ASYMPTOTIC ANALYSIS OF A SEMILINEAR ELLIPTIC EQUATION IN HIGHLY OSCILLATING THIN DOMAINS

MARCONE C. PEREIRA†

Abstract. In this work we are interested in the asymptotic behavior of a family of solutions of a semilinear elliptic problem with homogeneous Neumann boundary condition defined in a 2-dimensional bounded set which degenerates to the unit interval as a positive parameter ϵ goes to zero. Here we also allow that upper and lower boundaries from this singular region present highly oscillatory behavior with different orders and variable profile. Combining results from linear homogenization theory and nonlinear analyzes we get the limit problem showing upper and lower semicontinuity of the solutions at $\epsilon = 0$.

1. Introduction

In this paper we analyze the asymptotic behavior of the solutions of a semilinear elliptic problem with homogeneous Neumann boundary condition posed in a 2-dimensional bounded region R^ϵ which degenerates to the unit interval as the positive parameter $\epsilon \to 0$. Here we allow that upper and lower boundaries of this thin domain present highly oscillatory behavior with different orders of oscillations and variable profile following previous works as [6, 22].

The thin domain R^ϵ is given by

$$R^\epsilon = \{(x, y) \in \mathbb{R}^2 \; : \; x \in (0, 1) \text{ and } - \epsilon \, G_\epsilon(x) < y < \epsilon \, H_\epsilon(x)\} \tag{1.1}$$

where the functions G_ϵ and $H_\epsilon : (0, 1) \mapsto \mathbb{R}$ are suppose to be positive, smooth and uniformly bounded in $\epsilon > 0$. They set the lower and upper boundary of the thin domain R^ϵ which possesses thickness of order ϵ. Also, we take G_ϵ and H_ϵ depending on ϵ in such way that R^ϵ presents different orders of oscillations with variable profile. We establish this assuming

$$G_\epsilon(x) = G(x, x/\epsilon^\beta), \quad \text{for some } \beta > 1, \quad \text{and} \quad H_\epsilon(x) = H(x, x/\epsilon), \tag{1.2}$$

where G and $H : [0, 1] \times \mathbb{R} \mapsto (0, \infty)$ are smooth functions such that $y \to G(\cdot, y)$ and $y \to H(\cdot, y)$ are periodic in variable y, that is, there exist positive constants l_g and l_h such that $G(x, y + l_g) = G(x, y)$ and $H(x, y + l_h) = H(x, y)$ for all $(x, y) \in (0, 1) \times \mathbb{R}$. Note that the upper boundary of R^ϵ defined by $\epsilon \, H_\epsilon$ possesses same order of amplitude, period and thickness. On the other hand, the lower boundary given by $\epsilon \, G_\epsilon$ presents oscillation order larger than the compression order ϵ. For instance see Figure 1.

2010 *Mathematics Subject Classification.* 34B15, 35J61, 35J91.

Key words and phrases. Semilinear elliptic equations, upper semicontinuity, lower semicontinuity, thin domains.

†Partially supported by FAPESP 2013/22275-1, CNPq 302960/2014-7 and 471210/2013-7, Brazil.

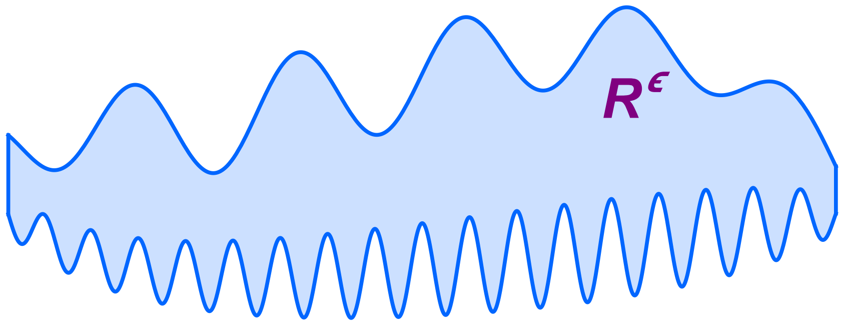

FIGURE 1. A thin domain with double oscillatory boundary.

In the thin domain R^ϵ we consider the following semilinear elliptic equation

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f(w^\epsilon) & \text{in } R^\epsilon \\ \partial_{\nu^\epsilon} w^\epsilon = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{1.3}$$

where ν^ϵ is the unit outward normal to ∂R^ϵ, ∂_{ν^ϵ} is the outwards normal derivative and the function $f : \mathbb{R} \mapsto \mathbb{R}$ is supposed to be a bounded $\mathcal{C}^2$-function satisfying

$$|f(w)| + |f'(w)| + |f''(w)| \leq C_f \text{ for all } w \in \mathbb{R}. \tag{1.4}$$

We note that from the point of view of investigating the asymptotic behavior of the solutions given by (1.3) to assume f bounded with bounded derivatives does not imply any restriction since we are interested in solutions uniformly bounded in L^∞ norms. See also [9].

Now, in order to analyze problem (1.3) we first perform a change of variables consisting in stretch R^ϵ in the y-direction by a factor of $1/\epsilon$. Following the pioneering works [15, 26], we set $x_1 = x, x_2 = y/\epsilon$ to transform R^ϵ into the region

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 \in (0,1) \text{ and } -G_\epsilon(x_1) < x_2 < H_\epsilon(x_1)\}. \tag{1.5}$$

By doing so, we get a domain which is not thin anymore although it still presents very oscillatory behavior. Indeed upper and lower boundaries of Ω^ϵ are the graph of the oscillating functions G_ϵ and H_ϵ. Consequently we transform problem (1.3) into

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f(u^\epsilon) & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases} \tag{1.6}$$

where $N^\varepsilon = (N_1^\varepsilon, N_2^\varepsilon)$ is the outward normal to the boundary $\partial\Omega^\epsilon$.

Observe the factor $1/\epsilon^2$ in front of the x_2-derivative stablishes a very fast diffusion in the vertical direction. In some way, we are substituting the thin domain R^ϵ with a non thin domain Ω^ϵ but with a very strong diffusion mechanism in the x_2-direction. The presence of this very strong diffusion mechanism implies that the solutions of (1.6) become homogeneous in the x_2-direction as $\epsilon \to 0$. Hence the limiting solution will not have a dependence in this direction being solution of a 1-dimensional limiting problem. Indeed, this is in agreement with the intuitive idea that (1.6) approaches an equation in a line segment.

It is known from [22] that the limit problem to (1.6) is given by

$$\begin{cases} -\dfrac{1}{p(x)}\,(q(x)\,u_x)_x + u = f(u), \quad x \in (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases} \tag{1.7}$$

where functions p and $q : (0,1) \mapsto (0,\infty)$ are smooth an positive given by

$$\begin{aligned} q(x) &= \frac{1}{l_h} \int_{Y^*(x)} \left\{ 1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2) \right\} dy_1 dy_2, \\ p(x) &= \frac{|Y^*(x)|}{l_h} + \frac{1}{l_g} \int_0^{l_g} G(x,y)\,dy - G_0(x), \\ G_0(x) &= \min_{y \in \mathbb{R}} G(x,y). \end{aligned} \tag{1.8}$$

$X(x)$ is the unique solution of the auxiliary problem

$$\begin{cases} -\Delta X(x) = 0 \text{ in } Y^*(x) \\ \partial_N X(x) = 0 \text{ on } B_2(x) \\ \partial_N X(x) = N_1 \text{ on } B_1(x) \\ X(x)\ l_h\text{-periodic on } B_0(x) \\ \int_{Y^*(x)} X(x)\ dy_1 dy_2 = 0 \end{cases} \tag{1.9}$$

in the representative cell $Y^*(x)$ given by

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < l_h, \quad -G_0(x) < y_2 < H(x, y_1)\}, \tag{1.10}$$

where B_0, B_1 and B_2 are lateral, upper and lower boundary of ∂Y^* for each $x \in (0,1)$. Note that the auxiliary solution X and the representative cell Y^* depend on x setting a non constant and positive *homogenized coefficient* q for the *homogenized equation* (1.7).

