \overline{Q}(z) = Q(z) \text{ for all } z, \\ \text{and} \\ \overline{r}(z) = r(z) \text{ for all } z. \end{cases}$$

At last, by the Uniqueness Theorem for the Coefficients of a Power Series (see Oliveira [7, p. 7; 6, p. 15], Beardon [2, pp. 112–113]), and by the identity principle for polynomials, we conclude that all the coefficients of the power series $Q(z)$ and of the remainder polynomial $r(z)$ are, in fact, real numbers.

The existence of $Q = Q(z)$ and $r = r(z)$ is proven.

- **The Uniqueness.** The uniqueness of the power series $Q(z)$ and of the remainder polynomial $r(z)$ are obvious consequences of the uniqueness proven in the previous section (section 2, The Complex Case).

The proof of the theorem is complete♣

In the next section we show how to compute explicitly the remainder $r = r(z)$. Such computation gives another proof of the uniqueness of $r(z)$ (and thus, it also gives another proof of the uniqueness of power series $Q(z)$).

4. Explicit Computation of the Remainder (and its uniqueness)

Theorem. *Let $A(z)$ and $Q(z)$ be two everywhere convergent complex power series, and p and r be two complex polynomials satisfying*

$$\begin{cases} A(z) = p(z)Q(z) + r(z), \text{ for all } z, \\ \text{with} \\ p \text{ monic and degree}(r) < \text{degree}(p) = m. \end{cases}$$

Let $\lambda_1, \ldots, \lambda_l$ be the distinct zeros of $p(z)$, with respective multiplicities $m_1, \ldots, m_l$. [We remark that $m_1 + \cdots + m_l = m$ and, obviously, $p(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_l)^{m_l}$.]

Then, there are m constants

$$C_{1,1}, \ldots, C_{1,m_1}, \ldots, C_{l,1}, \ldots, C_{l,m_l}$$

such that we have

$$\frac{A(z)}{p(z)} = Q(z) + \left[\frac{C_{1,1}}{z - \lambda_1} + \cdots + \frac{C_{1,m_1}}{(z - \lambda_1)^{m_1}}\right] + \cdots + \left[\frac{C_{l,1}}{z - \lambda_l} + \cdots + \frac{C_{l,m_l}}{(z - \lambda_l)^{m_l}}\right],$$

for all z outside $\{\lambda_1, \ldots, \lambda_l\}$. These m constants are given by

$$C_{j,k} = \frac{f_j^{(m_j - k)}(\lambda_j)}{(m_j - k)!},$$

where

$$f_j(z) = \frac{A(z)(z - \lambda_j)^{m_j}}{p(z)}, \text{ for each } j = 1, \ldots, l.$$

Proof. It is clear that we have

$$\frac{A(z)}{p(z)} = Q(z) + \frac{r(z)}{p(z)}, \text{ if } z \notin \{\lambda_1, \ldots, \lambda_l\}.$$

By applying the well known result The Partial Fraction Decomposition Theorem (see Gamelin [3, p. 179–181], Hairer & Wanner [4, pp. 119–122], Spivak [9, p. 378]) to the quotient $r(z)/p(z)$ it follows that there are m complex constants $C_{1,1}, \ldots, C_{1,m_1}, \ldots, C_{l,1}, \ldots, C_{l,m_l}$ such that we may write

$$\frac{r(z)}{p(z)} = \sum_{j=1}^{l} \left[\sum_{k=1}^{m_j} \frac{C_{j,k}}{(z - \lambda_j)^k}\right], \text{ for all } z \text{ outside } \{\lambda_1, \ldots, \lambda_l\}.$$

Let us show how to calculate the constants $C_{1,1}, \ldots, C_{1,m_1}$. The computations of the other constants are analogous.

Substituting the last equation into the previous one, we arrive at the identity

$$\frac{A(z)}{p(z)} = Q(z) + \sum_{j=1}^{l}\left[\sum_{k=1}^{m_j}\frac{C_{j,k}}{(z-\lambda_j)^k}\right], \text{ if } z \notin \{\lambda_1,\ldots,\lambda_l\}.$$

The left-hand side and the right-hand side of such identity have a singularity at the point $z = \lambda_1$, and this singularity has order m_1. We eliminate such singularity by multiplying both sides by $(z-\lambda_1)^{m_1}$. Then, highlighting the powers of $(z-\lambda_1)$ on the right-hand side we conveniently write

$$\begin{aligned}\frac{A(z)(z-\lambda_1)^{m_1}}{p(z)} &= (z-\lambda_1)^{m_1}\left[Q(z) + \sum_{2\leq j\leq l}\left(\sum_{1\leq k\leq m_j}\frac{C_{j,k}}{(z-\lambda_j)^k}\right)\right] \\ &\quad + C_{1,1}(z-\lambda_1)^{m_1-1} + C_{1,2}(z-\lambda_1)^{m_1-2} + \cdots + C_{1,m_1}(z-\lambda_1)^{m_1-m_1}.\end{aligned}$$

With obvious identifications, we simplify and rewrite this last equation as

$$\begin{aligned}f_1(z) &= (z-\lambda_1)^{m_1}Q_1(z) \\ &\quad + C_{1,1}(z-\lambda_1)^{m_1-1} + C_{1,2}(z-\lambda_1)^{m_1-2} + \cdots + C_{1,m_1-1}(z-\lambda_1)^1 + C_{1,m_1},\end{aligned}$$

with $f_1(z)$, and $Q_1(z)$, differentiable on a neighborhood of the point $z = \lambda_1$.

At this point, we become very happy to notice that all the derivatives of order $0, 1, \ldots, m_1 - 1$ of the function $(z-\lambda_1)^{m_1}Q_1(z)$ are zero at the point $z = \lambda_1$.

Hence, we have the set of identities

$$\begin{cases} f_1^{(0)}(\lambda_1) &= 0!C_{1,m_1}, \\ f_1^{(1)}(\lambda_1) &= 1!C_{1,m_1-1}, \\ f_1^{(2)}(\lambda_1) &= 2!C_{1,m_1-2}, \\ & \vdots \\ f_1^{(m_1-1)}(\lambda_1) &= (m_1-1)!C_{1,1}. \end{cases}$$

The proof is complete ♣

As already stated, at the end of the last section (Section 3), this explicit computation of the remainder gives another proof of the uniqueness of $r(z)$ and thus it also gives another proof of the uniqueness of power series $Q(z)$.

5. Comparing the two Divisions of a Power Series by a Polynomial

There are two most known divisions of a power series by a polynomial.

The one very often used is the Euclidean Division of a Power Series by a Polynomial, simply called Division of a Power Series by a polynomial. This is the division presented in this text.

The other division is the Long Division, usually applied for the division of a power series by another power series (see Oliveira [8]). This long division is usually applied for two power series that are not polynomials. A power series that is not a polynomial has infinite nonzero coefficients, and it is many times called an infinite power series to distinguish it from polynomials (a polynomial can be said to be "a finite power series", but such label is awkward and thus avoided at all costs).

For examples of the long division of a power series by a polynomial, as well as of the long division of a polynomial by a polynomial, see Oliveira [8].

Given an infinite power series and a polynomial we remark that while the "Euclidean division" gives to us a quotient (which is a power series) and a remainder (which is a polynomial, presumably simpler than the one we started with), the long division algorithm gives to us an exact quotient (hence, no remainder) that is a power series (most probably, an infinite power series).

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