NONLOCAL PROBLEMS IN THIN DOMAINS

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ABSTRACT. In this paper we consider nonlocal problems in thin domains. First, we deal with a nonlocal Neumann problem, that is, we study the behavior of the solutions to $f(x) = \int_{\Omega_1 \times \Omega_2} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))dy$ with $J_\epsilon(z) = J(z_1, \epsilon z_2)$ and $\Omega_1 \times \Omega_2 \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ a bounded domain. We find that there is a limit problem, that is, we show that $u^\epsilon \to u_0$ as $\epsilon \to 0$ in Ω and this limit function verifies $\int_{\Omega_2} f(x_1, x_2)\, dx_2 = |\Omega_2| \int_{\Omega_1} J(x_1 - y_1, 0)(U_0(y_1) - U_0(x_1))dy_1$, with $U_0(x_1) = \int_{\Omega_2} u_0(x_1, x_2)\, dx_2$. In addition, we deal with a double limit when we add to this model a rescale in the kernel with a parameter that controls the size of the support of J. We show that this double limit exhibits some interesting features.

We also study a nonlocal Dirichlet problem $f(x) = \int_{\mathbb{R}^N} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))\, dy$, $x \in \Omega$, with $u^\epsilon(x) \equiv 0$, $x \in \mathbb{R}^N \setminus \Omega$, and deal with similar issues. In this case the limit as $\epsilon \to 0$ is $u_0 = 0$ and the double limit problem commutes and also gives $v \equiv 0$ at the end.

1. INTRODUCTION

Our main goal in this paper is to study nonlocal problems with non-singular kernels in thin domains. We deal with Neumann or Dirichlet conditions.

The Neumann problem. We consider

$$(1.1) \qquad f(x) = \int_{\Omega_1 \times \Omega_2} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))dy$$

where

$$J_\epsilon(z) = J(z_1, \epsilon z_2),$$

$\epsilon > 0$ is a parameter, and $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is a bounded Lipschitz domain. We denote by $x = (x_1, x_2)$ a point in $\Omega_1 \times \Omega_2$, that is, $x_1 \in \Omega_1$, $x_2 \in \Omega_2$.

Here, and along the whole paper, the function J satisfies the following hypotheses

$$J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \text{ is non-negative with } J(0) > 0, \ J(-x) = J(x) \text{ for every } x \in \mathbb{R}^N, \text{ and}$$

$$(\mathbf{H}) \qquad \int_{\mathbb{R}^N} J(x)\, dx = 1.$$

On the other hand we only assume that $f \in L^2(\Omega)$.

It is worth noting that we are calling (1.1) as a nonlocal thin domain problem due to its equivalence with the equation

$$(1.2) \qquad h_\epsilon(z_1, z_2) = \frac{1}{\epsilon^{N_2}} \int_{\Omega_1 \times \epsilon \Omega_2} J(z - w)(v(w) - v(z))\, dw$$

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with $h_\epsilon(z_1, z_2) = f(z_1, z_2/\epsilon)$. This equivalence between (1.2) and (1.1) is a direct consequence of the change of variable

$$\Omega_1 \times \Omega_2 \ni (x_1, x_2) \mapsto (x_1, \epsilon x_2) \in \Omega_1 \times \epsilon \Omega_2.$$

Furthermore, we can see that (1.2), and then (1.1), are nonlocal singular problems since the bounded domain $\Omega_1 \times \epsilon \Omega_2$ degenerates to $\Omega_1 \times \{0\}$ when the positive parameter ϵ approaches to zero. We also are in agreement with references [8, 17, 14] using the factor $1/\epsilon^{N_2}$ in (1.2) to preserve the relative capacity of the open set $\Omega_1 \times \epsilon \Omega_2$ for small ϵ. The convenience of dealing with (1.1) is clear since its solutions u^ϵ are defined in the fixed domain $\Omega = \Omega_1 \times \Omega_2$.

Solutions to (1.1) are understood in a weak sense, that is,

$$
\begin{aligned}
(1.3) \quad \int_{\Omega_1 \times \Omega_2} f(x)\varphi(x)\,dx &= \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))\,dy\,\varphi(x)\,dx \\
&= -\frac{1}{2} \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))(\varphi(y) - \varphi(x))\,dydx,
\end{aligned}
$$

for every $\varphi \in L^2(\Omega)$. Note that taking $\varphi = 1$ we obtain that the condition

$$\int_{\Omega_1 \times \Omega_2} f(x)\,dx = 0$$

is necessary to have a solution. Also note that solutions are defined up to an additive constant and hence we will normalize them and look for solutions with zero mean value

$$\int_{\Omega_1 \times \Omega_2} u^\epsilon(x)\,dx = 0.$$

Now we are ready to state our first result. It says that there is a limit as $\epsilon \to 0$ of the solutions to our problem and that this limit, when we take its mean value in the y-direction, is a solution to a limit nonlocal problem in Ω_1 with a forcing term given by the mean value of f. Even when we consider the averages of solutions in Ω_2 we obtain strong convergence in $L^2(\Omega_1)$.

Theorem 1.1. *Let $\{u^\epsilon\}_{\epsilon>0}$ be a family of solutions of problem (1.1). Then, exists $u_0 \in L^2(\Omega)$ such that*

$$u^\epsilon \to u_0 \text{ strongly in } L^2(\Omega)$$

and

$$U^\epsilon \to U_0 \text{ strongly in } L^2(\Omega_1)$$

where the functions U^ϵ and U_0 are given by

$$(1.4) \qquad U^\epsilon(x_1) = \int_{\Omega_2} u^\epsilon(x_1, x_2)\,dx_2, \qquad and \qquad U_0(x_1) = \int_{\Omega_2} u_0(x_1, x_2)\,dx_2,$$

respectively. Furthermore, we have that U_0 satisfies the following nonlocal problem in Ω_1

$$(1.5) \qquad \tilde{f}(x_1) = \int_{\Omega_2} f(x_1, x_2)\,dx_2 = |\Omega_2| \int_{\Omega_1} J(x_1 - y_1, 0)(U_0(y_1) - U_0(x_1))dy_1.$$

Note that, since the kernel is smooth, there is no regularizing effect for this problem and therefore to obtain strong convergence in L^2-norm is not straightforward.

We call equation (1.5) as the *limit equation* of problem (1.1). Note its dependence on the open set Ω given by the coefficient $|\Omega_2|$ in the right side of the limit problem. If we denote by $\mu_\mathcal{O}(\varphi)$ the average of φ on a set $\mathcal{O}$, that is,

$$\mu_\mathcal{O}(\varphi) = \frac{1}{|\mathcal{O}|} \int_\mathcal{O} \varphi(\xi)\,d\xi,$$

we obtain from Theorem 1.1 that

$$\mu_{\Omega_2}(u^\epsilon) \to \mu_{\Omega_2}(u_0) \quad \text{in } L^2(\Omega_1)$$

where $\mu_{\Omega_2}(u_0) \in L^2(\Omega_1)$ is the unique solution of the nonlocal limit equation

$$\mu_{\Omega_2}(f) = |\Omega_2| \int_{\Omega_1} J(x_1 - y_1, 0)(\mu_{\Omega_2}(u_0)(y_1) - \mu_{\Omega_2}(u_0)(x_1))dy_1.$$

As a final remark concerning Theorem 1.1 we point out that the limit u_0 in general depends on x_2. This can be seen just by computing the derivative with respect to x_2 (that exists assuming that f is smooth) and showing that this derivative does not go to zero as $\epsilon \to 0$. We provide some details concerning this fact in Section 3. This fact has to be contrasted with what happens in parabolic and elliptic local problems posed in thin domains [8, 15], where the limit function u_0 does not depend on variable $x_2 \in \Omega_2$. Thus, the average convergence of the solutions u^ϵ becomes a nice way to describe the asymptotic behavior of problem (1.1) as $\epsilon \to 0$ (note that since in general the limit depends on x_2 we cannot avoid the use of averages to obtain solutions of the limit problem).

Concerning references for nonlocal problems with smooth kernels we refer to the book [1] and references therein. Let us point out that since we are integrating in Ω this problem is a nonlocal analogous to the classical elliptic problem for the Laplacian with homogeneous Neumann boundary conditions, that is,

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

In fact, in [6] is proved that solutions to the nonlocal problem (1.1) converge as a rescaling parameter that controls the size of the support of J goes to zero to the solution to the local problem (here we normalize as before and consider the unique solution that has zero mean value).

