AN OBSTACLE PROBLEM FOR NONLOCAL EQUATIONS IN PERFORATED DOMAINS

MARCONE C. PEREIRA AND JULIO D. ROSSI

Abstract. In this paper we analyze the behavior of solutions to a nonlocal equation of the form $J * u(x) - u(x) = f(x)$ in a perforated domain $\Omega \setminus A^\epsilon$ with $u = 0$ in $A^\epsilon \cup \Omega^c$ and an obstacle constraint, $u \geq \psi$ in $\Omega \setminus A^\epsilon$. We show that, assuming that the characteristic function of the domain $\Omega \setminus A^\epsilon$ verifies $\chi_\epsilon \rightharpoonup \mathcal{X}$ weakly* in $L^\infty(\Omega)$, there exists a weak limit of the solutions u^ϵ and we find the limit problem that is satisfied in the limit. When $\mathcal{X} \not\equiv 1$ in this limit problem an extra term appears in the equation as well as a modification of the obstacle constraint inside the domain.

1. Introduction

Let us take a family of open bounded sets $\Omega^\epsilon \subset \mathbb{R}^N$ satisfying $\Omega^\epsilon \subset \Omega$ for some fixed open bounded domain $\Omega \subset \mathbb{R}^N$ and a positive parameter ϵ. In this work we see Ω^ϵ as a perforated domain where the set

$$A^\epsilon = \Omega \setminus \Omega^\epsilon$$

describe the holes inside Ω.

If we denote by $\chi_\epsilon \in L^\infty(\mathbb{R}^N)$ the characteristic function of Ω^ϵ, we assume that there exists a function $\mathcal{X} \in L^\infty(\mathbb{R}^N)$, strictly positive inside Ω, such that

$$\chi_\epsilon \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{1.1}$$

This means that

$$\int_\Omega \chi_\epsilon(x)\, \varphi(x)\, dx \to \int_\Omega \mathcal{X}(x)\, \varphi(x)\, dx \qquad \text{as } \epsilon \to 0$$

for all $\varphi \in L^1(\Omega)$ and there exists a positive constant $c > 0$ such that

$$0 < c \leq \mathcal{X}(x) \leq 1 \quad \text{ for } x \in \Omega.$$

Notice that χ_ϵ as a characteristic function also satisfies

$$0 \leq \chi_\epsilon(x) \leq 1 \quad \text{ in } \mathbb{R}^N.$$

Now given a function $\psi \in L^2(\mathbb{R}^N)$ let us introduce the following unilateral convex sets

$$K_\epsilon = \left\{ \phi \in L^2(\mathbb{R}^N) \,:\, \phi(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega^\epsilon \text{ with } \phi \geq \psi \text{ in } \Omega^\epsilon \right\}$$

and

$$K = \left\{ \phi \in L^2(\mathbb{R}^N) \,:\, \phi(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ with } \phi \geq \mathcal{X}\psi \text{ in } \Omega \right\}.$$

Key words and phrases. perforated domains, nonlocal equations, Neumann problem, Dirichlet problem.
2010 *Mathematics Subject Classification.* 45A05, 45M05, 49J40.

In the convex set K_ϵ, the function ψ defines an obstacle constraint which oscillates according to the configuration of the holes A^ϵ since for all $\phi \in K_\epsilon$ we have $\phi(x) = 0$ wherever $x \in A^\epsilon$. Notice that the convex set K, which is the weak limit of the sets K_ϵ (note that any weak limit of a sequence $v^\epsilon \in K_\epsilon$ belongs to K), possesses as obstacle the function $\mathcal{X}\psi$ which depend on the limit of the characteristic functions χ_ϵ.

Consider now the following functionals I and I_0 defined for any $\phi \in L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and a fixed function $f \in L^2(\Omega)$

$$I(\phi) = \frac{1}{2}|||\phi|||^2 - \int_{\mathbb{R}^N} f\phi\, dx$$

and

$$I_0(\phi) = \frac{1}{2}|||\phi|||^2 + \frac{1}{2}\int_{\mathbb{R}^n} \mu\, \phi^2\, dx - \int_{\mathbb{R}^N} f\phi\, dx$$

where the coefficient $\mu \in L^\infty(\Omega)$ is given by

$$\mu(x) = \frac{1-\mathcal{X}(x)}{\mathcal{X}(x)}. \tag{1.2}$$

The norm $|||\cdot|||$ is set for any $\phi \in L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and is given by

$$|||\phi|||^2 = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\phi(y)-\phi(x))^2 dydx.$$

Notice that $|||\cdot|||$ is associated to the nonlocal Dirichlet problem

$$\int_{\mathbb{R}^N} J(x-y)(\phi(y)-\phi(x))dy = f(x) \quad \text{ a.e. in } \Omega \tag{1.3}$$

with

$$\phi(x) \equiv 0 \qquad x \in \mathbb{R}^N \setminus \Omega \tag{1.4}$$

where the function J that appears as the kernel in the integral equation is assumed to satisfy

$$(\mathbf{H_J}) \qquad \begin{gathered} J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \text{ is non-negative and compactly supported with } J(0) > 0, \\ J(-x) = J(x) \text{ for every } x \in \mathbb{R}^N, \text{ and } \int_{\mathbb{R}^N} J(x)\, dx = 1. \end{gathered}$$

As we can see in [1, Section 2.1.1], under these conditions $|||\cdot|||$ defines a norm which is equivalent to the usual L^2-norm, and then defines a coercive bilinear and continuous form in L^2. Indeed, if λ_1 is the first eigenvalue of the nonlocal Dirichlet problem (1.3) and (1.4), we know that satisfies

$$|||\phi|||^2 \geq \lambda_1 \|\phi\|^2_{L^2(\Omega)}$$

for any $\phi \in L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

We refer to [1, 2, 10, 11, 12, 14, 15, 17, 18] for references involving nonlocal operators with non-singular kernels. The mathematical interest of dealing with a non-singular kernel is due to the fact that, in general, these operators have no regularizing effect and therefore no general compactness tools are available.

