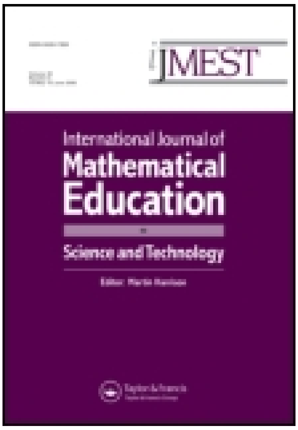

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A formula substituting the undetermined coefficients and the annihilator methods

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A formula substituting the undetermined coefficients and the annihilator methods

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This article presents an easy formula for obtaining a particular solution of a linear ordinary differential equation with constant real coefficients, $P(\frac{d}{dt})x = f$, where f is a function given by a linear combination of polynomials, trigonometrical and exponential real function products. The desired solution is obtained through solving straightforward lower triangular linear systems with constant coefficients. Three examples are given.

Keywords: ordinary differential equations; linear equations and systems; linear differential equations; complex exponential

AMS Subject Classifications: 34A30; 97D40; 30-01

1. Introduction

The objective of this article is to present a very simple formula that provides an alternative method to the method of undetermined coefficients [1–9] and also to the annihilator method [8–10], both very well known, of solving a linear ordinary differential equation with constant real coefficients, $P(\frac{d}{dt})x = f$, where f is a real function given by a linear combination of polynomials, trigonometrical functions and exponential functions products. This is accomplished by reducing the given differential equation to the trivial case in which f is a polynomial. Most text books develop the method of undetermined coefficients but not the annihilator method, considered to be slightly sophisticated. In most books, it is not proven that the undetermined coefficients strategy is a valid one and such approach is, in some texts, named 'the guessing method' or 'the lucky guess method'. In general, applying either the undetermined coefficients method or the annihilator method requires a large amount of computation.

It is worth pointing out that Ross [7] clearly explains why the undetermined coefficients strategy works. Moreover, Gupta [11] shows (without proof) a recursive algorithm for solving the same differential equations studied in this article. However, this article provides an easy formula that further simplifies the task of obtaining a particular solution for such equations. The method presented here uses this formula and has four advantages. First, the formula is rather easy. Second, at many times this method reduces the amount of computation necessary in comparison with either the undetermined coefficients method or the annihilator method. Third, the application of this method does not require calculations in the differential operators algebra. Fourth, the proof of the formula is accessible to first-year students and is also rather easy. In fact, this proof only uses the general Leibniz rule (for the n-th derivative of a

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product of two functions), the complex exponential function, polynomial integration and basic arithmetics.

As is well known, the general solution of the homogeneous equation associated with the given ordinary differential equation, $P(\frac{\mathrm{d}}{\mathrm{d}t})x = f$, is a consequence of the fundamental theorem of algebra [12], and can be found in the textbooks cited in the references.

The method presented in this article uses complex arithmetic and the complex exponential function. It is interesting pointing out the 'rising need for general competency in complex operations at this level (the undergraduate level) among members of the mathematical community', see [13, p. 23].

2. Preliminaries and two lemmas

Given an infinitely differentiable function f: $\mathbb{R}\rightarrow\mathbb{C}$, then $f^{(k)}$ indicates the k-th derivative of f. We also write, $f=f^{(0)}$, $f'=f^{(1)}$, $f''=f^{(2)}$, $f'''=f^{(3)}$, etc. Moreover, we indicate $C^{\infty}(\mathbb{R};\mathbb{C})=\{f\colon \mathbb{R}\rightarrow\mathbb{C}\colon f^{(m)}$ exists for all $m\in\mathbb{N}\}$.

Before proving the first result of this article, the rather intuitive Lemma 1, we consider the following illustrative example.