This relation between nonlocal and local problems gives rise to the following issue: we can rescale the kernel (with a parameter that we call δ and controls the size of the support of the kernel) and study this rescaled problems in a thin domain. That is, let us consider

$$f(x) = \frac{C(\epsilon)}{\delta^{N+2}} \int_{\Omega_1 \times \Omega_2} J\left(\frac{(x_1 - y_1)}{\delta}, \epsilon\frac{(x_2 - y_2)}{\delta}\right)(u^{\epsilon,\delta}(y) - u^{\epsilon,\delta}(x))dy.$$

Here $C(\epsilon)$ is the normalizing constant given by

(1.6) $$C(\epsilon) = \left(\frac{1}{2}\int_{\mathbb{R}^N} J(x_1, \epsilon x_2)\, x_{1_1}^2\, dx\right)^{-1}$$

where x_{1_1} is the first coordinate of $x_1 \in \Omega_1$. As we can see in Remark 4.1,

(1.7) $$\lim_{\epsilon \to 0} C(\epsilon) = 0.$$

If we take first the limit as $\delta \to 0$ and then as $\epsilon \to 0$ we get

Theorem 1.2. *Under the additional assumptions that $f \in \mathcal{C}^\alpha(\Omega)$ and that Ω has a boundary $C^{2+\alpha}$-regular, $0 < \alpha < 1$, as $\delta \to 0$ there exists a strong limit in $L^2(\Omega)$ to a solution to a local problem, that is,*

$$\lim_{\delta \to 0} u^{\epsilon,\delta} = v^\epsilon$$

where v^ϵ is the solution to

$$\begin{cases} \Delta_{x_1} v^\epsilon + \dfrac{1}{\epsilon^2} \Delta_{x_2} v^\epsilon = f, & in\ \Omega, \\[2mm] \dfrac{\partial v^\epsilon}{\partial n_1} + \dfrac{1}{\epsilon^2} \dfrac{\partial v^\epsilon}{\partial n_2} = 0, & on\ \partial\Omega, \end{cases}$$

with $\int_\Omega v^\epsilon = 0$. Moreover, there is a limit as $\epsilon \to 0$ of v^ϵ that we call v that is the solution to

$$\begin{cases} \Delta_{x_1} v = \dfrac{1}{|\Omega_2|} \displaystyle\int_{\Omega_2} f(x_1, x_2) dx_2, & in\ \Omega_1, \\[3mm] \dfrac{\partial v}{\partial n_1} = 0, & on\ \partial\Omega_1, \end{cases}$$

also satisfying $\int_{\Omega_1} v = 0$. Hence, we conclude that

$$\lim_{\epsilon \to 0} \left(\lim_{\delta \to 0} u^{\delta,\epsilon} \right) = v \quad in\ L^2(\Omega).$$

Note that here Δ_{x_i} denotes the Laplacian differential operator in the open set Ω_i, $i = 1, 2$.

On the other hand, when we reverse the order in which we take limits, that is we take first the limit as $\epsilon \to 0$ and then as $\delta \to 0$ we get

Theorem 1.3. *As $\epsilon \to 0$ there exists a strong limit in $L^2(\Omega)$, that is,*

$$\lim_{\epsilon \to 0} C(\epsilon)\, u^{\epsilon,\delta} = u^\delta$$

where u^δ is the solution to

$$(1.8) \qquad f(x) = \frac{1}{\delta^{N+2}} \int_{\Omega_1 \times \Omega_2} J\left(\frac{(x_1 - y_1)}{\delta}, 0 \right) (u^\delta(y) - u^\delta(x)) dy.$$

Then, if we take C_1 given by

$$C_1 = \left(\frac{1}{2} \int_{\mathbb{R}^N} J(x_1, 0)\, x_{1_1}^2\, dx \right)^{-1},$$

where x_{1_1} is the first coordinate of $x_1 \in \Omega_1$, there is a limit as $\delta \to 0$ of

$$U^\delta(x_1) := \int_{\Omega_2} \frac{u^\delta(x_1, x_2)}{C_1\, \delta^{N_2}} dx_2$$

that we call v, and can be characterized as the solution to

$$\begin{cases} \Delta_{x_1} v = \dfrac{1}{|\Omega_2|} \displaystyle\int_{\Omega_2} f(x_1, x_2)\, dx_2, & in\ \Omega_1, \\[3mm] \dfrac{\partial v}{\partial n_1} = 0, & on\ \partial\Omega_1, \end{cases}$$

with $\int_{\Omega_1} v = 0$. Hence, we conclude that

$$\lim_{\delta \to 0} \left(\int_{\Omega_2} \lim_{\epsilon \to 0} \frac{C(\epsilon)}{C_1 \, \delta^{N_2}} u^{\delta,\epsilon}(x_1, x_2) \, dx_2 \right) = v \quad \text{in } L^2(\Omega_1).$$

Due to Theorems 1.2 and 1.3, we obtain that the limits of $u^{\delta,\epsilon}$ depend on the order in which they are taken. Indeed, for any $f \neq 0$ in $L^2(\Omega)$, we get from (1.8) that u^δ is not the null function. Hence, from (1.7) and Theorem 1.3 we get

$$\lim_{\epsilon \to 0} \|u^{\delta,\epsilon}\|_{L^2(\Omega)} = \infty$$

for any $\delta > 0$. Thus,

$$v = \lim_{\epsilon \to 0} \left(\lim_{\delta \to 0} u^{\delta,\epsilon} \right) \neq \lim_{\delta \to 0} \left(\lim_{\epsilon \to 0} u^{\delta,\epsilon} \right) \quad \text{in } L^2(\Omega)$$

and we can not pass to the double limit in the solutions $u^{\delta,\epsilon}$.

The Dirichlet problem. Now, let us consider

(1.9) $$f(x) = \int_{\mathbb{R}^N} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x)) dy, \qquad x \in \Omega,$$

with

(1.10) $$u^\epsilon(x) \equiv 0, \qquad x \in \mathbb{R}^N \setminus \Omega,$$

where, as before,

$$J_\epsilon(z) = J(z_1, \epsilon z_2),$$

$\epsilon > 0$ is a parameter, and $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is a bounded Lipschitz domain. As in the Neumann problem (1.1), equations (1.9) and (1.10) are called nonlocal Dirichlet problem in thin domains due to their equivalence to the problems

$$h_\epsilon(z_1, z_2) = \frac{1}{\epsilon^{N_2}} \int_{\mathbb{R}^N} J(z - w)(v^\epsilon(w) - v^\epsilon(z)) \, dw$$

with $v^\epsilon(z) \equiv 0, \, z \in \mathbb{R}^N \setminus (\Omega_1 \times \epsilon \Omega_2)$, and $h_\epsilon(z_1, z_2) = f(z_1, z_2/\epsilon)$.

For this problem, we have the following result, which is in agreement with the local Dirichlet problem in thin domains [8, 15]:

Theorem 1.4. *Let $\{u^\epsilon\}_{\epsilon > 0}$ be a family of solutions of problem* (1.9) *satisfying* (1.10). *Then,*

$$u^\epsilon \to 0 \text{ strongly in } L^2(\Omega).$$

Now we just observe that if we introduce an extra parameter δ that controls the size of the support of the kernel, as we did for the Neumann case, we are lead to consider the following problem:

$$f(x) = \frac{C(\epsilon)}{\delta^{N+2}} \int_{\mathbb{R}^N} J\left(\frac{x_1 - y_1}{\delta}, \epsilon \frac{x_2 - y_2}{\delta}\right) (u^{\epsilon,\delta}(y) - u^{\epsilon,\delta}(x)) dy, \qquad x \in \Omega,$$

with $u^{\epsilon,\delta}$ satisfying (1.10) and $C(\epsilon)$ given by (1.6). For this problem we have

Theorem 1.5. *Under the additional conditions $f \in \mathcal{C}^\alpha(\Omega)$ and Ω with boundary $C^{2+\alpha}$-regular, $0 < \alpha < 1$, as $\delta \to 0$ there exists a strong limit in $L^2(\Omega)$, that is,*

$$\lim_{\delta \to 0} u^{\epsilon,\delta} = v^\epsilon$$

where v^ϵ is the solution to

$$
\begin{cases}
\Delta_{x_1} v^\epsilon + \dfrac{1}{\epsilon^2} \Delta_{x_2} v^\epsilon = f, & in \ \Omega, \\
v^\epsilon = 0, & on \ \partial\Omega.
\end{cases}
$$

Moreover, the limit as $\epsilon \to 0$ of v^ϵ is $v \equiv 0$. Hence, we conclude that

$$
\lim_{\epsilon \to 0} \left(\lim_{\delta \to 0} u^{\delta,\epsilon} \right) = 0 \quad in \ L^2(\Omega).
$$

If we reverse the order in which we take limits we obtain the same limit.