Now we are ready to state the main result of this paper.

Theorem 1.1. *Let $\{u^\epsilon\}_{\epsilon>0}$ be the family of functions given by the unique solutions of the minimization problems*

$$I(u^\epsilon) = \inf_{\phi \in K_\epsilon} I(\phi). \tag{1.5}$$

Then, there exists $u^* \in K$ such that

$$u^\epsilon \rightharpoonup u^* \text{ weakly in } L^2(\Omega)$$

as $\epsilon \to 0$ with u^* being the unique solution of the minimization problem

$$I_0(u^*) = \inf_{\phi \in K} I_0(\phi).$$

Notice that when $\mathcal{X} \equiv 1$ the effect of the holes disapear in the limit and we find that the limit problem has the same functional and the obstacle constraint is reduced to $u \geq \psi$ inside Ω. Indeed, we have the following result.

Corollary 1.1. *Under the assumptions of Theorem 1.1 with the additional hypothesis*

$$\mathcal{X} \equiv 1 \text{ in } \Omega,$$

we have that

$$u^\epsilon \to u^* \text{ strongly in } L^2(\Omega)$$

being u^* the unique minimizer of the problem

$$I(u^*) = \inf_{\phi \in K} I(\phi)$$

with $K = \{\phi \in L^2(\mathbb{R}^N) \,:\, \phi(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ with } \phi \geq \psi \text{ in } \Omega\}$.

Our results are valid under very general assumptions on the holes, namely we only require that the characteristic functions of the involved domains converge weakly.

An example. One simple example is the case where we have a bounded domain Ω from where we have removed a big number of periodic small balls (the holes). That is, we consider

$$\Omega^\epsilon = \Omega \setminus \cup_i B_{r^\epsilon}(x_i)$$

where $B_{r^\epsilon}(x_i)$ is a ball centered in $x_i \in \Omega$ of the form $x_i \in 2\epsilon\mathbb{Z}^N$ with radius $0 < r^\epsilon < \epsilon \leq 1$. In this situation we have a critical value for the size of the holes. In fact, if

$$r^\epsilon = C_0\, \epsilon$$

we obtain from Theorem 1.1 that the limit u^* is a solution to a minimization problem with an extra term that can be explicitly computed. In fact,

$$\mu = \frac{1 - \mathcal{X}}{\mathcal{X}}$$

where $\mathcal{X} \in L^\infty(\Omega)$ is just a positive constant, $\mathcal{X} = cte$, determined by the proportion of the cube which is occupied by the hole.

This follows from the fact that in this periodic case we obtain from [8, Theorem 2.6] that

$$\chi_\epsilon \rightharpoonup \mathcal{X} = |Q \setminus B|/|Q|$$

(here Q is the unit cube and B is a ball of radius C_0 inside the cube). Note that the terms $\mathcal{X}$ and $(1-\mathcal{X})$ that appear in the limit problem can be seen as the effect of the holes in the original problem and thus the coefficient ν that appears in the limit represents a kind of friction or drag caused by the perforations.

We also remark that in this critical case also the obstacle constraint is modified when passing to the limit, changing from $u \geq \psi$ to $u \geq \mathcal{X}\psi$. This is due to the fact that the measure of the holes in this critical case is of order one.

For holes that are smaller, that is, for radii

$$r^\epsilon \ll \epsilon,$$

we have that the effect of the holes disappear in the limit. This is due to the fact that we have $\mathcal{X} \equiv 1$ in this case. We remark that in this case we have strong convergence of u^ϵ in L^2 due to the fact that the convergence $\chi_\epsilon \to 1$ is strong in L^2.

Previous results. Now, let us end the introduction comparing our results with the ones that hold for the local Laplacian and the fractional Laplacian. First, for the local Laplacian, i.e., for the problem

$$\min \int_{\Omega^\epsilon} \frac{|\nabla u^\epsilon|^2}{2} - f u^\epsilon$$

with $u^\epsilon = 0$ on $\partial\Omega^\epsilon$ and $u^\epsilon \geq \psi$ in Ω^ϵ, the following results are obtained in the classical paper [6] (see also [5, 7, 8, 9, 13, 19, 21]). Assuming $N \geq 3$, the critical size of the holes is given by

$$r^\epsilon \sim \epsilon^{\frac{N}{N-2}}.$$

In this case also an extra term of the form $\int_\Omega \mu\,(v^*)^2$ appears in the limit functional, but there is no change in the obstacle constraint that remains as $v^* \geq \psi$.