Several are the works in the literature concerned with partial differential equations posed in thin domains with oscillatory boundaries. In [19] the authors have studied the solutions of linear elliptic equations in thin perforated domains. Stokes and Navier-Stokes problems are also consider in thin channels [20, 21]. In [10, 12] the asymptotic description of nonlinearly elastic thin films with fast-oscillating profile was obtained in a context of Γ-*convergence* [14].

Recently we have studied different classes of oscillating thin domains discussing limit problems and convergence properties [2, 4, 5, 23, 24]. See also [7] where the authors considere linear problems with varying period and [6, 8] where they deal with linear elliptic problems in thin channels presenting doubly oscillatory behavior and constant profile. In [22] we have gotten the upper semicontinuity of the attractors of the parabolic problem associated to problem (1.3) also proving the upper semicontinuity of the equilibria set.

Our main goal here is to show the lower semicontinuity of the family of solutions given by (1.3) under the assumption that all the solutions of the limit problem (1.7) are *hyperbolic*. Indeed we combine upper and lower semicontinuity to get continuity also approaching the problem with a kind of *fixed point* functional setting. In some sense, we are discussing here some conditions in order to approximate the singular equation (1.3) to the regular one (1.7) showing how the geometry and the oscillatory behavior of the thin domain affect the problem. Recall that a solution u of a nonlinear boundary value problem is called hyperbolic if the eigenvalues of the linearized equation around u are all different from zero.

Finally we note that different conditions in the lateral boundaries of the thin domain may be set preserving the Neumann type boundary condition in the upper and lower boundary of R^ϵ. Dirichlet or even Robin homogeneous can be set in the lateral boundaries of (1.6). The limit problem will preserve this boundary condition as a point condition. On the other hand, if we assume Dirichlet boundary condition in whole ∂R^ϵ, the family of solutions converges to the null function. See for example [25].

2. Basic facts and notations

We assume that functions G_ϵ and $H_\epsilon : (0,1) \to (0,\infty)$ satisfy the following assumption:
(H) there exist nonnegative constants G_0, G_1, H_0 and H_1 such that

$$0 < G_0 \le G_\epsilon(x) \le G_1 \quad \text{ and } \quad 0 < H_0 \le H_\epsilon(x) \le H_1,$$

for all $x \in (0,1)$ and $\epsilon \in (0,\epsilon_0)$, for some $\epsilon_0 > 0$, with G_ϵ and H_ϵ given by expressions (1.2) where the functions H and $G : [0,1] \times \mathbb{R} \mapsto (0,+\infty)$ are periodic in the second variable, in such a way that there exist positive constants l_g and l_h such that $G(x,y+l_g) = G(x,y)$ and $H(x,y+l_h) = H(x,y)$ for all $(x,y) \in [0,1] \times \mathbb{R}$. We also suppose G and H are piecewise C^1 with respect to the first variable, it means, there exists a finite number of $0 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$ such that G and H restricted to the set $(\xi_i, \xi_{i+1}) \times \mathbb{R}$ are C^1 with G, H, G_x, H_x, G_y and H_y uniformly bounded in $(\xi_i, \xi_{i+1}) \times \mathbb{R}$ having limits when we approach ξ_i and ξ_{i+1}.

Observe that hypothesis **(H)** set the geometric conditions on the thin domain R^ϵ, and consequently, on the oscillating domain Ω^ϵ where problems (1.3) and (1.6) are posed. Also, it is worth noting that any function defined in the unit interval $(0,1)$ can be seen as a function in R^ϵ or Ω^ϵ just extending it as a constant in the vertical direction.

In order to investigate the asymptotic behavior of solutions of the semilinear problem (1.6) we first need to study the associated linear elliptic equation

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2} \dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2} \dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases}. \tag{2.1}$$

Indeed, if we assume $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfying the uniform condition

$$\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C, \quad \forall \epsilon > 0, \tag{2.2}$$

for some $C > 0$ independent of ϵ, we get from [22, Theorem 4.1] the following outcome

Theorem 2.1. *Let u^ϵ be the solution of (2.1) with $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfying condition (2.2), and assume that the function*

$$\hat{f}^\epsilon(x) = \int_{-G_\epsilon(x)}^{H_\epsilon(x)} f^\epsilon(x,s)\, ds, \quad x \in (0,1), \tag{2.3}$$

satisfies that $\hat{f}^\epsilon \rightharpoonup \hat{f}$, w-$L^2(0,1)$, as $\epsilon \to 0$.

Then, there exists $\hat{u} \in H^1(0,1)$, such that

$$\|u^\epsilon - \hat{u}\|_{L^2(\Omega^\epsilon)} \to 0, \qquad as\ \epsilon \to 0, \tag{2.4}$$

where $\hat{u}$ is the unique solution of the Neumann problem

$$\int_0^1 \Big\{ q(x)\, u_x(x)\, \varphi_x(x) + p(x)\, u(x)\, \varphi(x) \Big\} dx = \int_0^1 \hat{f}(x)\, \varphi(x)\, dx \tag{2.5}$$

for all $\varphi \in H^1(0,1)$, where p and q are positive functions given by (1.8)*, X is the auxiliary solution defined in* (1.9) *and Y^* is the representative cell* (1.10)*.*

Remark 2.2. (*i*) *If q is continuous, (2.5) is the weak formulation of*

$$\begin{cases} \dfrac{1}{p(x)} \left(q(x)\, u_x(x)\right)_x + u(x) = f(x), \quad x \in (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases}$$

with $f(x) = \hat{f}(x)/p(x)$.

(*ii*) *Furthermore, if we initially assume that f^ϵ does not depend on the vertical variable y, that is, $f^\epsilon(x,y) = f_0(x)$, then it is not difficult to see that*

$$\hat{f^\epsilon}(x) = \left(H_\epsilon(x) + G_\epsilon(x)\right) f_0(x)$$

and so, due to the Average Theorem discussed for example in [5, Lemma 4.2]*, we have*

$$H_\epsilon(x) + G_\epsilon(x) \rightharpoonup \frac{1}{l_h} \int_0^{l_h} H(x,y)\, dy + \frac{1}{l_g} \int_0^{l_g} G(x,y)\, dy, \quad w^* - L^\infty(0,1),$$

as $\epsilon \to 0$, and then, $H_\epsilon(x) + G_\epsilon(x) \rightharpoonup p(x)$, $w^* - L^\infty(0,1)$, and $\hat{f}(x) = p(x) f_0(x)$.

(*iii*) *The variational formulation of* (2.1) *is find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that*

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2,\ \forall \varphi \in H^1(\Omega^\epsilon). \tag{2.6}$$

Thus, taking $\varphi = u^\epsilon$ in (2.6)*, we get that the solutions u^ϵ satisfy*

$$\Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|^2_{L^2(\Omega^\epsilon)} + \|u^\epsilon\|^2_{L^2(\Omega^\epsilon)} \le \|f^\epsilon\|_{L^2(\Omega^\epsilon)} \|u^\epsilon\|_{L^2(\Omega^\epsilon)}. \tag{2.7}$$

Consequently, it follows from (2.2) that

$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)}, \Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|_{L^2(\Omega^\epsilon)} \ and\ \frac{1}{\epsilon} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega^\epsilon)} \le C, \quad \forall \epsilon > 0. \tag{2.8}$$

Hence, we can combine (2.8)*,* (2.4) *and interpolation inequality* [16, Section 1.4] *to get*

$$\|u^\epsilon - \hat{u}\|_{H^\alpha(\Omega^\epsilon)} \to 0, \qquad as\ \epsilon \to 0, \qquad for\ all\ \alpha \in [0,1).$$

3. Abstract setting and existence of solutions

Now let us write the problems (1.6) and (1.7) in an abstract form. Here we adopt the convergence concept introduced in [27] and successfully applied in [1, 9, 13] to deal with boundary perturbation problems.