Theorem 1.6. *As $\epsilon \to 0$ and then $\delta \to 0$, $v \equiv 0$ is the strong limit in $L^2(\Omega)$, that is,*

$$
\lim_{\delta \to 0} \left(\lim_{\epsilon \to 0} u^{\delta,\epsilon} \right) = 0 \quad in \ L^2(\Omega).
$$

Remark 1.1. Note that, in contrast with Theorems 1.2 and 1.3, we get that the limit as $\delta \to 0$ and $\epsilon \to 0$ of $u^{\epsilon,\delta}$ commutes.

With the same methods and ideas we can handle the case of Ω being a general domain in $\mathbb{R}^{N_1+N_2}$ (not necessarily a product domain) getting different limit problems. In forthcoming papers we will discuss this problem in its full generality. Here, we prefer to present our results for a product domain to simplify the notation and the arguments involved.

We remark that here we deal nonlocal problems with a non-singular and compactly supported kernel. The main difficulty that this fact introduces in the problem is the lack of regularizing effect in the solutions. In fact, for example, solutions to (1.1) are as smooth as the forcing term f is. We leave the case of singular kernels (like the ones that appear considering the fractional Laplacian) for a future paper.

Let us end this introduction with a brief description of related references. Thin domains occur in applications and they can be found in mathematical models from many applied areas. For example, in ocean dynamics, one is dealing with fluid regions which are thin compared to the horizontal length scales. Other examples include lubrication, nanotechnology, blood circulation, material engineering, meteorology, etc; they are a part of a broader study of the behavior of various PDEs on thin $N-$dimensional domains, where $N \geq 2$. Many techniques and methods have been developed in order to understand the effect of the geometry and thickness of the domain on the solutions of such singular problems. From pioneering works to recent ones we can still mention [18, 13, 10, 5, 3] concerned with elliptic and parabolic equations, as well as [2, 9, 4, 11, 12] where the authors considered Stokes and Navier-Stokes equations from fluid mechanics.

The paper is organized as follows: in Section 2 we include the proof of our result concerning the limit as $\epsilon \to 0$ for the Neumann problem, Theorem 1.1; in Section 3 we show that in general the obtained limit u_0 depends on x_2; in Section 4 we deal with the double limit as $\epsilon \to 0$ and $\delta \to 0$ for the Neumann problem and prove Theorems 1.2 and 1.3; in Section 5 we start the analysis of the Dirichlet case and prove Theorem 1.4; finally, in Section 6 we deal with the rescales of the kernel for the Dirichlet case proving Theorems 1.5 and 1.6.

2. The Neumann problem. Proof of Theorem 1.1

First, we just observe that existence and uniqueness of a solution u^ϵ to our problem

$$f(x) = \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))\, dy$$

in the space

$$W = \left\{ u \in L^2(\Omega_1 \times \Omega_2) \, : \, \int_{\Omega_1 \times \Omega_2} u(x)\, dx = 0 \right\},$$

follows easily considering the variational problem

$$\min_{u \in W} \frac{1}{4} \int_\Omega \int_\Omega J_\epsilon(x-y)(u(y) - u(x))^2 dy\, dx - \int_\Omega f(x) u(x)\, dx.$$

In fact, from Lemma 2.1 (see below) we have that in W

$$\|u\|^2 := \int_\Omega \int_\Omega J_\epsilon(x-y)(u(y) - u(x))^2 dy\, dx$$

is a norm equivalent to the usual L^2−norm. From this fact, it is immediate that the functional involved in the minimization problem is lower semicontinuous, coercive and convex in W, and hence there is a unique minimizer in W. This unique minimizer (that we call u^ϵ) is a weak solution to our problem, that is, it verifies (1.3),

$$\int_{\Omega_1 \times \Omega_2} f(x)\varphi(x)\, dx = \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x)) dy\, \varphi(x)\, dx$$

$$= -\frac{1}{2} \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))(\varphi(y) - \varphi(x))\, dy dx$$

for every $\varphi \in L^2(\Omega)$.

Taking $\varphi = u^\epsilon$ we get

$$-\int_{\Omega_1 \times \Omega_2} f(x) u^\epsilon(x)\, dx = \frac{1}{2} \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))^2 \, dy dx$$

$$\geq \lambda_1^\epsilon \int_{\Omega_1 \times \Omega_2} (u^\epsilon(x))^2\, dx.$$

Here λ_1^ϵ is the first eigenvalue associated with this operator in the space W. This first eigenvalue is given by

$$\lambda_1^\epsilon = \inf_{u \in W} \frac{\dfrac{1}{2} \int_\Omega \int_\Omega J_\epsilon(x-y)(u(y) - u(x))^2 dy\, dx}{\displaystyle\int_\Omega u^2(x)\, dx}.$$

For a proof that λ_1^ϵ is strictly positive we refer to [1].

Therefore, using that $\lambda_1^\epsilon \geq c > 0$ with c independent of ϵ (see Lemma 2.1), we get that

$$\int_{\Omega_1 \times \Omega_2} (u^\epsilon(x))^2\, dx$$

is bounded by a constant that depends only on f but is independent of ϵ. Hence, along a subsequence if necessary,

$$(2.1) \qquad\qquad\qquad\qquad\qquad u^\epsilon \rightharpoonup u_0$$

weakly in $L^2(\Omega_1 \times \Omega_2)$ as $\epsilon \to 0$.

Now, let us take a test function that depends only on the first variable, that is $\varphi = \varphi(x_1)$ in (1.3). We obtain

$$\int_{\Omega_1} \varphi(x_1) \left(\int_{\Omega_2} f(x_1, x_2) \, dx_2 \right) dx_1 = \int_{\Omega_1 \times \Omega_2} f(x) \varphi(x_1) \, dx$$

$$= \int_{\Omega_1} \varphi(x_1) \int_{\Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x)) dy \, dx_2 \, dx_1.$$

Taking limit as $\epsilon \to 0$ we get

$$\int_{\Omega_1} \varphi(x_1) \left(\int_{\Omega_2} f(x_1, x_2) \, dx_2 \right) dx_1$$

(2.2)
$$= \int_{\Omega_1} \varphi(x_1) \int_{\Omega_1} J(x_1 - y_1, 0) \left[\int_{\Omega_2} \int_{\Omega_2} (u_0(y_1, y_2) - u_0(x_1, x_2)) \, dy_2 dx_2 \right] dy_1 \, dx_1$$

$$= |\Omega_2| \int_{\Omega_1} \varphi(x_1) \int_{\Omega_1} J(x_1 - y_1, 0) \left[\int_{\Omega_2} u_0(y_1, y_2) dy_2 - \int_{\Omega_2} u_0(x_1, x_2) \, dx_2 \right] dy_1 \, dx_1.$$

Then,

$$U_0(x_1) = \int_{\Omega_2} u_0(x_1, x_2) \, dx_2$$

is a solution to

$$\tilde{f}(x_1) = \int_{\Omega_2} f(x_1, x_2) \, dx_2 = |\Omega_2| \int_{\Omega_1} J(x_1 - y_1, 0)(U_0(y_1) - U_0(x_1)) dy_1.$$

Note that since

$$\int_{\Omega_1} \left(\int_{\Omega_2} u^\epsilon(x_1, x_2) \, dx_2 \right) dx_1 = \int_{\Omega_1 \times \Omega_2} u^\epsilon(x) \, dx = 0$$

we get

$$\int_{\Omega_1} U_0(x_1) \, dx_1 = \int_{\Omega_1 \times \Omega_2} u_0(x) \, dx = 0,$$

and then, U_0 is the unique solution of (1.5) since $J(\cdot, 0)$ also satisfies assumption (**H**). In fact, we have from [1] that $u_0 \in L^2(\Omega_1 \times \Omega_2)$ is the unique solution of

(2.3)
$$f(x) = \int_{\Omega_1 \times \Omega_2} J(x_1 - y_1, 0)(u_0(y) - u_0(x)) dy_1 dy_2$$

in the space W. In this way, we conclude that the sequence u^ϵ is weakly convergent in $L^2(\Omega)$ with limit u_0 as $\epsilon \to 0$.