For a similar problem with the fractional Laplacian (a nonlocal problem but with a singular kernel) we refer to [4] (see also [3]) where the authors show that the critical radii for the appearance of an extra term is

$$r^\epsilon \sim \epsilon^{\frac{N}{N-2s}}.$$

In this case again there is no change in the obstacle constraint that remains as $v^* \geq \psi$.

For nonlocal problems of this kind but without the obstacle constraint we refer to [20]. It is proved there that an extra term arises in the equation in the critical case, but there is no interplay with an obstacle. The fact that there is an obstacle involved makes the passage to the limit more difficult since we have to find what is called a corrector in the homogenization literature. This corrector takes the simple form $w^\epsilon = \chi_\epsilon/\mathcal{X}$ but it only converges weakly (and not strongly) to 1 as $\epsilon \to 0$. This fact creates new difficulties when passing to the limit in the functional to be minimized, see Section 4.

Therefore, we conclude that, in contrast to what happens for the Laplacian and the fractional Laplacian, for non-singular kernels the effect of the holes in the critical case affects the limit functional to be minimized and also the obstacle constraint that is satisfied in the limit. There is also a difference in the extra term that appears in the limit functional in the critical case, in our case we have $\int_\Omega \mu u^2$ and not $\int_\Omega \mu (u_-)^2$ as happens in [6] and [4]. This is due to the fact that we are taking $u^\epsilon = 0$ in the holes and the obstacle constraint $u^\epsilon \geq \psi$ in Ω^ϵ, the whole set minus the holes; while in [6] and [4] it is assumed that $u^\epsilon \geq \psi\chi_\epsilon$ in the whole Ω (note that this only implies $u^\epsilon \geq 0$ in the holes).

Note that in the critical case one has weak convergence in the natural space. In fact, for our problem we have weak convergence in L^2 of the sequence u^ϵ while for the Laplacian there is weak convergence in H^1 and for the fractional Laplacian the convergence is weak in H^s.

We remark that in our problem we are assuming only weak convergence of the characteristic functions of the involved domains Ω^ϵ, $\chi_\epsilon \rightharpoonup \mathcal{X}$ and we don't have any regularizing effect coming from the involved operator (that is only a bounded operator in L^2) these facts make the passage to limit a nontrivial task, since the ideas contained in the previously mentioned references for the Laplacian or the fractional Laplacian are not directly applicable in the present situation.

The paper is organized as follows: in Section 2 we collect some preliminary results; in Section 3 we present some results concerning the obstacle problem (in particular we prove that there is a unique solution) and, finally, in Section 4 we prove our main result.

2. Preliminary results

In this section we introduce some technical results which are needed to prove Theorem 1.1.

Proposition 2.1. *Let ϕ^ϵ and φ^ϵ be sequences in $L^2(\mathbb{R}^N)$ vanishing in $\mathbb{R}^N \setminus \Omega$ and satisfying*

$$\phi^\epsilon \rightharpoonup \phi \text{ and } \varphi^\epsilon \rightharpoonup \varphi \qquad \text{weakly in } L^2(\Omega)$$

as $\epsilon \to 0$, for some ϕ and φ in $L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ and $\varphi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.

Then

$$\int_{\mathbb{R}^N} \phi^\epsilon(x) \int_{\mathbb{R}^N} J(x-y)\varphi^\epsilon(y)\, dydx \to \int_{\mathbb{R}^N} \phi(x) \int_{\mathbb{R}^N} J(x-y)\varphi(y)\, dydx.$$

Proof. Notice that we conclude the proof if we show that

$$\begin{aligned} U_\epsilon(x) = \int_{\mathbb{R}^N} J(x-y)\varphi^\epsilon(y)\, dy = \int_\Omega J(x-y)\varphi^\epsilon(y)\, dy \\ \to \quad U_0(x) = \int_\Omega J(x-y)\varphi(y)\, dy = \int_{\mathbb{R}^N} J(x-y)\varphi(y)\, dy, \qquad \text{as } \epsilon \to 0, \end{aligned}$$

strongly in $L^2(\Omega)$.

To do that we first observe that the fact that $\varphi^\epsilon \rightharpoonup \varphi$ weakly in $L^2(\Omega)$ implies

$$U_\epsilon(x) \to U_0(x)$$

for all $x \in \Omega$. Also, we have that U_ϵ satisfies

$$|U_\epsilon(x)| \leq |\Omega|^{1/2} \|J\|_{L^\infty(\mathbb{R}^N)} \|\varphi^\epsilon\|_{L^2(\Omega)}.$$

Thus, due to Convergence Dominated Theorem, we get that

$$U_\epsilon \rightharpoonup U_0 \qquad \text{weakly in } L^2(\Omega).$$

Moreover, we have that

$$\|U_\epsilon\|_{L^2(\Omega)} \to \|U_0\|_{L^2(\Omega)}$$

since

$$|U_\epsilon(x)|^2 \leq |\Omega| \|J\|^2_{L^\infty(\mathbb{R}^N)} \|\varphi^\epsilon\|^2_{L^2(\Omega)}$$

and $U_\epsilon(x)^2 \to U_0(x)^2$ as $\epsilon \to 0$ wherever $x \in \Omega$.