3.1. **Existence of solutions.**

We first consider the family of Hilbert spaces $\{Z_\epsilon\}_{\epsilon>0}$ defined by $Z_\epsilon = L^2(\Omega^\epsilon)$ under the canonical inner product

$$(u, v)_\epsilon = \int_{\Omega^\epsilon} u(x_1, x_2)\, v(x_1, x_2)\, dx_1 dx_2$$

and by $Z_0 = L^2(0,1)$ the limiting Hilbert space with the inner product $(\cdot,\cdot)_0$ given by

$$(u, v)_0 = \int_0^1 p(x)\, u(x)\, v(x)\, dx$$

where

$$p(x) = \frac{|Y^*|}{l_h} + \frac{1}{l_g} \int_0^{l_g} G(x, y)\, dy - G_0(x)$$

is the positive function introduced in (1.8).

Next we look at the linear elliptic problem associated to (1.6) as an abstract equation $L_\epsilon u = f^\epsilon$ where $L_\epsilon : \mathcal{D}(L_\epsilon) \subset Z_\epsilon \mapsto Z_\epsilon$ is given by

$$\begin{aligned} \mathcal{D}(L_\epsilon) &= \left\{ u \in H^2(\Omega^\epsilon) \,:\, \frac{\partial u}{\partial x_1} N_1^\epsilon + \frac{1}{\epsilon^2} \frac{\partial u}{\partial x_2} N_2^\epsilon = 0 \text{ on } \partial\Omega^\epsilon \right\} \\ L_\epsilon u &= -\frac{\partial^2 u}{\partial {x_1}^2} - \frac{1}{\epsilon^2} \frac{\partial^2 u}{\partial {x_2}^2} + u, \quad u \in \mathcal{D}(L_\epsilon). \end{aligned} \tag{3.1}$$

It follows from [15, Corollary A.9] that L_ϵ is selfadjoint and positive with compact resolvent.

Analogously, we associate the limit elliptic problem given by (1.7) to the limit linear operator $L_0 : \mathcal{D}(L_0) \subset Z_0 \mapsto Z_0$ defined by

$$\begin{aligned} \mathcal{D}(L_0) &= \left\{ u \in H^2(0,1) \,:\, u'(0) = u'(1) = 0 \right\} \\ L_0 u &= -\frac{1}{p(x)} \left(q(x) u_x \right)_x + u, \quad u \in \mathcal{D}(L_0) \end{aligned} \tag{3.2}$$

where p and q are the homogenized coefficients introduced in (1.8). Due to [22, Remark 4.3], it is clear that L_0 is a positive selfadjoint operator with compact resolvent. In fact, by (3.1) and (3.2), we have established here a family of selfadjoint and positive linear operators $\{L_\epsilon\}_{\epsilon\geq 0}$ with compact resolvent.

Now let us consider the Nemitiskii map F_ϵ corresponding to nonlinearity f

$$F_\epsilon : Z_\epsilon^\alpha \mapsto Z_\epsilon : u^\epsilon \to f(u^\epsilon(x)), \qquad \forall x \in \Omega^\epsilon, \tag{3.3}$$

wherever $\epsilon > 0$, and for $\epsilon = 0$

$$F_0 : Z_0^\alpha \mapsto Z_0 : u \to f(u(x)), \qquad \forall x \in (0,1), \tag{3.4}$$

where

$$Z_\epsilon^\alpha := H^\alpha(\Omega^\epsilon),\ \epsilon > 0, \qquad \text{and} \qquad Z_0^\alpha := H^\alpha(0,1) \text{ at } \epsilon = 0.$$

In this work we will usually take $\alpha \in [0,1)$ since the oscillatory behavior of the domains Ω^ϵ does not allow convergence of the solutions in H^1-norm [11].

Lemma 3.1. *For each $\epsilon \geq 0$ and $0 < \alpha < 1$, we have that the Nemitiskii map F_ϵ satisfies*

a) *There exists $K > 0$, independent of ϵ, such that*

$$\|F_\epsilon(u^\epsilon)\|_{L^\infty(Z_\epsilon^\alpha, Z_\epsilon)} \leq K \qquad \forall u^\epsilon \in Z_\epsilon^\alpha.$$

b) *There exists $\theta \in (0,1]$ and $L > 0$, independent of ϵ, such that $F_\epsilon \in \mathcal{C}^{1+\theta}(Z_\epsilon^\alpha, Z_\epsilon)$,*

$$\|F_\epsilon(u^\epsilon) - F_\epsilon(v^\epsilon)\|_{Z_\epsilon} \leq L\|u^\epsilon - v^\epsilon\|_{Z_\epsilon^\alpha}$$
$$\|DF_\epsilon(u^\epsilon) - DF_\epsilon(v^\epsilon)\|_{\mathcal{L}(Z_\epsilon^\alpha, Z_\epsilon)} \leq L\|u^\epsilon - v^\epsilon\|^\theta_{Z_\epsilon^\alpha}$$

for all u^ϵ, $v^\epsilon \in Z_\epsilon^\alpha$ and $\epsilon \in [0, \epsilon_0)$.

c) *There exists $C > 0$, independent of ϵ, such that*

$$\|F_\epsilon(u^\epsilon) - F_\epsilon(v^\epsilon) - DF_\epsilon(v^\epsilon)(u^\epsilon - v^\epsilon)\|_{Z_\epsilon} \leq C\|u^\epsilon - v^\epsilon\|^{1+\alpha}_{Z_\epsilon^\alpha}$$

for all u^ϵ, $v^\epsilon \in Z_\epsilon^\alpha$ and $\epsilon \in [0, \epsilon_0)$.

Proof. The proof of itens a) and b) is essentially the same one given in [3, Lemma 3.3.1] since the embedding $Z_\epsilon^\alpha \hookrightarrow L^{\frac{2}{1-\alpha}}(\Omega^\epsilon)$ can be taken independent of ϵ for each $\alpha \in (0,1)$. Indeed, if Ω^ϵ is a family of Lipschitz domains uniformly bounded in $\epsilon \geq 0$, then, it follows from [17, Remark 6.8] that there exists $C > 0$, independent of ϵ, such that

$$\|u^\epsilon\|_{L^{\frac{2}{1-\alpha}}(\Omega^\epsilon)} \leq C\|u^\epsilon\|_{H^\alpha(\Omega^\epsilon)}, \qquad \text{for all } u^\epsilon \in H^\alpha(\Omega^\epsilon). \tag{3.5}$$

The exponent $\frac{2}{1-\alpha}$ is called *fractional critical exponent*. Here we show item c). First note that Mean Value Theorem implies

$$|f(u^\epsilon(x)) - f(v^\epsilon(x)) - f'(v^\epsilon(x))(u^\epsilon(x) - v^\epsilon(x))| = |(f'(\xi(x)) - f'(v^\epsilon(x)))(u^\epsilon(x) - v^\epsilon(x))|$$

for some $\xi(x)$ between $u^\epsilon(x)$ and $v^\epsilon(x)$. On the one hand, we get from (1.4) that

$$|f'(\xi(x)) - f'(v^\epsilon(x))| \leq 2C_f.$$

On the other hand, we have once more by Mean Value Theorem that

$$|(f'(\xi(x)) - f'(v^\epsilon(x)))(u^\epsilon(x) - v^\epsilon(x))| \leq C_f|u^\epsilon(x) - v^\epsilon(x)|.$$