Consequently, if we denote

$$U^\epsilon(x_1) = \int_{\Omega_2} u^\epsilon(x_1, x_2) \, dx_2,$$

it follows that

(2.4)
$$U^\epsilon \rightharpoonup U_0, \quad \text{as } \epsilon \to 0,$$

weakly in $L^2(\Omega_1)$. Indeed, for all $\varphi \in L^2(\Omega_1)$,

$$\int_{\Omega_1} \varphi \, U^\epsilon \, dx_1 = \int_{\Omega_1 \times \Omega_2} \varphi(x_1) \, u^\epsilon(x_1, x_2) \, dx_1 dx_2$$

$$\rightarrow \int_{\Omega_1 \times \Omega_2} \varphi(x_1) \, u_0(x_1, x_2) \, dx_1 dx_2 = \int_{\Omega_1} \varphi \, U_0 \, dx_1.$$

Now we are ready to proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. The existence of $u_0 \in L^2(\Omega)$ such that $u^\epsilon \rightharpoonup u_0$ and $U^\epsilon \rightharpoonup U_0$ weakly in $L^2(\Omega)$ and $L^2(\Omega_1)$ respectively has been proved in (2.1)-(2.4). Thus, since we are working in Hilbert spaces, we will complete the proof showing the convergence of the norm.

First, we show that

$$\|u^\epsilon\|_{L^2(\Omega)} \rightarrow \|u_0\|_{L^2(\Omega)}.$$

Taking $\phi = u^\epsilon$ in (1.3), we get

$$\int_\Omega \left(\int_\Omega J_\epsilon(x-y) \, dy \right) u^\epsilon(x)^2 \, dx = \int_\Omega \int_\Omega J_\epsilon(x-y) u^\epsilon(y) u^\epsilon(x) \, dy dx - \int_\Omega f(x) u^\epsilon(x) dx.$$

From weak convergence, it is clear that

$$(2.5) \qquad \int_\Omega f(x) u^\epsilon(x) dx \rightarrow \int_\Omega f(x) u_0(x) dx \qquad \text{as } \epsilon \rightarrow 0.$$

Moreover, we have

$$(2.6) \qquad A_\epsilon(x) = \int_\Omega J_\epsilon(x-y) \, dy \; \rightarrow \; \int_\Omega J(x_1 - y_1, 0) \, dy = A_0(x) \qquad \text{in } L^\infty(\Omega)$$

with

$$|\Omega| \|J\|_\infty \geq A_0(x) \geq m > 0 \qquad \forall x \in \Omega.$$

Indeed, since we assume $J(0) > 0$, there exit $\epsilon_0 > 0$ and $m > 0$ such that

$$(2.7) \qquad |\Omega| \|J\|_\infty \geq \int_\Omega J_\epsilon(x-y) \, dy \geq m > 0, \text{ whenever } \epsilon \in [0, \epsilon_0].$$

Also, J is a continuous function in $\mathbb{R}^N$, and then, for any compact set $K \subset \mathbb{R}^N$, given $\delta > 0$, there exists $\epsilon_1 > 0$ such that

$$|A_\epsilon(x) - A_0(x)| \leq \int_\Omega |J(x_1 - y_1, \epsilon(x_2 - y_2)) - J(x_1 - y_1, 0)| \, dz \leq \delta \, |\Omega|$$

whenever $\epsilon |x_2 - y_2| \leq \epsilon_1$ and $(x - y) \in K$. In this way we get (2.6).

Now let us show that

$$(2.8) \qquad \int_\Omega \int_\Omega J_\epsilon(x-y) u^\epsilon(y) u^\epsilon(x) \, dy dx \rightarrow \int_\Omega \int_\Omega J(x_1 - y_1, 0) u_0(y) u_0(x) \, dy dx, \qquad \text{as } \epsilon \rightarrow 0.$$

In order to do so, let us consider

$$O^\epsilon(x) = \int_\Omega J_\epsilon(x-y) u^\epsilon(y) \, dy, \quad x \in \Omega.$$

Since $u^\epsilon \rightharpoonup u_0$ weakly in $L^2(\Omega)$ and $J_\epsilon(x - \cdot) \rightarrow J(x_1 - \cdot, 0)$ in $L^2(\Omega)$ for all $x \in \Omega$, it is not difficult to see that $O^\epsilon(x) \rightarrow O_0(x)$ for all $x \in \Omega$ where

$$O_0(x) = \int_\Omega J(x_1 - y_1, 0) u_0(y) \, dy.$$

Hence, since $|O^\epsilon(x)| \leq \|J\|_\infty \|u^\epsilon\|_{L^2(\Omega)}$ with $\|u^\epsilon\|_{L^2(\Omega)}$ uniformly bounded in ϵ, it follows from the Dominated Convergence Theorem that

$$(2.9) \qquad\qquad\qquad\qquad O^\epsilon \rightharpoonup O_0$$

weakly in $L^2(\Omega)$ as $\epsilon \to 0$.

Moreover, we have $\{O^\epsilon(x)\}^2 \to \{O_0(x)\}^2$ and $|O^\epsilon(x)|^2 \leq \|J\|_\infty^2 \|u^\epsilon\|_{L^2(\Omega)}^2$ for all $x \in \Omega$. Thus, arguing as in (2.9) we get

$$(2.10) \qquad\qquad\qquad \|O^\epsilon\|_{L^2(\Omega)} \to \|O_0\|_{L^2(\Omega)}.$$

We conclude from (2.9) and (2.10) that $O^\epsilon \to O_0$ strongly in $L^2(\Omega)$, which implies

$$\int_\Omega \int_\Omega J_\epsilon(x-y) u^\epsilon(y) u^\epsilon(x) \, dy \, dx = \int_\Omega O^\epsilon(x) u^\epsilon(x) \, dx$$

$$\to \int_\Omega O_0(x) u_0(x) \, dx = \int_\Omega \int_\Omega J(x_1 - y_1, 0) u_0(y) u_0(x) \, dy \, dx, \qquad \text{as } \epsilon \to 0.$$

Then, from (2.5) and (2.8), we obtain

$$\int_\Omega \left(\int_\Omega J_\epsilon(x-y) \, dy \right) u^\epsilon(x)^2 \, dx \to \int_\Omega \int_\Omega J(x_1 - y_1, 0) u_0(y) u_0(x) \, dy \, dx$$

$$(2.11)$$

$$- \int_\Omega f(x) u_0(x) \, dx.$$

Now, observe that

$$\int_\Omega \left(\int_\Omega J(x_1 - y_1, 0) \, dy \right) u_0(x)^2 \, dx = \int_\Omega \int_\Omega J(x_1 - y_1, 0) u_0(y) u_0(x) \, dy \, dx$$

$$- \int_\Omega f(x) u_0(x) dx$$

since $u_0 \in L^2(\Omega)$ satisfies the limit equation (2.3). Then, it follows from (2.11) that

$$\int_\Omega \left(\int_\Omega J_\epsilon(x-y) \, dy \right) u^\epsilon(x)^2 \, dx \to \int_\Omega \left(\int_\Omega J(x_1 - y_1, 0) \, dy \right) u_0(x)^2 \, dx,$$

as $\epsilon \to 0$, that is,

$$(2.12) \qquad\qquad \int_\Omega A_\epsilon(x) \, u^\epsilon(x)^2 \, dx \to \int_\Omega A_0(x) \, u_0(x)^2 \, dx.$$

On the other hand, notice that

$$\int_\Omega A_0(x) \left\{ u^\epsilon(x)^2 - u_0(x)^2 \right\} dx = \int_\Omega \left\{ A_\epsilon(x) \, u^\epsilon(x)^2 - A_0(x) \, u_0(x)^2 \right\} dx$$

$$- \int_\Omega \left\{ A_\epsilon(x) - A_0(x) \right\} u^\epsilon(x)^2 \, dx$$

which implies from (2.6) and (2.12) that

$$\int_\Omega A_0(x) \left\{ u^\epsilon(x)^2 - u_0(x)^2 \right\} dx \to 0, \qquad \text{as } \epsilon \to 0.$$

Hence, since A_0 is strictly positive, we get

$$\|u^\epsilon\|_{L^2(\Omega)} \to \|u_0\|_{L^2(\Omega)}$$

and we conclude that $u^\epsilon \to u_0$ strongly in $L^2(\Omega)$.