Example

(E1) Let us show that there exists a real polynomial solution $p=p(t)$ of the linear ordinary differential equation $x'''+4x''+2x'=t^4$, where $x=x(t)$: $\mathbb{R}\rightarrow\mathbb{R}$. Using the substitution $y=x'$, we proceed by looking for a real polynomial solution of the differential equation $y''+4y'+2y=t^4$, $t\in\mathbb{R}$. If $y=y(t)$ is such solution, then it follows at once that y has degree 4. Thus, we have that $y(t)=at^4+bt^3+ct^2+\mathrm{d}t+e$, where a, b, c, d and e are real numbers. Substituting this expression for $y(t)$ into the equation $y''+4y'+2y=t^4$ and identifying the coefficients of t^4, t^3, t^2, t, and also the constant terms, in this order, respectively, we obtain the trivial lower triangular linear system

$$\begin{cases} 2a = 1, \\ 16a + 2b = 0, \\ 12a + 12b + 2c = 0, \\ 0a + 6b + 8c + 2d = 0, \\ 0a + 0b + 2c + 4d + 2e = 0. \end{cases}$$

Thus, we have that $y(t) = \frac{t^4}{2} - 4t^3 + 21t^2 - 72t + 123$. Finally, we conclude that $p(t) = \frac{t^5}{10} - t^4 + 7t^3 - 36t^2 + 123t = t\big(\frac{t^4}{10} - t^3 + 7t^2 - 36t + 123\big)$ is a real polynomial solution of the initial differential equation. □

Lemma 1: *Let us consider the linear ordinary differential inhomogeneous equation with coefficients* $a_j\in\mathbb{C}$, $0\le j\le n$, *with at least one* $a_j\neq 0$, $1\le j\le n$,

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = R, \quad x = x(t),\ t\in\mathbb{R},$$

where the function $R=R(t)=b_n t^n+\cdots+b_1 t+b_0$ *is a nonzero polynomial in the real variable* t, *with coefficients* $b_j\in\mathbb{C}$, $0\le j\le n$. *We claim that this equation has a polynomial solution* $Q=Q(t)$: $\mathbb{R}\rightarrow\mathbb{C}$. *We also claim that,*

(a) *If* $a_0\neq 0$, *then we have degree*$(Q)=$*degree*(R).

(b) *If* $k = \max\{j\colon a_l = 0,\ l \le j\}$ *exists, then we can admit* $Q = t^{k+1}Q_1$, *where* Q_1 *is a polynomial satisfying degree*$(Q_1) =$ *degree*(R).

(c) *If all the above coefficients are real, then* Q *and* Q_1 *are real polynomials.*

Proof: If k exists, then it is quite clear that $k < n$.

(a) Let us solve the pair of equations

$$\begin{cases} Q(t) = c_n t^n + \cdots + c_1 t + c_0, \\ a_0 Q + a_1 Q' + \cdots + a_j Q^{(j)} + \cdots + a_n Q^{(n)} = R, \end{cases} \tag{1.1}$$

through identifying the coefficient of the monomial t^{n-l}, where $l \le n$, at the terms $a_j Q^{(j)}$, where $j \le l$, by noticing that at the other similar terms the coefficient of the monomial t^{n-l} is zero. Fix the term $a_j Q^{(j)}$, where $j \le l$, a factor of the desired coefficient comes up with the calculation

$$c_{n-l+j}\frac{d^j}{dt^j}\left\{ t^{n-l+j}\right\} = c_{n-l+j}(n-l+j)(n-l+j-1)\cdots(n-l+1)t^{n-l}.$$

Hence, the coefficient looked for is $a_j c_{n-l+j}\frac{(n-l+j)!}{(n-l)!}$.
Therefore, substituting in (1.1) the expression $Q(t) = c_n t^n + \cdots + c_1 t + c_0$, and its derivatives, we conclude that the coefficient of the monomial t^{n-l} satisfies the identity (in which the sum is written in decreasing order on j, where j decreases from $j = l$ to $j = 0$)

$$a_l c_n \frac{n!}{(n-l)!} + \cdots + a_j c_{n-l+j}\frac{(n-l+j)!}{(n-l)!} + \cdots + a_0 c_{n-l} = b_{n-l},\ 0 \le l \le n. \tag{1.2}$$