Thus, if we call $\gamma^\epsilon(x) = \min\{1, |u^\epsilon(x) - v^\epsilon(x)|\}$, we conclude

$$|f(u^\epsilon(x)) - f(v^\epsilon(x)) - f'(v^\epsilon(x))(u^\epsilon(x) - v^\epsilon(x))| \leq 2C_f|\gamma^\epsilon(x)||(u^\epsilon(x) - v^\epsilon(x))|. \tag{3.6}$$

Also, $\|\gamma^\epsilon\|_{L^\infty(\Omega^\epsilon)} \leq 1$ and $\|\gamma^\epsilon\|_{L^2(\Omega^\epsilon)} \leq \|u^\epsilon - v^\epsilon\|_{L^2(\Omega^\epsilon)}$ implies $\|\gamma^\epsilon\|_{L^p(\Omega^\epsilon)} \leq \|u^\epsilon - v^\epsilon\|^{2/p}_{L^2(\Omega^\epsilon)}$ for any $p > 2$. Hence, due to (3.5) and (3.6), there exists $\hat{C} > 0$ independent of ϵ such that

$$\begin{aligned}\|F_\epsilon(u^\epsilon) - F_\epsilon(v^\epsilon) - DF_\epsilon(v^\epsilon)(u^\epsilon - v^\epsilon)\|^2_{L^2(\Omega^\epsilon)} &\leq 2C_f\|\gamma^\epsilon(u^\epsilon - v^\epsilon)\|^2_{L^2(\Omega^\epsilon)} \\ &\leq 2C_f\|\gamma^\epsilon\|^2_{L^{\frac{2}{\alpha}}(\Omega^\epsilon)}\|u^\epsilon - v^\epsilon\|^2_{L^{\frac{2}{1-\alpha}}(\Omega^\epsilon)} \leq \hat{C}\|u^\epsilon - v^\epsilon\|^{2(1+\alpha)}_{Z_\epsilon^\alpha}\end{aligned}$$

for all $\alpha \in (0,1)$. ∎

Now we can write problems (1.6) and (1.7) as $L_\epsilon u = F_\epsilon(u)$ for each $\epsilon \geq 0$. More, we have that u^ϵ is a solution of (1.6) or (1.7), if only if $u^\epsilon \in Z^\alpha_\epsilon$ satisfies $u^\epsilon = L_\epsilon^{-1} F_\epsilon(u^\epsilon)$. Then, u^ϵ must be a fixed point of the nonlinear map

$$L_\epsilon^{-1} \circ F_\epsilon : Z^\alpha_\epsilon \mapsto Z^\alpha_\epsilon. \tag{3.7}$$

Consequently, the existence of solutions of problems (1.6) and (1.7) follows from Schauder's Fixed Point Theorem.

We introduce the set $\mathcal{E}_\epsilon$ in order to denote the family of solutions given by equations (1.6) and (1.7) for each $\epsilon \geq 0$, that is,

$$\mathcal{E}_\epsilon = \{u^\epsilon \in Z_\epsilon \,:\, L_\epsilon u^\epsilon = F_\epsilon(u^\epsilon)\}.$$

Note that $\mathcal{E}_\epsilon$ is the *equilibria set* of the parabolic equation associated to (1.6) and (1.7). Here we investigate the asymptotic behavior of $\mathcal{E}_\epsilon$ at $\epsilon = 0$.

We also recall that a solution u of a boundary value problem is called hyperbolic if the eigenvalues of the linearized equation around u are all different from zero. In particular, we call a solution u of (1.7) hyperbolic if $\lambda = 0$ is not an eigenvalue of the eigenvalue problem

$$\begin{cases} \dfrac{1}{p(x)} \left(q(x)\, v_x\right)_x + v = f'(u)\, v + \lambda v, \quad x \in (0,1), \\ v_x(0) = v_x(1) = 0. \end{cases}$$

Finally we observe that assumption (1.4) implies $\mathcal{E}_\epsilon$ is uniform bound in L^∞.

Proposition 3.2. *Let $u^\epsilon \in H^1(\Omega^\epsilon)$ be a solution of* (1.6) *with nonlinearity f satisfying* (1.4)*. Then, there exists $K > 0$, independent of $\epsilon > 0$, such that*

$$\|u^\epsilon\|_{L^\infty(\Omega^\epsilon)} \leq K.$$

Proof. Let $u^\epsilon \in H^1(\Omega^\epsilon)$ be a solution of (1.6). Then, for all $\varphi \in H^1(\Omega^\epsilon)$,

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f(u^\epsilon) \varphi dx_1 dx_2. \tag{3.8}$$

Now let us take $\varphi = U^\epsilon = (u^\epsilon - k)^+$ in (3.8) for some $k > 0$ where f^+ denotes the positive part of a function f. Hence, adding and subtracting k in an appropriated way, we obtain

$$\|U^\epsilon\|^2_{L^2(\Omega^\epsilon)} + \Big\| \frac{\partial U^\epsilon}{\partial x_1} \Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2} \Big\| \frac{\partial U^\epsilon}{\partial x_2} \Big\|^2_{L^2(\Omega^\epsilon)} = \int_{\Omega^\epsilon} \left(f(u^\epsilon) - k\right) U^\epsilon dx_1 dx_2.$$

Consequently, if we pick $k \geq C_f$ given in (1.4), we get

$$\|U^\epsilon\|_{H^1(\Omega^\epsilon)} = 0, \quad \text{for any } \epsilon > 0,$$

and then, $u^\epsilon \leq k$. Finally we can argue in a similar way for $-u^\epsilon$ getting the desired result. □

3.2. **E-convergence notion.** Now we introduce the concept of *compact convergence* to discuss the convergence of the solutions u^ϵ of (1.6). For this, we consider the family of linear continuous operators $E_\epsilon : Z_0 \mapsto Z_\epsilon$

$$(E_\epsilon u)(x_1, x_2) = u(x_1) \text{ on } \Omega^\epsilon$$

for each $u \in Z_0$. Since

$$p_\epsilon(x) = H_\epsilon(x) + G_\epsilon(x) \rightharpoonup p(x), \quad w^* - L^\infty(0,1), \tag{3.9}$$

we have

$$\begin{aligned} \|E_\epsilon u\|^2_{Z_\epsilon} &= \int_{\Omega^\epsilon} u^2(x_1)\, dx_1 dx_2 = \int_0^1 \{H_\epsilon(x_1) + G_\epsilon(x_1)\}\, u^2(x_1)\, dx_1 \\ &\to \|u\|_{Z_0} \text{ as } \epsilon \to 0. \end{aligned}$$

Analogously, if we set $E_\epsilon : L^1_0 \to L^1_\epsilon$ with $L^1_\epsilon = D(L_\epsilon)$ taking in L^1_0 the norm $\|u\|_{Z^1_0} = \|-u_{xx} + u\|_{Z_0}$ we have

$$\|E_\epsilon u\|_{L^1_\epsilon} \to \|u\|_{L^1_0}.$$

Consequently, since

$$\sup_{0\leqslant\epsilon\leqslant 1} \{\|E_\epsilon\|_{\mathcal{L}(Z_0,Z_\epsilon)}, \|E_\epsilon\|_{\mathcal{L}(L^1_0,L^1_\epsilon)}\} < \infty,$$

we get by interpolation that

$$C = \sup_{\epsilon>0} \|E_\epsilon\|_{\mathcal{L}(Z^\alpha_0,Z^\alpha_\epsilon)} < \infty \text{ for } 0 \leqslant \alpha \leqslant 1.$$

Remark 3.3. *Note that E_ϵ is a kind of inclusion operator from Z^α_0 into Z^α_ϵ.*

Now we are in condition to introduce the concept of E-convergence.