We finish our proof showing that

$$\|U^\epsilon\|_{L^2(\Omega_1)} \to \|U_0\|_{L^2(\Omega_1)}$$

as $\epsilon \to 0$. Again, this is a consequence of the Dominated Convergence Theorem since $U^\epsilon \rightharpoonup U_0$ weakly in $L^2(\Omega_1)$ with

$$\{U^\epsilon(x_1)\}^2 \le \left(\int_{\Omega_1} u^\epsilon(x_1, x_2)\, dx_2\right)^2 \le |\Omega_2| \|u^\epsilon(x_1, \cdot)\|^2_{L^2(\Omega_2)} \quad \text{a.e. } \Omega_1,$$

and

$$\int_{\Omega_1} \|u^\epsilon(x_1, \cdot)\|^2_{L^2(\Omega_2)}\, dx_1 = \|u^\epsilon\|^2_{L^2(\Omega)} \to \|u_0\|^2_{L^2(\Omega)} \quad \text{as } \epsilon \to 0.$$

$\square$

Remark 2.1. Let us point out that for the more general situation

$$f^\epsilon(x) = \int_\Omega J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))\, dy$$

with $f^\epsilon \to f$ strongly in $L^2(\Omega)$ we can reach the same results described in Theorem 1.1 since $\|f^\epsilon\|_{L^2(\Omega)}$ remains uniformly bounded in $\epsilon > 0$ and

$$\int_\Omega f^\epsilon(x)\, u^\epsilon(x)\, dx \to \int_\Omega f(x)\, u_0(x)\, dx$$

even when $u^\epsilon \to u_0$ weakly in $L^2(\Omega)$.

Now, let us show that the first eigenvalue associated to this nonlocal problem also converges as $\epsilon \to 0$ to the first eigenvalue of the associated limit operator. The fact that this limit eigenvalue is positive implies that λ_1^ϵ (with ϵ small) is bounded below by a positive constant that is independent of ϵ.

Lemma 2.1. *Let $\{\lambda_1^\epsilon\}_{\epsilon>0}$ be the family of first eigenvalues introduced in (1.3).*

Then, there exists $\lambda_1 > 0$, the first eigenvalue of the operator $T_0 : W_0 \mapsto W_0$ given by

$$(T_0 u)(x_1) := -|\Omega_2| \int_{\Omega_1} J(x_1 - y_1, 0)(u(y_1) - u(x_1))dy_1, \quad x_1 \in \Omega_1,$$

with

$$W_0 = \left\{u \in L^2(\Omega_1) \,:\, \int_{\Omega_1} u(x_1)\, dx_1 = 0\right\},$$

and it holds that

$$\lambda_1^\epsilon \to \lambda_1 \quad \text{as } \epsilon \to 0.$$

Consequently, there exists $c > 0$ independent of ϵ such that $\lambda_\epsilon^1 > c$ for all $\epsilon > 0$.

Proof. First, we note that, due to [1, Lemma 3.5]

$$0 \le \lambda_\epsilon^1 \le \min_{x \in \bar\Omega} \int_\Omega J_\epsilon(x - y)dy \le |\Omega| \|J\|_\infty, \quad \forall \epsilon > 0.$$

Thus, we can extract a subsequence, still denoted by λ_1^ϵ, such that $\lambda_1^\epsilon \to \lambda_1 \ge 0$. Furthermore, we can take a sequence of eigenfunctions $\phi^\epsilon \in L^2(\Omega)$ with $\|\phi^\epsilon\|_{L^2(\Omega)} = 1$ and

$$\phi^\epsilon \rightharpoonup \phi_0$$

weakly in $L^2(\Omega)$ as $\epsilon \to 0$. Note that, for all $\varphi \in L^2(\Omega)$,

$$(2.13) \qquad -\lambda_1^\epsilon \int_\Omega \varphi \, \phi^\epsilon \, dx = \int_\Omega \varphi(x) \left[\int_\Omega J_\epsilon(x-y)(\phi^\epsilon(y) - \phi^\epsilon(x)) dy \right] dx.$$

Moreover, we can assume that $\phi_0 \in L^2(\Omega)$ is not the null function. In fact, if we take $\varphi = \phi^\epsilon$ in (2.13), we get by (2.6) and $\|\phi^\epsilon\|_{L^2(\Omega)} = 1$ that

$$\int_\Omega A_\epsilon(x) \, \phi^\epsilon(x)^2 \, dx - \lambda_1^\epsilon = \int_\Omega \int_\Omega J_\epsilon(x-y) \, \phi^\epsilon(x) \, \phi^\epsilon(y) \, dy dx.$$

Hence, if we have $\phi_0 \equiv 0$ in Ω, we can pass to the limit obtaining

$$\lambda_1^\epsilon \to \lambda_1 \geq m > 0,$$

since the function A_ϵ satisfies (2.7). Thus, we have $\lambda_1 > 0$ and nothing more to prove.

Then, let us assume $\phi_0 \neq 0$ in $L^2(\Omega)$. Passing to the limit in (2.13), we obtain

$$(2.14) \qquad -\lambda_1 \int_\Omega \varphi \, \phi_0 \, dx = \int_\Omega \varphi(x) \left[\int_\Omega J(x_1 - y_1, 0)(\phi_0(y) - \phi_0(x)) dy \right] dx, \quad \forall \varphi \in L^2(\Omega).$$

Arguing as in (2.2), we get from (2.14) that

$$-\lambda_1 \int_{\Omega_1} \varphi \, \Phi_0 \, dx_1 = |\Omega_2| \int_{\Omega_1} \varphi \left[\int_{\Omega_1} J(x_1 - y_1, 0)(\Phi_0(y_1) - \Phi_0(x_1)) dy_1 \right] dx_1, \quad \forall \varphi \in L^2(\Omega_1),$$

where

$$\Phi_0(x_1) = \int_{\Omega_2} \phi_0(x_1, x_2) dx_2.$$

Since $J(\cdot, 0)$ satisfies hypothesis **(H)**, it follows from [1, Proposition 3.4] that λ_1 is the first eigenvalue of T_0 which is positive. We complete the proof observing that $\lambda_1^\epsilon \to \lambda_1$ is an arbitrary subsequence. $\qquad\square$

3. Remarks on the limit solution

Here we discuss the dependence of the limit function u_0 given by Theorem 1.1 on the second variable $x_2 \in \Omega_2$. In order to do so, we will assume that the forcing term f is a smooth function, and the kernel J is of class $\mathcal{C}^1$ with bounded derivatives. Under these assumptions, u^ϵ is also smooth and we can differentiate expression (1.1) with respect to x_2 obtaining

$$\frac{\partial f}{\partial x_2}(x) = \epsilon \int_\Omega \frac{\partial J_\epsilon}{\partial x_2}(x-y)(u^\epsilon(y) - u^\epsilon(x)) \, dy - \int_\Omega J_\epsilon(x-y) \frac{\partial u^\epsilon}{\partial x_2}(x) \, dy,$$

which implies

$$(3.1) \qquad \frac{\partial u^\epsilon}{\partial x_2}(x) = \left(\int_\Omega J_\epsilon(x-y) \, dy \right)^{-1} \left(\epsilon \int_\Omega \frac{\partial J_\epsilon}{\partial x_2}(x-y)(u^\epsilon(y) - u^\epsilon(x)) \, dy - \frac{\partial f}{\partial x_2}(x) \right).$$

Then, since $\|J_\epsilon\|_\infty$ and $\|\partial J_\epsilon/\partial x_2\|_\infty$ are uniformly bounded by a constant independent of ϵ and $u^\epsilon \to u_0$ in $L^2(\Omega)$, we get

$$(3.2) \qquad \frac{\partial u^\epsilon}{\partial x_2}(x) \to - \left(\int_\Omega J(x_1 - y_1, 0) \, dy \right)^{-1} \frac{\partial f}{\partial x_2}(x) \quad \text{a.e. } \Omega$$

as $\epsilon \to 0$. Moreover, it follows from

$$\left| \int_\Omega \frac{\partial J_\epsilon}{\partial x_2}(x-y)(u^\epsilon(y) - u^\epsilon(x))\, dy \right| \leq \left\| \frac{\partial J_\epsilon}{\partial x_2}(x-\cdot) \right\|_{L^2(\Omega)} \left(\|u^\epsilon\|_{L^2(\Omega)} + |\Omega|^{1/2} |u^\epsilon(x)| \right)$$

that

$$\left\| \int_\Omega \frac{\partial J_\epsilon}{\partial x_2}(x-y)(u^\epsilon(y) - u^\epsilon(x))\, dy \right\|_{L^2(\Omega)} \leq 2|\Omega| \left\| \frac{\partial J_\epsilon}{\partial x_2} \right\|_\infty \|u^\epsilon\|_{L^2(\Omega)}.$$