Consequently, as we are working in a Hilbert space, we conclude

$$U_\epsilon \to U_0 \quad \text{strongly in } L^2(\Omega)$$

proving the result. □

Proposition 2.2. *Let w^ϵ be any sequence in $L^2(\mathbb{R}^N)$ with $w^\epsilon(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$.*

If $w^\epsilon \rightharpoonup w$ for some $w \in L^2(\mathbb{R}^N)$ as $\epsilon \to 0$ and $w^\epsilon(x) = 0$ a.e. in the holes A^ϵ, then

$$\liminf_{\epsilon > 0} |||w^\epsilon|||^2 \geq |||w|||^2 + \int_{\mathbb{R}^N} \mu w^2 \, dx.$$

Proof. Take any test function $\phi \in L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and consider

$$\begin{aligned} |||w^\epsilon - \frac{\chi_\epsilon}{\mathcal{X}}\phi|||^2 &= |||w^\epsilon|||^2 + |||\frac{\chi_\epsilon}{\mathcal{X}}\phi|||^2 \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(w^\epsilon(y) - w^\epsilon(x)) \left(\frac{\chi_\epsilon}{\mathcal{X}}(y)\phi(y) - \frac{\chi_\epsilon}{\mathcal{X}}(x)\phi(x)\right) dydx. \end{aligned}$$

Then, as $|||w^\epsilon - (\chi_\epsilon/\mathcal{X})\,\phi||| \geq 0$, we have

$$\begin{aligned} |||w^\epsilon|||^2 &\geq -|||\frac{\chi_\epsilon}{\mathcal{X}}\phi|||^2 \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(w^\epsilon(y) - w^\epsilon(x)) \left(\frac{\chi_\epsilon}{\mathcal{X}}(y)\phi(y) - \frac{\chi_\epsilon}{\mathcal{X}}(x)\phi(x)\right) dydx. \end{aligned} \tag{2.6}$$

First, from the fact that $(\chi_\epsilon(x))^2 = \chi_\epsilon(x)$, we get

$$|||\frac{\chi_\epsilon}{\mathcal{X}}\phi|||^2 = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(\frac{\chi_\epsilon}{\mathcal{X}^2}(y)\phi(y)^2 + \frac{\chi_\epsilon}{\mathcal{X}^2}(x)\phi(x)^2 - 2\frac{\chi_\epsilon}{\mathcal{X}}(y)\phi(y)\frac{\chi_\epsilon}{\mathcal{X}}(x)\phi(x)\right) dydx.$$

Hence, as $\epsilon \to 0$, we can pass to the limit using Proposition 2.1 to obtain

$$\begin{aligned} |||\frac{\chi_\epsilon}{\mathcal{X}}\phi|||^2 &\to \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(\frac{\phi^2(y)}{\mathcal{X}(y)} + \frac{\phi^2(x)}{\mathcal{X}(x)} - 2\phi(y)\phi(x)\right) dydx \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\Big[(\phi(y) - \phi(x))^2 + \\ &\qquad + \phi^2(y)\left(\frac{1-\mathcal{X}(y)}{\mathcal{X}(y)}\right) + \phi^2(x)\left(\frac{1-\mathcal{X}(x)}{\mathcal{X}(x)}\right)\Big] dydx \\ &= |||\phi|||^2 + \int_{\mathbb{R}^N} \mu\,\phi^2\,dx \end{aligned} \tag{2.7}$$

since $\chi_\epsilon \rightharpoonup \mathcal{X}$ weakly in $L^\infty(\mathbb{R}^N)$ and the function J is even.

Now let us evaluate the integral

$$Q_\epsilon = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(w^\epsilon(y)-w^\epsilon(x)) \left(\frac{\chi_\epsilon}{\mathcal{X}}(y)\phi(y) - \frac{\chi_\epsilon}{\mathcal{X}}(x)\phi(x)\right) dydx.$$

Since we are assuming $w^\epsilon(x) = 0$ in A^ϵ, we have $w^\epsilon(x)\chi_\epsilon(x) = w^\epsilon(x)$, and then,

$$\begin{aligned} Q_\epsilon &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(w^\epsilon(y)\frac{\chi_\epsilon}{\mathcal{X}}(y)\phi(y) - w^\epsilon(y)\frac{\chi_\epsilon}{\mathcal{X}}(x)\phi(x)\right) dydx \\ &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(\frac{w^\epsilon}{\mathcal{X}}(y)\phi(y) - w^\epsilon(y)\frac{\chi_\epsilon}{\mathcal{X}}(x)\phi(x)\right) dydx. \end{aligned}$$

Consequently, we can pass to the limit in Q_ϵ getting

$$\begin{aligned} Q_\epsilon &\to 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(w(y)\frac{\phi(y)}{\mathcal{X}(y)} - w(y)\phi(x)\right) dydx \\ &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left[w(y)\left(\phi(y)-\phi(x)\right) + w(y)\phi(y)\left(\frac{1-\mathcal{X}(y)}{\mathcal{X}(y)}\right)\right] dydx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(w(y)-w(x))(\phi(y)-\phi(x))dydx + 2\int_{\mathbb{R}^N} \mu\, w\, \phi\, dx \end{aligned}$$

for any test function ϕ.