Definition 3.4. *We say that a sequence of elements $\{u^\epsilon\}_{\epsilon>0}$ with $u^\epsilon \in Z_\epsilon$ is* E-convergent *to $u \in Z_0$, if $\|u^\epsilon - E_\epsilon u\|_{Z_\epsilon} \to 0$ as $\epsilon \to 0$. We write $u^\epsilon \xrightarrow{E} u$.*

Definition 3.5. *A sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in Z_{\epsilon_n}$ is said to be* E-precompact *if for any subsequence $\{u_{n'}\}$ there exist a subsequence $\{u_{n''}\}$ and $u \in Z_0$ such that $u_{n''} \xrightarrow{E} u$ as $n'' \to \infty$.*

Definition 3.6. *We say that a family of operators $\{B_\epsilon \in \mathcal{L}(Z_\epsilon) : \epsilon > 0\}$* E-converges *to $B \in \mathcal{L}(Z_0)$ as $\epsilon \to 0$, if $B_\epsilon f^\epsilon \xrightarrow{E} Bf$ whenever $f^\epsilon \xrightarrow{E} f \in Z_0$. We write $B_\epsilon \xrightarrow{EE} B$.*

Definition 3.7. *We say that a family of compact operators $\{B_\epsilon \in \mathcal{L}(Z_\epsilon) : \epsilon > 0\}$ converges compactly to a compact operator $B \in \mathcal{L}(Z_0)$, if for any family $\{f^\epsilon\}_{\epsilon>0}$ with $\|f^\epsilon\|_{Z_\epsilon} \leq 1$, we have that the family $\{B_\epsilon f^\epsilon\}$ is E-precompact and $B_\epsilon \xrightarrow{EE} B$. We write $B_\epsilon \xrightarrow{CC} B$.*

Next we note this notion of convergence can also be extended to sets.

Definition 3.8. *Let $\mathcal{O}_\epsilon \subset Z^\alpha_\epsilon$, $\epsilon \in [0,1]$, and $\mathcal{O}_0 \subset Z^\alpha_0$, $\alpha \in [0,1]$.*

(a) *We say that the family of sets $\{\mathcal{O}_\epsilon\}_{\epsilon\in[0,1]}$ is E-upper semicontinuous or just upper semicontinuous at $\epsilon = 0$ if*

$$\sup_{w^\epsilon\in\mathcal{O}_\epsilon} \left[\inf_{w\in\mathcal{O}_0} \left\{ \|w^\epsilon - E_\epsilon w\|_{Z^\alpha_\epsilon} \right\} \right] \to 0, \text{ as } \epsilon \to 0.$$

(b) *We say that the family of sets $\{\mathcal{O}_\epsilon\}_{\epsilon\in[0,1]}$ is E-lower semicontinuous or just lower semicontinuous at $\epsilon = 0$ if*

$$\sup_{w\in\mathcal{O}_0} \left[\inf_{w^\epsilon\in\mathcal{O}_\epsilon} \left\{ \|w^\epsilon - E_\epsilon w\|_{Z^\alpha_\epsilon} \right\} \right] \to 0, \text{ as } \epsilon \to 0.$$

(c) *If $\{\mathcal{O}_\epsilon\}_{\epsilon\in[0,1]}$ is upper and lower semicontinuous at $\epsilon = 0$, we say that is continuous.*

We also recall an useful characterization of upper and lower semicontinuity of sets:

(a) If any sequence $\{u^\epsilon\} \subset \mathcal{O}_\epsilon$ has a E-convergent subsequence with limit belonging to $\mathcal{O}_0$, then $\{\mathcal{O}_\epsilon\}$ is E-upper semicontinuous at zero.
(b) If $\mathcal{O}_0$ is a compact set and for each $u \in \mathcal{O}$ there exists a sequence $\{u^\epsilon\} \subset \mathcal{O}_\epsilon$ such that u^ϵ is E-convergent to u, then $\{\mathcal{O}_\epsilon\}$ is E-lower semicontinuous at zero.

4. Convergence results

On the framework introduced in section 3.2 we have from [22, Corollary 5.7] that the family of compact operators $\{L_\epsilon^{-1} \in \mathcal{L}(Z_\epsilon)\}_{\epsilon>0}$ converges compactly to the compact operator $L_0^{-1} \in \mathcal{L}(Z_0)$ as $\epsilon \to 0$. Indeed, it follows from [22, Corollary 5.9] that there exist $\epsilon_0 > 0$, and a function $\vartheta : (0, \epsilon_0) \mapsto (0, \infty)$, with $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$, such that

$$\|L_\epsilon^{-1} - E_\epsilon L_0^{-1} M_\epsilon\|_{\mathcal{L}(Z_\epsilon)} \leq \vartheta(\epsilon), \quad \forall \epsilon \in (0, \epsilon_0), \tag{4.1}$$

where $M_\epsilon : L^r(\Omega^\epsilon) \mapsto L^r(0,1)$ is the projection operator

$$(M_\epsilon f^\epsilon)(x) = \frac{1}{p_\epsilon(x)} \int_{-G_\epsilon(x)}^{H_\epsilon(x)} f^\epsilon(x,s)\, ds \quad x \in (0,1), \quad 1 \leq r \leq \infty,$$

which satisfies $\|M_\epsilon f^\epsilon\|_{L^r(0,1)} \leq C \|f^\epsilon\|_{L^r(\Omega^\epsilon)}$ for some $C > 0$ depending only on r, G_0, H_0, G_1 and H_1. Inequality (4.1) is a direct consequence of Theorem 2.1 and means the *convergence of the resolvent* operators L_ϵ^{-1}.

In this section, we show the continuity of the family $\{\mathcal{E}_\epsilon\}_{\epsilon \geq 0}$ at $\epsilon = 0$. In order to do that, we first discuss some properties of the nonlinear maps $L_\epsilon^{-1} F_\epsilon$.

Lemma 4.1. *Let $L_\epsilon^{-1} F_\epsilon$ be the map defined in* (3.7) *for $\epsilon \in [0,1)$. Then*

i) *$L_\epsilon^{-1} F_\epsilon$ is a compact operator.*
ii) *$L_\epsilon^{-1} F_\epsilon$ converges compactaly to $L_0^{-1} F_0$, that is, if $\|u^\epsilon - E_\epsilon u\|_{Z_\epsilon^\alpha} \to 0$ as $\epsilon \to 0$, then*

$$\|L_\epsilon^{-1} F_\epsilon(u^\epsilon) - E_\epsilon L_0^{-1} F_0(u)\|_{Z_\epsilon^\alpha} \to 0.$$

Proof. First we observe that i), for each $\epsilon \geq 0$ fixed, is a consequence of the continuity of $F_\epsilon : Z_\epsilon^\alpha \mapsto Z_0$ and $L_\epsilon^{-1} \in \mathcal{L}(Z_0)$, as well as, the compact imbedding of $Z_\epsilon^\alpha \hookrightarrow Z_0$ for $\alpha > 0$.