Consequently, we can get from (3.1) that

$$\frac{\partial u^\epsilon}{\partial x_2} \to -A^{-1} \frac{\partial f}{\partial x_2} \quad \text{in } L^2(\Omega)$$

where $A \in L^\infty(\Omega)$ is the positive function given by

(3.3) $$A = \lim_{\epsilon \to 0} \int_\Omega J_\epsilon(\cdot - y)\, dy \quad \text{in } L^\infty(\Omega),$$

that is,

$$A(x) = \int_\Omega J(x_1 - y_1, 0)\, dy, \quad \text{for } x \in \Omega_1 \times \Omega_2.$$

Note that (3.3) is a consequence of the smoothness of J and the boundedness of Ω since

$$\int_\Omega \left\{ J_\epsilon(x-y) - J(x_1 - y_1, 0) \right\} dy \leq \epsilon \left\| \frac{\partial J}{\partial x_2} \right\|_\infty \left(|\Omega|\, |x_2| + \int_\Omega |y_2|\, dy \right).$$

On the other hand, the function u_0 satisfies the limit problem (2.3), and under the assumption that f and J are smooth, we can also differentiate this expression obtaining

$$\frac{\partial f}{\partial x_2}(x) = - \int_\Omega J(x_1 - y_1, 0) \frac{\partial u_0}{\partial x_2}(x)\, dy,$$

that is,

(3.4) $$\frac{\partial u_0}{\partial x_2}(x) = -A^{-1} \frac{\partial f}{\partial x_2}(x) \quad \text{in } \Omega.$$

Then, it follows from (3.2) and (3.4) that

(3.5) $$\frac{\partial u^\epsilon}{\partial x_2} \to \frac{\partial u_0}{\partial x_2} \quad \text{in } L^2(\Omega)$$

as $\epsilon \to 0$.

Therefore, $\partial u_0 / \partial x_2$ will vanish almost everywhere in Ω, if and only if $\partial f / \partial x_2$ does since $A \in L^\infty(\Omega)$ is strictly positive.

Remark 3.1. Note that (3.4) and (3.5) show a contrast between the elliptic local problems and nonlocal ones posed in thin domains. Indeed, as we can see in [8, 17, 14, 15], the limit solutions of homogeneous Neumann local problems necessarily does not depend on the shrinking variable. Here this happens only with the additional assumption

$$\frac{\partial f}{\partial x_2} \equiv 0 \text{ a.e. } \Omega,$$

that is, just when we take forcing terms f independent of the second variable x_2.

Finally, let us analyze the convergence of the function $\partial u^\epsilon / \partial x_1$ in order to get convergence in $H^1(\Omega)$ of u^ϵ. Arguing as in (3.1), we obtain

$$(3.6) \qquad \frac{\partial u^\epsilon}{\partial x_1}(x) = \left(\int_\Omega J_\epsilon(x - y)\, dy \right)^{-1} \left(\int_\Omega \frac{\partial J_\epsilon}{\partial x_1}(x - y)(u^\epsilon(y) - u^\epsilon(x))\, dy - \frac{\partial f}{\partial x_1}(x) \right).$$

Thus, to pass to the limit in (3.6), we just need to consider

$$\left(\int_\Omega J_\epsilon(x - y)\, dy \right)^{-1} \left(\int_\Omega \frac{\partial J_\epsilon}{\partial x_1}(x - y)(u^\epsilon(y) - u^\epsilon(x))\, dy \right)$$

since, due to (3.3), we have

$$\left(\int_\Omega J_\epsilon(x - y)\, dy \right)^{-1} \frac{\partial f}{\partial x_1} \to A^{-1} \frac{\partial f}{\partial x_1} \quad \text{in } L^2(\Omega)$$

as $\epsilon \to 0$.

Now, using that $\|\partial J_\epsilon / \partial x_1\|_\infty$ is uniformly bounded and $u^\epsilon \to u_0$ in $L^2(\Omega)$, we can proceed as in (2.9) and (2.10) to show that

$$\int_\Omega \frac{\partial J_\epsilon}{\partial x_1}(x - y)\, u^\epsilon(y)\, dy \to \int_\Omega \frac{\partial J}{\partial x_1}(x_1 - y_1, 0)\, u_0(y)\, dy$$

and

$$\left(\int_\Omega \frac{\partial J_\epsilon}{\partial x_1}(x - y)\, dy \right) u^\epsilon \to \left(\int_\Omega \frac{\partial J}{\partial x_1}(x_1 - y_1, 0)\, dy \right) u_0 \quad \text{in } L^2(\Omega).$$

Consequently, we obtain from (3.6) that

$$(3.7) \qquad \frac{\partial u^\epsilon}{\partial x_1} \to A^{-1} \left(\int_\Omega \frac{\partial J}{\partial x_1}(\cdot_1 - y_1, 0)(u_0(y) - u_0(\cdot))\, dy - \frac{\partial f}{\partial x_1} \right) \quad \text{in } L^2(\Omega).$$

On the other hand, we can differentiate (2.3) to get

$$(3.8) \qquad \frac{\partial u_0}{\partial x_1} = A(x)^{-1} \left(\int_\Omega \frac{\partial J}{\partial x_1}(x_1 - y_1, 0)(u_0(y) - u_0(x))\, dy - \frac{\partial f}{\partial x_1} \right) \quad \text{in } \Omega.$$

Hence, due to (3.7) and (3.8), we can also conclude that

$$(3.9) \qquad \frac{\partial u^\epsilon}{\partial x_1} \to \frac{\partial u_0}{\partial x_1} \quad \text{in } L^2(\Omega)$$

as $\epsilon \to 0$ obtaining the following result:

Proposition 3.1. *Let $f \in H^1(\Omega)$ and $J \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ satisfying hypothesis* (**H**). *Then, if u^ϵ is the solution of (1.1), there exists $u_0 \in H^1(\Omega)$ such that the functions U^ϵ and U_0 introduced in (1.4) satisfy*

$$U^\epsilon \to U_0 \quad \text{in } H^1(\Omega_1).$$

Proof. It is a direct consequence of Theorem 1.1, (3.5) and (3.9) since

$$\frac{\partial U^\epsilon}{\partial x_1}(x_1) = \int_{\Omega_2} \frac{\partial u^\epsilon}{\partial x_1}(x_1, x_2)\, dx_2.$$

$\square$

4. Rescaling the kernel. The Neumann case.

For a given J satisfying assumption $\mathbf{(H)}$, we consider the following rescaled thin domain kernel

$$J_{\delta,\epsilon}(x) = C \frac{1}{\delta^{N+2}} J_\epsilon \left(\frac{x}{\delta} \right), \quad x \in \Omega$$

where

$$J_\epsilon(x) = J(x_1, \epsilon x_2), \quad (x_1, x_2) \in \Omega_1 \times \Omega_2$$

and

(4.1) $$C = C(\epsilon) = \left(\frac{1}{2} \int_{\mathbb{R}^N} J(x_1, \epsilon x_2) x_{1_1}^2 \, dx \right)^{-1}.$$

Here, x_{1_1} is the first coordinate of $x_1 \in \Omega_1$, and $N = N_1 + N_2$ with $N_i = \dim \Omega_i$, $i = 1, 2$.