Therefore, if we choose an appropriate subsequence w^ϵ (still denoted by w^ϵ) such that

$$\lim_{\epsilon\to 0} |||w^\epsilon|||^2 = \liminf_{\epsilon\to 0} |||w^\epsilon|||^2$$

we obtain from (2.6) that

$$\begin{aligned} \liminf_{\epsilon\to 0} |||w^\epsilon|||^2 \geq & -|||\phi|||^2 - \int_{\mathbb{R}^N} \mu\, \phi^2\, dx + 2\int_{\mathbb{R}^N} \mu\, w\, \phi\, dx \\ & + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(w(y)-w(x))\ (\phi(y)-\phi(x))\ dydx. \end{aligned} \tag{2.8}$$

Thus, we conclude the proof taking $\phi = w$ in (2.8). □

Remark 2.1. Notice that the classical weak lower semicontinuity of the norm implies

$$\liminf_{\epsilon > 0} |||w^\epsilon|||^2 \geq |||w|||^2$$

for any sequence w^ϵ weak convergent to w. Thus the additional condition $w^\epsilon(x) = 0$ in the holes A^ϵ improves this last inequality implying that at the limit a new term depending on the coefficient term μ appears.

3. The obstacle problem

Let I be the functional

$$I(\phi) = \frac{1}{2}|||\phi|||^2 - \int_{\mathbb{R}^N} \phi f\, dx$$

defined for any $\phi \in L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and some $f \in L^2(\Omega)$.

Also, consider the following unilateral convex set

$$\mathcal{K} = \{\phi \in L^2(\mathbb{R}^N) \,:\, \phi(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ with } \phi \geq \psi\}.$$

Here the obstacle function ψ is any one in $L^2(\mathbb{R}^N)$ satisfying $\psi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ in order to guarantee that the convex set $\mathcal{K}$ is not empty.

Theorem 3.1. *There exists unique $u \in \mathcal{K}$ such that*

$$I(u) = \inf_{\phi \in \mathcal{K}} I(\phi). \tag{3.9}$$

This optimal u is characterized by the following variational inequality

$$-\int_{\mathbb{R}^N} (v(x) - u(x)) \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))\, dydx \geq \int_{\mathbb{R}^N} f(v(x) - u(x))\, dx,$$

for all $v \in \mathcal{K}$.

Moreover, the map $f \mapsto u$ is continuous in the sense that

$$\|u_1 - u_2\|_{L^2(\Omega)} \leq \frac{1}{\lambda_1} \|f_1 - f_2\|_{L^2(\Omega)}$$

where u_i is the minimizer corresponding to f_i and λ_1 is the first eigenvalue of the nonlocal Dirichlet problem (1.3) *and* (1.4).

Proof. First we prove that u is a minimizer of (3.9) if and only if satisfies the following variational inequality

$$-\int_{\mathbb{R}^N} (v(x) - u(x)) \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))\, dydx \geq \int_{\mathbb{R}^N} f(v(x) - u(x))\, dx, \tag{3.10}$$

for all $v \in \mathcal{K}$.

In fact, if $v \in \mathcal{K}$ and $0 < \xi < 1$, we have $u + \xi(v - u) \in \mathcal{K}$, and then,

$$I(u + \xi(v - u)) \geq I(u). \tag{3.11}$$

Let us denote $\varphi = v - u$. Hence, we get from (3.11) that

$$\frac{1}{2}|||u + \xi\varphi|||^2 - \frac{1}{2}|||u|||^2 \geq -\xi \int_{\mathbb{R}^N} f\varphi\, dx$$

which is equivalent to

$$\frac{\xi^2}{2}|||\varphi|||^2 + \frac{\xi}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))(\varphi(y) - \varphi(x))\, dydx \geq \xi \int_{\mathbb{R}^N} f\varphi\, dx.$$

Multiplying this inequality by the positive number ξ^{-1} and taking $\xi \to 0$, we obtain

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))(\varphi(y) - \varphi(x))\, dydx \geq \int_{\mathbb{R}^N} f\varphi\, dx.$$

Consequently, since

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} & J(x-y)(u(y) - u(x))(\varphi(y) - \varphi(x))\, dydx \\ & = -\int_{\mathbb{R}^N} \varphi(x) \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))\, dydx, \end{aligned}$$

we get the desired variational inequality (3.10).

Now the result follows from [16, Theorem 2.7] since

$$a(u,v) = -\int_{\mathbb{R}^N} v(x) \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))\, dydx$$

defines a bilinear form for all functions u and $v \in L^2(\mathbb{R}^N)$ vanishing in $\mathbb{R}^N \setminus \Omega$. Indeed we have that the form a is continuous and coercive with

$$a(v,v) = |||v|||^2 \geq \lambda_1 \|v\|^2_{L^2(\Omega)}.$$

Thus, the result follows. □

Now let us consider the functional

$$I_0(\phi) = \frac{1}{2}|||\phi|||^2 + \frac{1}{2}\int_{\mathbb{R}^N} \mu\phi^2\, dx - \int_{\mathbb{R}^N} \phi f\, dx$$

defined for any $\phi \in L^2(\mathbb{R}^N)$ with $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and some $f \in L^2(\Omega)$.

Since the coefficient μ introduced in (1.2) is non negative, we can proceed as in the proof of Theorem 3.1 to obtain that u^* is the minimizer of I_0, if and only if satisfies the variational inequality

$$\begin{aligned} -\int_{\mathbb{R}^N} (v(x)-u^*(x)) \int_{\mathbb{R}^N} J(x-y)(u^*(y)-u^*(x))\, dydx + \int_{\mathbb{R}^N} \mu\, u^* v\, dx \\ \geq \int_{\mathbb{R}^N} f(v(x)-u^*(x))\, dx \end{aligned} \tag{3.12}$$

for any $v \in K = \{\phi \in L^2(\mathbb{R}^N) \,:\, \phi(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ with } \phi \geq \mathcal{X}\psi \text{ in } \Omega\}$.