Next we prove ii). To do so, let us denote

$$w^\epsilon = L_\epsilon^{-1} F_\epsilon(u^\epsilon). \tag{4.2}$$

Hence, $w^\epsilon \in H^2(\Omega^\epsilon)$ and satisfies the equation $L_\epsilon w^\epsilon = F_\epsilon(u^\epsilon)$. Consequently, we have

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial w^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial w^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + w^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f(u^\epsilon) \varphi dx_1 dx_2, \ \forall \varphi \in H^1(\Omega^\epsilon). \tag{4.3}$$

Now let $\{g^\epsilon\}_{\epsilon \geq 0} \subset L^2(0,1)$ be the following family of functions

$$g^\epsilon(x) = \int_{-G_\epsilon(x)}^{H_\epsilon(x)} f(u^\epsilon(x,s)) ds, \text{ for } \epsilon > 0, \quad \text{and} \quad g(x) = p(x)\, f(u(x)), \text{ at } \epsilon = 0,$$

where p is the function given by (1.8). Since $u^\epsilon \xrightarrow{E} u$, we have $g^\epsilon \rightharpoonup g$, w-$L^2(0,1)$. In fact,

$$\int_0^1 g^\epsilon(x) \varphi(x) dx = \int_{\Omega^\epsilon} (f(u^\epsilon) - f(u)) \varphi(x_1)\, dx_1 dx_2 + \int_{\Omega^\epsilon} f(u) \varphi(x_1) dx_1 dx_2,$$

with the last integral satisfying

$$\begin{aligned}\int_{\Omega^\epsilon} f(u)\varphi(x_1)dx_1dx_2 &= \int_0^1 f(u)\varphi(x_1)\int_{-G_\epsilon(x_1)}^{H_\epsilon(x_1)} dx_2 = \int_0^1 (H_\epsilon(x)+G_\epsilon(x))f(u)\varphi\, dx \\ &\to \int_0^1 p\, f(u)\,\varphi\, dx, \text{ as } \epsilon\to 0,\end{aligned}$$

due to (3.9). Also, from (1.4) we have

$$\int_{\Omega^\epsilon} |f(u^\epsilon) - f(u)\,\varphi|\, dx_1dx_2 \le \|f'\|_\infty \|u^\epsilon - E_\epsilon u\|_{Z_\epsilon}\|\varphi\|_{Z_\epsilon} \to 0, \text{ as } \epsilon \to 0.$$

Therefore,

$$g^\epsilon \rightharpoonup g, \quad w - L^2(0,1). \tag{4.4}$$

Now let us observe that (4.3) implies $L_\epsilon w^\epsilon = g^\epsilon$ for all $\epsilon > 0$. Then, we can conclude from (4.3), (4.4) and Theorem 2.1 that there exist $w \in H^1(0,1)$, such that,

$$\|w^\epsilon - E_\epsilon w\|_{Z_\epsilon} = \|w^\epsilon - w\|_{Z_\epsilon} \to 0, \text{ as } \epsilon \to 0, \tag{4.5}$$

where w is the unique solution of problem (2.5) with $\hat{f} = pf(u) \in L^2(0,1)$. Consequently, we obtain from Remark 2.2, (4.2) and (4.5) that

$$L_0 w = f(u) \qquad \text{and} \qquad \|L_\epsilon^{-1}F_\epsilon(u^\epsilon) - L_0^{-1}F_0(u)\|_{Z_\epsilon} \to 0. \tag{4.6}$$

Finally we observe that proof can be concluded if we show that w^ϵ is uniformly bounded in Z^1_ϵ. In fact, if w^ϵ is uniformly bounded in Z^1_ϵ, then $w^\epsilon - E_\epsilon w$ is also. Thus, we can extract a convergent subsequence of $\{w^\epsilon - E_\epsilon w\}_{\epsilon>0}$ in Z^α_ϵ for any $\alpha \in [0,1)$. Therefore, we can conclude from (4.5) and (4.6) that $\|w^\epsilon - E_\epsilon w\|_{Z^\alpha_\epsilon} \to 0$, and then, $\|L_\epsilon^{-1}F_\epsilon(u^\epsilon) - L_0^{-1}F_0(u)\|_{Z^\alpha_\epsilon} \to 0$ as $\epsilon \to 0$ for all $\alpha \in [0,1)$.

Thus, let us prove w^ϵ is uniformly bounded in Z^1_ϵ. If we take $\varphi = w^\epsilon$ in (4.3), we get

$$\Big\|\frac{\partial w^\epsilon}{\partial x_1}\Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial w^\epsilon}{\partial x_2}\Big\|^2_{L^2(\Omega^\epsilon)} + \|w^\epsilon\|^2_{L^2(\Omega^\epsilon)} \le \|f(u^\epsilon)\|_{L^2(\Omega^\epsilon)}\|w^\epsilon\|_{L^2(\Omega^\epsilon)}.$$

Hence, since f is bounded,

$$\|w^\epsilon\|^2_{L^2(\Omega^\epsilon)} + \Big\|\frac{\partial w^\epsilon}{\partial x_1}\Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial w^\epsilon}{\partial x_2}\Big\|^2_{L^2(\Omega^\epsilon)} \le C_f\,(G_1 + H_1)^{1/2}\|w^\epsilon\|_{L^2(\Omega^\epsilon)},$$

and then, we obtain that there exists $K > 0$, independent of ϵ, such that

$$\|w^\epsilon\|_{L^2(\Omega^\epsilon)}, \Big\|\frac{\partial w^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega^\epsilon)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial w^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega^\epsilon)} \le K, \quad \forall \epsilon \in (0,1].$$

□

Now we are in condition to get the continuity of the equilibria set $\mathcal{E}_\epsilon$ at $\epsilon = 0$. First we show the upper semicontinuity using the compactness of the problems (1.6). Next, assuming all solutions of (1.7) are hyperbolic, we obtain the lower semicontinuity as a consequence of Lemma 4.1 and [27, Theorem 3].

Proposition 4.2. *Let u^ϵ be a family of solutions of problem* (1.6) *satisfying* $\|u^\epsilon\|_{L^\infty(\Omega^\epsilon)} \leq R$ *for some positive constant R independent of ϵ.*

Then there exists a subsequence, still denoted by u^ϵ, and $u \in H^1(0,1)$, solution of (1.7), *such that*
$$\|u^\epsilon - E_\epsilon u\|_{Z^\alpha_\epsilon} \to 0, \ \text{ as } \epsilon \to 0,$$
for any $\alpha \in [0,1)$.

Proof. The variational formulation of (1.6) is find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that
$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f(u^\epsilon)\varphi dx_1 dx_2, \ \forall \varphi \in H^1(\Omega^\epsilon). \tag{4.7}$$
Thus, if we take $\varphi = u^\epsilon$ in (4.7), we get
$$\Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|^2_{L^2(\Omega^\epsilon)} + \|u^\epsilon\|^2_{L^2(\Omega^\epsilon)} \leq \|f(u^\epsilon)\|_{L^2(\Omega^\epsilon)}\|u^\epsilon\|_{L^2(\Omega^\epsilon)}. \tag{4.8}$$
Hence, since $\|u^\epsilon\|_{L^\infty(\Omega^\epsilon)} \leq R$, that there exists $C > 0$, independent of ϵ, such that
$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)}, \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega^\epsilon)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega^\epsilon)} \leq C, \quad \forall \epsilon > 0. \tag{4.9}$$

Now, let us take $\Omega_0 = \{(x_1, x_2) \,:\, x_1 \in (0,1), \ -G_0 < x_2 < H_0\} \subset \Omega^\epsilon$ for all $\epsilon \geq 0$. Since Ω_0 is an open set independent of ϵ and $u^\epsilon|_{\Omega_0} \in H^1(\Omega_0)$ is uniformly bounded, we can extract a subsequence, still denoted by u^ϵ, such that for some $u \in H^1(\Omega_0)$
$$\begin{gathered} u^\epsilon \rightharpoonup u \quad w - H^1(\Omega_0), \\ u^\epsilon \to u \quad s - H^\alpha(\Omega_0), \text{ for all } \alpha \in [0,1), \\ \frac{\partial u^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega_0). \end{gathered} \tag{4.10}$$

Observe that $u(x_1, x_2)$ does not depend on the variable x_2. Indeed, for all $\varphi \in \mathcal{C}_0^\infty(\Omega_0)$, we have from (4.10) that
$$\int_{\Omega_0} u \,\frac{\partial \varphi}{\partial x_2}\, dx_1 dx_2 = \lim_{\epsilon \to 0} \int_{\Omega_0} P_\epsilon u^\epsilon \,\frac{\partial \varphi}{\partial x_2}\, dx_1 dx_2 = -\lim_{\epsilon \to 0}\int_{\Omega_0} \frac{\partial P_\epsilon u^\epsilon}{\partial x_2}\,\varphi\, dx_1 dx_2 = 0,$$
and then, $u(x_1, x_2) = u(x_1)$ for all $(x_1, x_2) \in \Omega_0$ implying $u \in H^1(0,1)$.