From the expressions given by (1.2) follows the trivial matricial equation,

$$\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ a_1 n & a_0 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ a_2\frac{n!}{(n-2)!} & a_1\frac{(n-1)!}{(n-2)!} & a_0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ a_l\frac{n!}{(n-l)!} & a_{l-1}\frac{(n-1)!}{(n-l)!} & . & a_j\frac{(n-l+j)!}{(n-l)!} & . & . & a_0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & a_0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & a_0 & 0 \\ a_n n! & . & . & . & . & . & . & . & a_2 2! & a_1 & a_0 \end{bmatrix} \begin{bmatrix} c_n \\ c_{n-1} \\ c_{n-2} \\ . \\ . \\ . \\ c_{n-l} \\ . \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ . \\ . \\ . \\ b_{n-l} \\ . \\ b_2 \\ b_1 \\ b_0 \end{bmatrix}.$$

(b) In this case, the differential equation is $a_n x^{(n)} + \cdots + a_{k+1}x^{(k+1)} = R$. Thus, by (a), the differential equation $a_n y^{(n-k-1)} + \cdots + a_{k+1}y = R$ has a polynomial solution $y(t) = S(t)$, with degree(S) = degree(R). Integrating $(k+1)$-times the polynomial $y = S(t)$ and choosing, at each time, zero for independent term, we reach the desired polynomial solution $Q = t^{k+1}Q_1$, where Q_1 is a polynomial satisfying degree$(Q_1) =$ degree(R).

(c) It is trivial. □

Notation: In the next two results, we consider a differential operator,

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = a_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0 I,$$

where $t \in \mathbb{R}$, $a_j \in \mathbb{R}$, $0 \le j \le n$, and I is the identity operator over $C^\infty(\mathbb{R};\mathbb{C})$. We also consider the characteristic polynomial of $P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$,

$$p(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0, \text{ where } \lambda \in \mathbb{C}.$$

Lemma 2: *If* $Q = Q(t) \in C^\infty(\mathbb{R};\, \mathbb{C})$ *and* $\gamma \in \mathbb{C}$, *then we have*

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[Q(t)e^{\gamma t}] = \left[\frac{p^{(n)}(\gamma)}{n!}Q^{(n)} + \cdots + \frac{p''(\gamma)}{2!}Q'' + \frac{p'(\gamma)}{1!}Q' + \frac{p(\gamma)}{0!}Q\right]e^{\gamma t}.$$

Proof: Employing the general Leibniz rule, a short calculation shows that

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)[Q(t)e^{\gamma t}] = \sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}[Q(t)e^{\gamma t}] = \left[\sum_{k=0}^{n} a_k \sum_{j=0}^{k}\binom{k}{j}Q^{(j)}(t)\gamma^{k-j}\right]e^{\gamma t}.$$

Hence, changing the order of summation we conclude that

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{k}\binom{k}{j}Q^{(j)}\gamma^{k-j} = \sum_{j=0}^{n}\left[\sum_{k=j}^{n} a_k \frac{k!}{(k-j)!}\gamma^{k-j}\right]\frac{Q^{(j)}}{j!} = \sum_{j=0}^{n} p^{(j)}(\gamma)\frac{Q^{(j)}}{j!}.$$

□

3. Main result

Let R be a nonzero real polynomial and $(\gamma,\, \delta)$ be a pair of numbers such that

- γ is complex and δ is real, with the condition that $\delta = 0$ if γ is real.