Performing the change of variable

$$z_1 = \frac{(x_1 - y_1)}{\delta} \quad \text{and} \quad z_2 = \frac{\epsilon(x_2 - y_2)}{\delta},$$

and using Taylor expansion, we obtain

$$\int_{\mathbb{R}^N} J_{\delta,\epsilon}(x-y)(u(y) - u(x)) dy$$

$$= C \frac{1}{\delta^{2+N}} \int_{\mathbb{R}^N} J\left(\frac{x_1 - y_1}{\delta}, \frac{\epsilon(x_2 - y_2)}{\delta} \right) (u(y) - u(x)) \, dy$$

(4.2) $$= \frac{C}{\delta^2 \epsilon^{N_2}} \int_{\mathbb{R}^N} J(z) \left(u(x_1 - \delta z_1, x_2 - \delta z_2/\epsilon) - u(x) \right) dz$$

$$= \frac{C}{2\epsilon^{N_2}} \left[\sum_{i=1}^{N_1} \partial_i^2 u(x) + \frac{1}{\epsilon^2} \sum_{i=1+N_1}^{N_2} \partial_i^2 u(x) \right] \int_{\mathbb{R}^N} J(z) z_{1_1}^2 dz + O(\delta)$$

$$= \Delta_{x_1} u(x) + \frac{1}{\epsilon^2} \Delta_{x_2} u(x) + O(\delta)$$

since

$$\frac{1}{2\epsilon^{N_2}} \int_{\mathbb{R}^N} J(z) z_{1_1}^2 \, dz = \frac{1}{2} \int_{\mathbb{R}^N} J(z_1, \epsilon z_2) z_{1_1}^2 \, dz,$$

the constant $C = C(\epsilon)$, depending on $\epsilon > 0$, is given by (4.1), and, for all $i = 1, 2, ..., N$,

$$\int_{\mathbb{R}^N} J(z) z_i \, dz = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} J(z) z_i^2 \, dz = \int_{\mathbb{R}^N} J(z) z_1^2 \, dz < \infty.$$

Remark 4.1. Observe that

$$\lim_{\epsilon \to 0} C(\epsilon) = 0$$

since

$$\int_{\mathbb{R}^N} J(x_1, \epsilon x_2) x_{1_1}^2 \, dx = \frac{1}{\epsilon^{N_2}} \int_{\mathbb{R}^N} J(x) x_{1_1}^2 \, dx \to \infty, \quad \text{as } \epsilon \to 0.$$

Expression (4.2) makes a connection between the non-local problem set by the kernel $J_{\delta,\epsilon}$ and the following boundary value problem

(4.3) $$\begin{cases} \Delta_{x_1} v^\epsilon(x) + \dfrac{1}{\epsilon^2} \Delta_{x_2} v^\epsilon(x) = f(x), & x \in \Omega \\[2mm] \dfrac{\partial v^\epsilon}{\partial \eta_1} + \dfrac{1}{\epsilon^2} \dfrac{\partial v^\epsilon}{\partial \eta_2} = 0, & x \in \partial \Omega \end{cases}$$

where $\eta = (\eta_1, \eta_2)$ is the unit forward normal vector of $\partial\Omega$. As we can see in [8, 16, 17], problem (4.3) is that one used to study homogenous Neumann boundary conditions to the Laplacian operator in thin domains.

From Lax-Milgram Theorem, we obtain for each $\epsilon > 0$ and $f \in L^2(\Omega)$ with $\int_\Omega f(x)\,dx = 0$ that (4.3) possesses unique solution in the Hilbert space

$$H = \left\{ v \in H^1(\Omega) \ : \ \int_\Omega v\,dx = 0 \right\}.$$

Moreover, it is known from [16] that there exists $v \in H$, independent of the second variable x_2, (that is, $v(x_1, x_2) = v(x_1)$ a.e. $(x_1, x_2) \in \Omega_1 \times \Omega_2$), weak solution of the N_1-dimensional equation

(4.4)
$$\begin{cases} \Delta_{x_1} v(x_1) = \dfrac{1}{|\Omega_2|} \displaystyle\int_{\Omega_2} f(x_1, s)\,ds, & x_1 \in \Omega_1 \\[2mm] \dfrac{\partial v^\epsilon}{\partial \eta_1} = 0, & x_1 \in \partial\Omega_1 \end{cases}$$

satisfying

$$\|v^\epsilon - v\|_{H^1(\Omega)} \to 0, \quad \text{as } \epsilon \to 0.$$

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and $C^{2+\alpha}$-regular region, and $f \in \mathcal{C}^\alpha(\Omega)$ for some $0 < \alpha < 1$ with $\int_\Omega f(x)\,dx = 0$.*

Then, if v^ϵ, $v \in H$ are given by (4.3) and (4.4) respectively, we have

$$\|v^\epsilon - v\|_{L^2(\Omega)} \to 0, \quad \text{as } \epsilon \to 0.$$

Proof. We refer to [16]. □

Now, under the assumptions of Proposition 4.1, we can argue as in [1, Section 3.2.2] to obtain from (4.2) that the solutions $u^{\delta,\epsilon}$ given by the non-local problem

(4.5)
$$\int_\Omega J_{\delta,\epsilon}(x-y)(u^{\delta,\epsilon}(y) - u^{\delta,\epsilon}(x))dy = f(x), \quad x \in \Omega$$

satisfies

$$\|u^{\delta,\epsilon} - v^\epsilon\|_{L^2(\Omega)} \to 0, \quad \text{as } \delta \to 0$$

for each $\epsilon > 0$ fixed. We sate this result as follows:

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and $C^{2+\alpha}$-regular region, and $f \in \mathcal{C}^\alpha(\Omega)$ for some $0 < \alpha < 1$ with $\int_\Omega f(x)\,dx = 0$.*

Then, if $u^{\delta,\epsilon}$, v^ϵ are given by (4.5) and (4.3) respectively, we have

$$\|u^{\delta,\epsilon} - v^\epsilon\|_{L^2(\Omega)} \to 0, \quad \text{as } \delta \to 0.$$

Now, we just observe that Theorem 1.2 follows from the previous two propositions.

Proof of Theorem 1.2. It follows from Propositions 4.1 and 4.2 that we have

$$\lim_{\epsilon \to 0} \left(\lim_{\delta \to 0} u^{\delta,\epsilon} \right) = v \quad \text{in } L^2(\Omega),$$

as we wanted to show. □

On the other hand, it is not clear that there exists

$$\lim_{\delta \to 0} \left(\lim_{\epsilon \to 0} u^{\delta,\epsilon} \right)$$

since

$$C(\epsilon) \to 0, \quad \text{as } \epsilon \to 0,$$

and the functions $u^{\delta,\epsilon}$ satisfies

$$\int_\Omega f(x)\varphi(x)\,dx = \frac{C(\epsilon)}{\delta^{2+N}} \int_\Omega \varphi(x) \int_\Omega J\left(\frac{x_1 - y_1}{\delta}, \frac{\epsilon(x_2 - y_2)}{\delta} \right) (u^{\delta,\epsilon}(y) - u^{\delta,\epsilon}(x))\,dydx$$

for all $\varphi \in H$, δ and $\epsilon > 0$, and some $f \in H$ fixed. Therefore, we are lead to look for the limit

$$\lim_{\epsilon \to 0} C(\epsilon)\, u^{\epsilon,\delta}.$$

Note that

$$w^{\epsilon,\delta} := C(\epsilon)\, u^{\epsilon,\delta}$$

verifies

$$\int_\Omega f(x)\varphi(x)\,dx = \frac{1}{\delta^{2+N}} \int_\Omega \varphi(x) \int_\Omega J\left(\frac{x_1 - y_1}{\delta}, \frac{\epsilon(x_2 - y_2)}{\delta} \right) (w^{\delta,\epsilon}(y) - w^{\delta,\epsilon}(x))\,dydx.$$

Hence, arguing exactly as in Section 2 we obtain the following result:

Proposition 4.3. *There is a strong limit in $L^2(\Omega)$, that is,*

$$\lim_{\epsilon \to 0} C(\epsilon)\, u^{\epsilon,\delta} = u^\delta$$

as $\epsilon \to 0$. Here u^δ is the unique solution to

(4.6) $$f(x) = \frac{1}{\delta^{N+2}} \int_{\Omega_1 \times \Omega_2} J\left(\frac{(x_1 - y_1)}{\delta}, 0 \right) (u^\delta(y) - u^\delta(x))dy$$

with $\int_\Omega u^\delta\,dx = 0$.

Now, our aim is to take the limit as $\delta \to 0$ in problem (4.6). To this end, we first recall that the power of δ that appear in front of the integral term is not $\delta^{-(N_1+2)}$ but $\delta^{-(N_1+N_2+2)}$ (note that Ω_1 is a N_1-dimensional domain). Therefore, we are naturally lead to consider

$$z^\delta := \frac{u^\delta}{C_1 \delta^{N_2}}.$$

Here

$$C_1 = \left(\frac{1}{2} \int_{\mathbb{R}^N} J(x_1, 0)\, x_{1_1}^2\, dx \right)^{-1}$$

is just a normalizing constant to obtain the Laplacian in the limit.