Also, from (4.10) we have that the restriction of u^ϵ to coordinate axis x_1 converges to u. If $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 \,:\, x_1 \in (0,1)\}$, then
$$u^\epsilon|_\Gamma \to u \quad s - H^\alpha(\Gamma), \quad \forall \alpha \in [0, 1/2). \tag{4.11}$$

Now, using (4.11) with $\alpha = 0$, we obtain the L^2-convergence of u^ϵ to u in Ω^ϵ. In fact, due to (4.11), we have
$$\begin{aligned} \|u^\epsilon|_\Gamma - u\|^2_{L^2(\Omega_\epsilon)} &= \int_0^1 \int_{-G_\epsilon(x_1)}^{H_\epsilon(x_1)} |u^\epsilon(x_1, 0) - u(x_1)|^2 \, dx_2 dx_1 \\ &\leq (G_1 + H_1)\, \|u^\epsilon|_\Gamma - u\|^2_{L^2(\Gamma)} \to 0, \text{ as } \varepsilon \to 0. \end{aligned}$$

Moreover,

$$|u^\epsilon(x_1,x_2) - u^\epsilon(x_1,0)|^2 = \left| \int_0^{x_2} \frac{\partial u^\epsilon}{\partial x_2}(x_1,s)\,ds \right|^2 \le \left(\int_0^{x_2} \left| \frac{\partial u^\epsilon}{\partial x_2}(x_1,s) \right|^2 ds \right) |x_2|.$$

Then, integrating in Ω^ϵ and using (4.9), we get

$$\begin{aligned} \|u^\epsilon - u^\epsilon|_\Gamma\|^2_{L^2(\Omega^\epsilon)} &\le \int_0^1 \int_{-G_\epsilon(x_1)}^{H_\epsilon(x_1)} \left(\int_0^{x_2} \left| \frac{\partial u^\epsilon}{\partial x_2}(x_1,s) \right|^2 ds \right) |x_2|\, dx_2 dx_1 \\ &\le (G_1 + H_1)^2 \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|^2_{L^2(\Omega^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

Hence, since

$$\|u^\epsilon - u\|_{L^2(\Omega^\epsilon)} \le \|u^\epsilon - u^\epsilon|_\Gamma\|_{L^2(\Omega^\epsilon)} + \|u^\epsilon|_\Gamma - u\|_{L^2(\Omega^\epsilon)},$$

we obtain that $u^\epsilon \xrightarrow{E} u$. Consequently we get

$$\|u^\epsilon - E_\epsilon u\|_{Z^\alpha_\epsilon} \to 0, \text{ as } \epsilon \to 0, \tag{4.12}$$

for any $\alpha \in [0,1)$, since $u^\epsilon \xrightarrow{E} u$ and $\|u^\epsilon - E_\epsilon u\|_{H^1(\Omega^\epsilon)}$ is uniformly bounded in ϵ.

Finally we conclude the proof using *ii*) from Lemma 4.1. Since u^ϵ is a fixed point of $L_\epsilon^{-1} F_\epsilon$ if and only if is a solution of (1.6), we get from (4.12) and Lemma 4.1 that

$$\|u^\epsilon - E_\epsilon L_0^{-1} F_0(u)\|_{Z^\alpha_\epsilon} = \|L_\epsilon^{-1} F_\epsilon(u^\epsilon) - E_\epsilon L_0^{-1} F_0(u)\|_{Z^\alpha_\epsilon} \to 0.$$

Therefore, $u = L_0^{-1} F_0(u)$, and then, $u \in H^1(0,1)$ is a solution of the limit problem (1.7) proving the proposition.

□

Proposition 4.3. *Let u be a hyperbolic solution of problem* (1.7) *satisfying $\|u\|_{L^\infty(0,1)} \le R$ for some positive constant R.*

Then there exists a sequence of solutions u^ϵ of problem (1.6) *such that for any $\alpha \in [0,1)$*

$$\|u^\epsilon - E_\epsilon u\|_{Z^\alpha_\epsilon} \to 0, \text{ as } \epsilon \to 0.$$

Proof. First we note that if u is a hyperbolic solution of (1.7), then it is isolated. Hence, there exists $\delta > 0$ such that u is the unique solution of (1.7) in $B_0(u,\delta) \subset Z_0^\alpha$, where $B_0(u,\delta)$ is the open ball of radius δ centered at u. Also, its *fixed point index*, relatively to map $L_0^{-1} F_0$, satisfies $|\text{ind}(u, L_0^{-1} F_0)| = 1$. We refer to [18] for an appropriated definition of fixed point index. Now, since the family of compact operators $L_\epsilon^{-1} F_\epsilon$ satisfies items i) and ii) of Lemma 4.1, it follows from [27, Theorem 3] that there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ the operator $L_\epsilon^{-1} F_\epsilon$ has at least one fixed point $u^\epsilon \in Z_\epsilon^\alpha$ satisfying $\|u^\epsilon - E_\epsilon u\|_{Z^\alpha_\epsilon} \to 0$ as $\epsilon \to 0$. Finally we observe that E_ϵ is called a *connection system* in [27]. The proof of the proposition is completed. □

Theorem 4.4. *If the solutions of the limiting problem* (1.7) *are hyperbolic, then the family of steady state solutions $\mathcal{E}_\epsilon$ is continuous at $\epsilon = 0$.*

Proof. Let u be a solution of (1.7), then $u \in H^1(0,1)$ and satisfies
$$\int_0^1 \{q(x)u_x\varphi_x + p(x)u\varphi\}dx = \int_0^1 p(x)f(u)\varphi dx, \quad \forall \varphi \in H^1(0,1).$$
Taking $\varphi = u$ we get
$$\int_0^1 \{q(x){u_x}^2 + p(x)u^2\}dx \leq \|p\|_\infty \|f(u)\|_{Z_0}\|u\|_{Z_0}.$$
Note p and q are positive functions. Then, it follows from (1.4) that there exists $K > 0$ depending just on the functions H, G and f such that $\|u\|_{H^1(0,1)} \leq K$. Thus, the equilibria set $\mathcal{E}_0$ is discrete and bounded in $H^1(0,1)$ since we are assuming all solutions of (1.7) are hyperbolic. Then $\mathcal{E}_0$ is finite and we conclude the proof due to Proposition 4.2 and 4.3. □

We still can prove a uniqueness result.