Theorem 3: *The equation* $P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)x = R(t)e^{\gamma t + i\delta}$ *has a particular solution* $Q(t)e^{\gamma t + i\delta}$, *where* $Q(t)$ *is an arbitrary polynomial satisfying the differential equation,*

$$\frac{p^{(n)}(\gamma)}{n!}Q^{(n)} + \cdots + \frac{p'(\gamma)}{1!}Q' + \frac{p(\gamma)}{0!}Q = R. \tag{3.1}$$

Moreover,

(a) *If* $\gamma \in \mathbb{R}$, *then we can suppose that* Q *is real. In such case, the real function* $x(t) = Q(t)e^{\gamma t}$ *is a solution of the initial equation.*

(b) *If* $\gamma \notin \mathbb{R}$, *then* $z(t) = Q(t)e^{\gamma t + i\delta}$ *is a complex solution. If* $\gamma = \alpha + \beta i$, *where* α, $\beta \in \mathbb{R}$, *then the functions* $x(t) = \mathrm{Re}[z(t)]$ *and* $y(t) = \mathrm{Im}[z(t)]$ *satisfy*

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)x = R(t)e^{\alpha t}\cos(\beta t + \delta), \quad P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = R(t)e^{\alpha t}\sin(\beta t + \delta).$$

(c) *If* $p(\gamma) \neq 0$, *then we have degree*$(Q) =$ *degree*(R).

(d) *If* γ *is a root of multiplicity* k of the characteristic polynomial, *then we can choose a polynomial* $Q(t) = t^k Q_1(t)$, *with degree*$(Q_1) =$ *degree*(R).

Proof: Let us employ Lemma 2. Searching for a solution $Q(t)e^{\gamma t+i\delta}$ of the initial differential equation, we find the identity

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[Q(t)e^{\gamma t+i\delta}\right] = \left[\frac{p^{(n)}(\gamma)}{n!}Q^{(n)} + \cdots + \frac{p'(\gamma)}{1!}Q' + p(\gamma)Q\right]e^{\gamma t+i\delta} = R(t)e^{\gamma t+i\delta}.$$

Thus, we deduce Equation (3.1) at once.

In addition, employing Lemma 1 and solving a trivial lower triangular linear system we promptly obtain a polynomial solution $Q(t)$ of (3.1).

Next, we verify the properties (a)–(d).

(a) If γ is real, from Lemma 1 it follows that we can suppose Q also real.
(b) It follows immediately.
(c) It follows immediately.
(d) If γ is a root of multiplicity k of p, then Equation (3.1) becomes,

$$\frac{p^{(n)}(\gamma)}{n!}Q^{(n)} + \cdots + \frac{p^{(k)}(\gamma)}{k!}Q^{(k)} = R(t), \quad p^{(k)}(\gamma) \neq 0,$$

which has a polynomial solution $y = Q^{(k)}$, with degree$(Q^{(k)})$ = degree(R). Integrating k-times the function $y(t)$, but choosing at each time the independent term as zero, we obtain a suitable polynomial solution of (3.1).

□

Examples:

(E2) Let us solve the differential equation $x'' - 2x' + 2x = t^2e^t \sin(3t+5)$, where $x = x(t)\colon \mathbb{R} \to \mathbb{R}$.

The characteristic polynomial $p(\lambda)$ satisfies,

$$p(\lambda) = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1, \quad p'(\lambda) = 2(\lambda - 1), \text{ and } p''(\lambda) = 2\ .$$

The general solution of the homogeneous associated equation is

$$c_1e^t \cos t + c_2e^t \sin t, \qquad c_1, c_2 \in \mathbb{R}.$$

Employing Theorem 3, we deduce that the complex equation

$$z'' - 2z' + 2z = t^2e^{(1+3i)t+5i}$$

has a solution $z(t) = Q(t)e^{(1+3i)t+5i}$, with Q a polynomial that satisfies

$$Q'' + p'(1+3i)Q' + p(1+3i)Q = t^2. \qquad \text{(E2.1)}$$

Substituting $p(1+3i) = -8$ and $p'(1+3i) = 6i$ in Equation (E2.1), we obtain

$$Q'' + 6iQ' - 8Q = t^2.$$

Thus, we have degree$(Q) = 2$ and $Q(t) = -\frac{t^2}{8} + at + b$, with $a, b \in \mathbb{C}$. A short computation reveals that