Also, we can associate to inequality (3.12) the following bilinear form

$$a_0(u,v) = -\int_{\mathbb{R}^N} v(x) \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))\, dydx + \int_{\mathbb{R}^N} \mu\, uv\, dx$$

defined for any u and $v \in L^2(\mathbb{R}^N)$ vanishing in $\mathbb{R}^N \setminus \Omega$. Observe that a_0 is continuous and coercive since

$$a_0(u,u) = a(u,u) + \int_{\mathbb{R}^N} \mu\, u^2\, dx \geq \lambda_1 \|u\|^2_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega).$$

Therefore, we obtain from [16, Theorem 2.7] the following result.

Theorem 3.2. *There exists unique $u^* \in K$ such that*

$$I_0(u^*) = \inf_{\phi \in K} I_0(\phi).$$

The minimizer u^* is characterized by the variational inequality

$$\begin{aligned} -\int_{\mathbb{R}^N} (v(x)-u^*(x)) \int_{\mathbb{R}^N} J(x-y)(u^*(y)-u^*(x))\, dydx + \int_{\mathbb{R}^N} \mu\, u^* v\, dx \\ \geq \int_{\mathbb{R}^N} f(v(x)-u^*(x))\, dx \end{aligned}$$

for any $v \in K$.

Moreover, the map $f \mapsto u^*$ is continuous in the sense

$$\|u_1^* - u_2^*\|_{L^2(\Omega)} \leq \frac{1}{\lambda_1} \|f_1 - f_2\|_{L^2(\Omega)}$$

where u_i is the minimizer corresponding to f_i and λ_1 is the first eigenvalue of the nonlocal Dirichlet problem (1.3) *and* (1.4)*.*

4. Convergence results

In this section we prove our main results, Theorem 1.1, and then, Corollary 1.1. We also introduce a corrector result to improve the convergence obtained in Theorem 1.1.

Proof of Theorem 1.1. First we observe that the existence and uniqueness of the family of minimizers u^ϵ to the problem (1.5) is a consequence of Theorem 3.1. Note that in this case, we are taking as obstacle $\chi_\epsilon \psi$ where χ_ϵ is the characteristic function of the perforated domain Ω^ϵ which satisfies condition (1.1), and the function ψ is any fixed one in $L^2(\mathbb{R}^N)$.

Moreover, we have that there exists $C > 0$ such that

$$\|u^\epsilon\|_{L^2(\Omega)} \leq C, \quad \forall \epsilon > 0.$$

In fact, it follows from the variational inequality (3.10) that

$$|||u^\epsilon|||^2 \leq - \int_{\mathbb{R}^N} f(v - u^\epsilon)\, dx - \int_{\mathbb{R}^N} v(x) \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))\, dydx$$

for all $v \in K_\epsilon$. Hence, as the non-negative part of ψ, which we denote by ψ_+, belong to K_ϵ for all $\epsilon > 0$, we obtain for all $\epsilon > 0$ that

$$\lambda_1 ||u^\epsilon||^2_{L^2(\Omega)} \leq ||u^\epsilon||_{L^2(\Omega)} \left[||f||_{L^2(\Omega)} + ||\psi_+||_{L^2(\Omega)} \left(|\Omega|^{1/2} \|J\|_{L^\infty(\mathbb{R}^N)} + 1 \right) \right] - \int_{\mathbb{R}^N} f \psi_+ \, dx.$$

Thus, up to a subsequence, the exists $u^* \in L^2(\mathbb{R}^N)$ with $u^*(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ such that

$$u^\epsilon \rightharpoonup u^* \quad \text{weakly in } L^2(\Omega).$$

Now, our aim is to show that u^* is the minimizer of the functional I_0 proving Theorem 1.1. For any test function ϕ in the convex set K, let us take the following sequence

$$v^\epsilon = \frac{\chi_\epsilon}{\mathcal{X}} \, \phi.$$

Since $\phi \in K$, it is easy to see that

$$v^\epsilon \in K_\epsilon \quad \forall \epsilon > 0.$$

Also, it follows from (2.7) that

$$|||v^\epsilon|||^2 \to |||\phi|||^2 + \int_{\mathbb{R}^N} \mu \, \phi^2 \, dx \quad \text{ as } \epsilon \to 0.$$

Therefore, we obtain

$$\limsup_{\epsilon > 0} I(u^\epsilon) \leq \lim_{\epsilon \to 0} I(v^\epsilon) \leq \inf_{\phi \in K} I_0(\phi).$$

On the other hand, as a consequence of $u^\epsilon(x) = 0$ in A^ϵ and the weak convergence $u^\epsilon \rightharpoonup u^*$, we get from Proposition 2.2 that

$$\liminf_{\epsilon>0} |||u^\epsilon|||^2 \geq |||u^*|||^2 + \int_{\mathbb{R}^N} \mu\,(u^*)^2\,dx.$$