Corollary 4.5. *Let $u \in \mathcal{E}_0$ be hyperbolic and $\alpha \in (0,1)$. Then there exists $\delta > 0$ and $\epsilon_0 > 0$ such that, for each $\epsilon \in (0,\epsilon_0)$, there exists one and only one $u^\epsilon \in \mathcal{E}_\epsilon$ with $\|u^\epsilon - E_\epsilon u\|_{Z_\epsilon^\alpha} < \delta$.*

Proof. From Proposition 4.3, for $\delta > 0$ small enough, there exists $\epsilon_0 > 0$ such that $u^\epsilon \in \mathcal{E}_\epsilon$ and $\|u^\epsilon - E_\epsilon u\|_{Z_\epsilon^\alpha} < \delta$ for all $\epsilon \in (0,\epsilon_0)$. In order to prove the result, we just have to show the uniqueness of the fixed point u^ϵ proving $\|w^\epsilon - L_\epsilon^{-1}F_\epsilon(w^\epsilon)\|_{Z_\epsilon^\alpha} > 0$ wherever $\|u^\epsilon - w^\epsilon\|_{Z_\epsilon^\alpha} < \delta$ and $u^\epsilon \neq w^\epsilon$. Since u^ϵ is hyperbolic, it follows from Lemma 3.1, [22, Corollary 5.7] and [1, Lemma 4.7] that, there exist $M > 0$ and $m > 0$, independent of ϵ, such that $\|L_\epsilon^{-1}\|_{Z_\epsilon} \leq M$ and $\|I - L_\epsilon^{-1}DF_\epsilon(u^\epsilon)\|_{Z_\epsilon^\alpha} \geq m$. Then, adding and subtracting appropriated terms, we get from Lemma 3.1 that
$$\begin{aligned}\|w^\epsilon - L_\epsilon^{-1}F_\epsilon(w^\epsilon)\|_{Z_\epsilon^\alpha} \;\geq\;& \|(w^\epsilon - u^\epsilon) - L_\epsilon^{-1}DF_\epsilon(u^\epsilon)(w^\epsilon - u^\epsilon)\|_{Z_\epsilon^\alpha} \\ & - \|L_\epsilon^{-1}\left(F_\epsilon(w^\epsilon) - F_\epsilon(u^\epsilon) - DF_\epsilon(u^\epsilon)(w^\epsilon - u^\epsilon)\right)\|_{Z_\epsilon^\alpha} \\ & \geq (m - MC\|w^\epsilon - u^\epsilon\|_{Z_\epsilon^\alpha}^\alpha)\|w^\epsilon - u^\epsilon\|_{Z_\epsilon^\alpha}.\end{aligned}$$
Then, if we take δ small enough, we obtain $\|w^\epsilon - L_\epsilon^{-1}F_\epsilon(w^\epsilon)\|_{Z_\epsilon^\alpha} > 0$, proving the result. □

References

[1] J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz; *Dynamics in dumbbell domains I. Continuity of the set of equilibria*; Journal of Diff. Equations 231 (2006) 551-597.

[2] J. M. Arrieta, A. N. Carvalho, M. C. Pereira and R. P. Silva; *Semilinear parabolic problems in thin domains with a highly oscillatory boundary*; Nonlinear Analysis: Th. Meth. Appl. 74 (2011) 5111-5132.

[3] J. M. Arrieta and E. Santamaría; *Distance of Attractors of Evolutionary Equations*; Thesis Doctoral, Facultad de Ciencias Matemticas, Universidad Complutense de Madrid, Madrid (2014).

[4] J. M. Arrieta and M. C. Pereira; *Homogenization in a thin domain with an oscillatory boundary*; J. Math. Pures et Appl. 96 (2011) 29-57.

[5] J. M. Arrieta and M. C. Pereira; *The Neumann problem in thin domains with very highly oscillatory boundaries*; J. Math. Anal. Appl. 404 (2013) 86-104.

[6] J. M. Arrieta and M. Villanueva-Pesqueira; *Thin domains with doubly oscillatory boundaries*; Math. Meth. Appl. Sci. 37 (2) (2014) 158-166.

[7] J. M. Arrieta and M. Villanueva-Pesqueira; *Locally periodic thin domains with varying period*; C.R. Acad. Sci. Paris, Ser. I 352 (2014) 397-403.

[8] J. M. Arrieta and M. Villanueva-Pesqueira; *Fast and Slow Boundary Oscillations in a Thin Domain*; Advances in differential equations and applications 4 (2014) 13-22.

[9] J. M. Arrieta and S. M. Bruschi; *Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation*; Math. Models and Meth. in Appl. Sciences 17 (10) (2007) 1555-1585.

[10] M. Baía and E. Zappale; *A note on the 3D-2D dimensional reduction of a micromagnetic thin film with nonhomogeneous profile*; Appl. Anal. 86 (5) (2007) 555-575.

[11] A. Bensoussan, J.-L. Lions and G. Papanicolaou; *Asymptotic Analysis for Periodic Structures*; North-Holland (1978).

[12] A. Braides, I. Fonseca, G. Francfort; *3D-2D asymptotic analysis for inhomogeneous thin films*; Indiana Univ. Math. J. 49 (4) (2000) 1367-1404.

[13] A. N. Carvalho and S. Piskarev; *A general approximation scheme for attractors of abstract parabolic problems*; Numerical Functional Analysis and Optimization 27 (7-8) (2006) 785-829.

[14] G. Dal Maso; *An introduction to Γ-convergence*; Birkhăuser, Boston (1993).

[15] J. K. Hale and G. Raugel; *Reaction-diffusion equation on thin domains*; J. Math. Pures et Appl. (9) 71 (1) (1992) 33-95.

[16] D. B. Henry; *Geometric Theory of Semilinear Parabolic Equations*; Lecture Notes in Math. 840, Springer-Verlag (1981).

[17] E. Di Nezza, G. Palatucci and E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*; Bull. Sci. Math. 136 (2012) 521-573.

[18] M.A. Krasnoselskii and P.P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, New York, 1984.

[19] T. A. Mel'nyk and A. V. Popov; *Asymptotic analysis of boundary value and spectral problems in thin perforated domains with rapidly changing thickness and different limiting dimensions*; Mat. Sb. 203 (8) (2012) 97-124.

[20] J. Casado-Díaz, M. Luna-Laynez, F. J. Suárez-Grau; *Asymptotic behavior of the Navier-Stokes system in a thin domain with Navier condition on a slightly rough boundary*; SIAM J. Math. Anal. 45 (2) (2013) 1641-1674.

[21] I. Pazanin, F. J. Suárez-Grau; *Analysis of the thin film flow in a rough domain filled with micropolar fluid*; Computers & Mathematics with Applications 68 (12) (2014), 1915-1932.

[22] M. C. Pereira; *Parabolic problems in highly oscillating thin domains*; Annali di Matematica Pura ed Applicata 194 (4) (2015) 1203-1244.

[23] M. C. Pereira and R. P. Silva; *Error estimates for a Neumann problem in highly oscillating thin domains*; Discrete and Continuous Dyn. Systems 33 (2) (2013) 803-817.

[24] M. C. Pereira and R. P. Silva; *Correctors for the Neumann problem in thin domains with locally periodic oscillatory structure*; Quarterly of Applied Mathematics 73 (2015) 537-552.

[25] M. C. Pereira and R. P. Silva; *Remarks on the p-Laplacian on thin domains*; Progress in Nonlinear Diff. Eq. and Their Appl.(2015) 389-403.

[26] M. Prizzi and K. P. Rybakowski; *Some recent results on thin domain problems*; Topological Meth. in Nonlinear Analysis 14 (1999) 239-255.

[27] G. Vainikko; *Approximative methods for nonlinear equations (two approaches to the convergence problem)*; Nonlinear Analysis: Theory, Methods and Applications 2 (1978) 647-687.

(Marcone Corrêa Pereira) Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090, São Paulo - SP, Brazil

E-mail address: `marcone@ime.usp.br`