$$Q(t) = -\frac{t^2}{8} - \frac{3it}{16} + \frac{7}{64}.$$

Hence, we obtain

$$\begin{cases} z(t) = \left[-\frac{t^2}{8} - \frac{3it}{16} + \frac{7}{64} \right] e^t \, [\cos(3t+5) + i \sin(3t+5)], \\ \mathrm{Im}[z(t)] = e^t \left[-\frac{t^2 \sin(3t+5)}{8} - \frac{3t\cos(3t+5)}{16} + \frac{7\sin(3t+5)}{64} \right]. \end{cases}$$

Therefore, the general solution of the given differential equation is

$$x(t) \;=\; c_1 e^t \cos t + c_2 e^t \sin t \;+\; e^t \left[\left(-\frac{t^2}{8} + \frac{7}{64} \right) \sin(3t+5) - \frac{3t}{16} \cos(3t+5) \right],$$

where c_1 and c_2 are arbitrary real constants. □

(E3) Let us solve the differential equation $x''' - 5x'' + 3x' + 9x = t^5 e^{3t}$, where $x = x(t)\colon \mathbb{R} \to \mathbb{R}$.
The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 5\lambda^2 + 3\lambda + 9 = (\lambda - 3)^2 (\lambda + 1).$$

The general solution of the associated homogeneous equation is

$$c_1 e^{3t} + c_2 t e^{3t} + c_3 e^{-t}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Employing Theorem 3, we deduce that the given differential equation has a particular solution $Q(t)e^{3t}$, with Q a polynomial satisfying

$$\frac{p'''(3)}{3!} Q''' + \frac{p''(3)}{2!} Q'' + \frac{p'(3)}{1!} Q' + \frac{p(3)}{0!} Q = t^5.$$

Since $p' = 3\lambda^2 - 10\lambda + 3$, $p'' = 6\lambda - 10$, and $p''' = 6$, we easily obtain

$$Q''' + 4Q'' = t^5.$$

Substituting $y = Q''$ into this last equation, we arrive at the differential equation $y' + 4y = t^5$, which has a solution $y = \frac{t^5}{4} + at^4 + bt^3 + ct^2 + dt + e$, for constants a, b, c, d, $e \in \mathbb{R}$. A short calculation shows that

$$y(t) = \frac{t^5}{4} - \frac{5t^4}{16} + \frac{5t^3}{16} - \frac{15t^2}{64} + \frac{15t}{128} - \frac{15}{512}.$$

Hence, by integrating $y(t)$, we can choose

$$Q(t) = \frac{t^7}{168} - \frac{t^6}{96} + \frac{t^5}{64} - \frac{5t^4}{256} + \frac{5t^3}{256} - \frac{15t^2}{1024}.$$

Therefore, the general solution of the given differential equation is

$$x(t) = c_1 e^{3t} + c_2 t e^{3t} + c_3 e^{-t} + \left(\frac{t^7}{168} - \frac{t^6}{96} + \frac{t^5}{64} - \frac{5t^4}{256} + \frac{5t^3}{256} - \frac{15t^2}{1024} \right) e^{3t}, \quad c_i's \in \mathbb{R}.$$

□

If the function f, in the differential equation $P(\frac{d}{dt})x = f$, is given by a sum $\sum_{j=1}^{N} f_j$, where each function $f_j(t)$ is a polynomial, trigonometrical and exponential real function product, we proceed by finding a particular solution x_j of the equation $P(\frac{d}{dt})x = f_j$, $1 \leq j \leq N$. It is clear that the function $x = x_1 + \cdots + x_N$ is a solution of the equation $P(\frac{d}{dt})x = f$.

Remark: The method employed in this article can be adapted to similar difference equations, see [7].

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Sampling distributions for introductory statistics students using internet polling software

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