Hence

$$I_0(u^*) \leq \liminf_{\epsilon>0} I(u^\epsilon) \leq \limsup_{\epsilon>0} I(u^\epsilon) \leq \inf_{\phi\in K} I_0(\phi), \tag{4.13}$$

and then,

$$I_0(u^*) = \inf_{\phi\in K} I_0(\phi)$$

concluding the proof. □

Remark 4.1. As a consequence of inequality (4.13), we also have

$$\lim_{\epsilon\to 0} I(u^\epsilon) = I_0(u^*), \qquad \text{as } \epsilon \to 0.$$

Hence, since $u^\epsilon \rightharpoonup u^*$ weakly in $L^2(\Omega)$ implies

$$\lim_{\epsilon\to 0} \int_\Omega u^\epsilon f\,dx = \int_\Omega u^* f\,dx,$$

we obtain from the definition of the functionals I and I_0 that

$$|||u^\epsilon|||^2 \to |||u^*|||^2 + \int_{\mathbb{R}^N} \mu\,(u^*)^2\,dx. \tag{4.14}$$

This is known as the convergence of the energy.

In order to improve the weak convergence given in Theorem 1.1 to a strong one in L^2, we use the function

$$w^\epsilon(x) = \frac{\chi_\epsilon}{\mathcal{X}}(x), \quad x \in \Omega, \tag{4.15}$$

which is called the corrector in homogenization theory. Since $\mathcal{X}$ is strictly positive, it is clear that w^ϵ is well defined. Also, since $\chi_\epsilon \rightharpoonup \mathcal{X}$ weakly* in $L^\infty(\Omega)$, we have

$$w^\epsilon \rightharpoonup 1, \qquad \text{weakly}^* \text{ in } L^\infty(\Omega).$$

We have the following result.

Corollary 4.1. *Let u^ϵ and u^* be the functions that appear in Theorem 1.1.*

Then, if w^ϵ is the corrector introduced in (4.15)*, we have*

$$|||u^\epsilon - w^\epsilon u^*||| \to 0, \qquad \textit{as } \epsilon \to 0.$$

Proof. We pass to the limit here arguing as in Proposition 2.2. Note that

$$\begin{aligned} |||u^\epsilon - w^\epsilon u^*|||^2 &= |||u^\epsilon|||^2 + |||w^\epsilon u^*|||^2 \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u^\epsilon(y) - u^\epsilon(x))(w^\epsilon(y)u^*(y) - w^\epsilon(x)u^*(x))\,dydx. \end{aligned} \tag{4.16}$$

From (2.7) with $\phi = u^*$ and (4.14), we obtain

$$|||u^\epsilon|||^2 + |||w^\epsilon u^*|||^2 \quad \to \quad 2\left(|||u^*|||^2 + \int_{\mathbb{R}^N} \mu\,(u^*)^2\,dx\right) \tag{4.17}$$

as $\epsilon \to 0$.

On the other hand,

$$\begin{aligned}
D_\epsilon &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u^\epsilon(y)-u^\epsilon(x))(w^\epsilon(y)u^*(y)-w^\epsilon(x)u^*(x))\,dydx \\
&= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(u^\epsilon(y)\frac{\chi_\epsilon}{\mathcal{X}}(y)u^*(y) - u^\epsilon(y)\frac{\chi_\epsilon}{\mathcal{X}}(x)u^*(x)\right)dydx \\
&= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(\frac{u^\epsilon(y)}{\mathcal{X}(y)}u^*(y) - u^\epsilon(y)\frac{\chi_\epsilon}{\mathcal{X}}(x)u^*(x)\right)dydx
\end{aligned}$$

since $u^\epsilon(x)\chi_\epsilon(x) = u^\epsilon(x)$ (note that $u^\epsilon(x) = 0$ wherever $x \in A^\epsilon$).

Hence, we can use Proposition 2.1 to pass to the limit in D_ϵ getting

$$\begin{aligned}
D_\epsilon &\to 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,u^*(y)\left(\frac{u^*(y)}{\mathcal{X}(y)} - u^*(x)\right)dydx \\
&= -2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,u^*(x)\left(u^*(y) - \frac{u^*(x)}{\mathcal{X}(x)}\right)dydx \\
&= -2\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,u^*(x)\,(u^*(y)-u^*(x))\,dydx - \int_{\mathbb{R}^N}\mu\,(u^*)^2\,dx\right] \\
&= 2\left(|||u^*|||^2 + \int_{\mathbb{R}^N}\mu\,(u^*)^2\,dx\right).
\end{aligned}$$

Consequently, the result follows from (4.16) and (4.17). □

Finally, we show Corollary 1.1.

Proof of Corollary 1.1. First we note that when $\mathcal{X} = 1$ we have strong convergence of $\chi_\epsilon \to \mathcal{X}$ since in this case it holds that

$$|\Omega| = \int_\Omega \mathcal{X}^2 \le \lim_{\epsilon\to 0}\int_\Omega \chi_\epsilon \le \int_\Omega \mathcal{X}^2 = |\Omega|.$$

Hence, we have convergence of the norms $\|\chi_\epsilon\|_{L^2(\Omega)} \to \|\mathcal{X}\|_{L^2(\Omega)}$ and therefore we get strong convergence in L^2.

Now, using the strong convergence of χ_ϵ to $\mathcal{X} = 1$ the previous arguments can be used to show that $u^\epsilon \to u^*$ strongly in $L^2(\Omega)$ with u^* a solution to the minimization problem

$$I(u^*) = \inf_{\phi\in K} I(\phi), \tag{4.22}$$

with $K = \{\phi \in L^2(\mathbb{R}^N) \,:\, \phi(x) \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ with } \phi \ge \psi \text{ in } \Omega\}$.

In fact, from our previous arguments we have weak convergence $u^\epsilon \rightharpoonup u^*$ with u^* a solution to (4.22) (note that $\mu = 0$ due to the fact that $\mathcal{X} = 1$). Now, we notice that we have

$$I(u^*) \le \liminf_{\epsilon>0} I(u^\epsilon) \le \limsup_{\epsilon>0} I(u^\epsilon) \le I(u^*),$$

and since it holds that

$$\lim_{\epsilon\to 0}\int_\Omega u^\epsilon f\,dx = \int_\Omega u^* f\,dx$$

we get

$$\lim_{\epsilon\to 0}|||u^\epsilon|||^2 = |||u^*|||^2.$$

From this fact we obtain strong convergence in $|||\cdot|||$-norm that is equivalent to strong convergence in L^2 since the two norms are equivalent. □

Acknowledgements.

The first author (MCP) is partially supported by CNPq 302960/2014-7 and 471210/2013-7, FAPESP 2015/17702-3 (Brazil) and the second author (JDR) by MINCYT grant MTM2016-68210 (Spain).

References

[1] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. Toledo, Nonlocal Diffusion Problems. Mathematical Surveys and Monographs, vol. 165. AMS, 2010.
[2] G. Barles, E. Chasseigne and C. Imbert, *On the Dirichlet problem for second-order elliptic integro-differential equations.* Indiana Univ. Math. J. 57 (2008), no. 1, 213–246.
[3] L. A. Caffarelli and A. Mellet, *Random homogenization of an obstacle problem.* Ann. Inst. H. Poincar Anal. Non Linaire 26 (2009), no. 2, 375–395.
[4] L. A. Caffarelli and A. Mellet, *Random homogenization of fractional obstacle problems.* Netw. Heterog. Media 3 (2008), no. 3, 523–554.
[5] C. Calvo-Jurado, J. Casado-Díaz and M. Luna-Laynez, *Homogenization of nonlinear Dirichlet problems in random perforated domains*, Nonlinear Analysis 133 (2016) 250–274.
[6] D. Cioranescu and F. Murat, *A strange term coming from nowhere*, Progress in Nonl. Diff. Eq. and Their Appl. 31 (1997) 45–93.
[7] D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki, *The periodic unfolding method in domains with holes*, SIAM J. Math. Anal. 44 (2) (2012) 718–760.
[8] D. Cioranescu and P. Donato, An Introduction to Homogenization. Oxford lecture series in mathematics and its applications, vol.17, Oxford University Press, 1999.
[9] D. Cioranescu and J. Saint Jean Paulin, *Homogenization in open sets with holes.* J. Math. Anal. Appl. 71 (1979), no. 2, 590–607.
[10] E. Chasseigne, P. Felmer, J. D. Rossi and E. Topp, *Fractional decay bounds for nonlocal zero order heat equations.* Bull. Lond. Math. Soc. 46 (2014), no. 5, 943–952.
[11] C. Cortazar, M. Elgueta and J. D. Rossi. *Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions.* Israel J. Math. 170(1), (2009), 53–60.
[12] C. Cortazar, M. Elgueta, J. D. Rossi and N. Wolanski. *How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems.* Archive for Rational Mechanics and Analysis. 187(1), (2008), 137–156.
[13] R. Courant and D. Hilbert, Methods of mathematical physics, Volume I. Interscience, New York, 1953.
[14] Q. Du, M. Gunzburger, R. B. Lehoucq and K. Zhou, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws.* Math. Models Methods Appl. Sci. 23 (2013), no. 3, 493–540.
[15] P. Felmer and E. Topp, *Uniform equicontinuity for a family of zero order operators approaching the fractional Laplacian.* Comm. Partial Differential Equations 40 (2015), no. 9, 1591–1618.
[16] A. Friedman, Variational Principles and Free-boundary Problems. John Wiley & Sons, 1982.
[17] J. García Melián and J. D. Rossi. *On the principal eigenvalue of some nonlocal diffusion problems.* J. Differential Equations. 246(1), (2009), 21–38.

[18] R. B. Lehoucq and S. A. Silling, *Force flux and the peridynamic stress tensor*. J. Mech. Phys. Solids 56 (2008), no. 4, 1566–1577.
[19] J. Necas, Les méthodes directes en théorie des équations elliptiques. Masson, Paris, 1967.
[20] M. C. Pereira and J. D. Rossi. *Nonlocal problems in perforated domains.* Preprint. Available at http://mate.dm.uba.ar/∼jrossi/PD_Final_version.pdf.
[21] J. Rauch and M. Taylor, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal. 18 (1975), 27–59.

Julio D. Rossi
Dpto. de Matemáticas, FCEyN,
Universidad de Buenos Aires,
Ciudad Universitaria Pab 1, 1428, Buenos Aires, Argentina.

E-mail address: `jrossi@dm.uba.ar`
Web page: `http://mate.dm.uba.ar/∼jrossi/`

Marcone C. Pereira
Dpto. de Matemática Aplicada, IME,
Universidade de São Paulo,
Rua do Matão 1010, São Paulo - SP, Brazil.

E-mail address: `marcone@ime.usp.br`
Web page: `www.ime.usp.br/∼marcone`