The functions z^δ are solutions to

$$f(x) = \frac{C_1}{\delta^{N_1+2}} \int_{\Omega_1 \times \Omega_2} J\left(\frac{(x_1 - y_1)}{\delta}, 0 \right) (z^\delta(y) - z^\delta(x))dy.$$

Hence, if we integrate in Ω_2 we get

$$\int_{\Omega_2} f(x_1, x_2)\,dx_2$$

$$= |\Omega_2| \frac{C_1}{\delta^{N_1+2}} \int_{\Omega_1} J\left(\frac{(x_1 - y_1)}{\delta}, 0 \right) \left(\int_{\Omega_2} z^\delta(y_1, y_2)\,dy_2 - \int_{\Omega_2} z^\delta(x_1, x_2)\,dx_2 \right) dy_1.$$

Therefore, using again [1, Section 3.2.2] we obtain,

Proposition 4.4. *There is a strong limit in $L^2(\Omega_1)$,*

$$\lim_{\delta \to 0} \int_{\Omega_2} \frac{u^\delta(x_1, x_2)}{C_1 \, \delta^{N_2}} dx_2 = v(x_1).$$

This limit v is the unique solution to

$$\begin{cases} \Delta_{x_1} v = \dfrac{1}{|\Omega_2|} \displaystyle\int_{\Omega_2} f(x_1, x_2) dx_2, & in \ \Omega_1, \\[4mm] \dfrac{\partial v}{\partial n_1} = 0, & on \ \partial\Omega_1, \end{cases}$$

with $\int_{\Omega_1} v = 0$.

This fact ends the proof of Theorem 1.3.

Proof of Theorem 1.3. We just use Propositions 4.3 and 4.4 to obtain

$$\lim_{\delta \to 0} \left(\int_{\Omega_2} \lim_{\epsilon \to 0} \frac{C(\epsilon)}{C_1 \, \delta^{N_2}} u^{\delta,\epsilon}(x_1, x_2) dx_2 \right) = v \quad in \ L^2(\Omega_1).$$

as we wanted to show.   □

5. The Dirichlet problem. Proof of Theorem 1.4

In this case existence and uniqueness of a solution u^ϵ to our problem

$$f(x) = \int_{\mathbb{R}^N} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x)) dy, \qquad x \in \Omega,$$

with

$$u^\epsilon(x) = 0, \qquad x \in \mathbb{R}^N \setminus \Omega,$$

in the space

$$W = \left\{ u \in L^2(\Omega_1 \times \Omega_2) \right\},$$

follows easily considering the variational problem

$$\min_{u \in W} \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_\epsilon(x - y)(u(y) - u(x))^2 dy \, dx - \int_{\Omega} fu.$$

In fact, what we obtain is a weak solution, that is, u^ϵ verifies

$$\int_{\Omega} f(x)\varphi(x) \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x)) dy \varphi(x) \, dx$$

$$= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))(\varphi(y) - \varphi(x)) \, dy dx$$

for every $\varphi \in L^2(\Omega)$.

Taking, as we did for the Neumann case, $\varphi = u^\epsilon$, we get

$$-\int_{\Omega} f(x)u^\epsilon(x) \, dx = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_\epsilon(x - y)(u^\epsilon(y) - u^\epsilon(x))^2 \, dy dx$$

(5.1)

$$\geq \beta_1^\epsilon \int_{\Omega} (u^\epsilon(x))^2 \, dx.$$

Here β_1^ϵ is given by

$$(5.2) \qquad \beta_1^\epsilon = \inf_{u \in W} \frac{\dfrac{1}{2} \displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_\epsilon(x-y)(u(y)-u(x))^2 dy\, dx}{\displaystyle\int_\Omega u^2(x)\, dx}.$$

For a proof that β_1^ϵ is strictly positive we refer to [1].

Since

$$\left| \int_\Omega f(x)\, u^\epsilon(x)\, dx \right| \le \|f\|_{L^2(\Omega)} \|u^\epsilon\|_{L^2(\Omega)},$$

it follows from (5.1) that

$$\|u^\epsilon\|_{L^2(\Omega)} dx \le \frac{1}{\beta_1^\epsilon} \|f\|_{L^2(\Omega)}.$$

Consequently, if we show

$$(5.3) \qquad \beta_1^\epsilon \to \infty \quad \text{as } \epsilon \to 0,$$

we prove Theorem 1.4, that is,

$$u^\epsilon \to 0 \quad \text{in } L^2(\Omega).$$

Thus, let us show (5.3). From (5.2), we can take $\phi^\epsilon \in L^2(\Omega)$ with $\|\phi^\epsilon\|_{L^2(\Omega)} = 1$ such that, for all $\varphi \in L^2(\Omega)$,

$$(5.4) \qquad -\beta_1^\epsilon \int_\Omega \varphi\, \phi^\epsilon\, dx = \int_\Omega \varphi(x) \left[\int_{\mathbb{R}^N} J_\epsilon(x-y)(\phi^\epsilon(y) - \phi^\epsilon(x)) dy \right] dx$$

with

$$\phi^\epsilon = 0, \qquad x \in \mathbb{R}^N \setminus \Omega.$$

We can rewrite (5.4) obtaining

$$(5.5) \qquad \begin{aligned} -\beta_1^\epsilon \int_\Omega \varphi\, \phi^\epsilon\, dx &= \int_\Omega \varphi(x) \left(\int_\Omega J_\epsilon(x-y)\, \phi^\epsilon(y)\, dy \right) dx \\ &\quad - \int_\Omega \varphi(x)\, \phi^\epsilon(x) \left(\int_{\mathbb{R}^N} J_\epsilon(x-y)\, dy \right) dx \\ &= \int_\Omega \varphi(x) \left(\int_\Omega J_\epsilon(x-y)\, \phi^\epsilon(y)\, dy \right) dx - \frac{1}{\epsilon^{N_2}} \int_\Omega \varphi(x)\, \phi^\epsilon(x)\, dx \end{aligned}$$

since by hypothesis (**H**) we have

$$\int_{\mathbb{R}^N} J_\epsilon(x)\, dx = \frac{1}{\epsilon^{N_2}}.$$

Hence, if we set w^ϵ in $L^2(\Omega)$ by

$$w^\epsilon = \frac{\phi^\epsilon}{\epsilon^{N_2}},$$

we obtain from (5.5) that

$$(5.6) \qquad -\epsilon^{N_2} \beta_1^\epsilon \int_\Omega \varphi\, w^\epsilon\, dx = \epsilon^{N_2} \int_\Omega \varphi(x) \left(\int_\Omega J_\epsilon(x-y) w^\epsilon(y)\, dy \right) dx - \int_\Omega \varphi\, w^\epsilon\, dx$$

for any $\varphi \in L^2(\Omega)$.

Now, let us consider the following linear operator $S_\epsilon : L^2(\Omega) \mapsto L^2(\Omega)$ given by

$$S_\epsilon(w)(x) = \epsilon^{N_2} \int_\Omega J_\epsilon(x - y)w(y)\, dy - w(x), \qquad x \in \Omega.$$

Therefore, since

$$\left\| \int_\Omega J_\epsilon(\cdot - y)w(y)\, dy \right\|_{L^2(\Omega)} \leq \|J\|_\infty\, |\Omega|\, \|w\|_{L^2(\Omega)}$$

for any w, it is not difficult to see that the operator S_ϵ strongly converges to $-Id$ in the space of the bounded linear operators in $L^2(\Omega)$, that is,

(5.7) $$S_\epsilon \to -Id \quad \text{in } \mathcal{B}_{L^2(\Omega)},$$

as $\epsilon \to 0$. Observe that here we are denoting by Id the identity operator, and by $\mathcal{B}_{L^2(\Omega)}$ the space of bounded linear operators in $L^2(\Omega)$.

Due to (5.6), we have that

$$\lambda^\epsilon = -\epsilon^{N_2}\, \beta_1^\epsilon$$

is an eigenvalue of S_ϵ. Then, there exists $\psi^\epsilon \in L^2(\Omega)$ with $\|\psi^\epsilon\|_{L^2(\Omega)} = 1$ such that

$$S_\epsilon(\psi^\epsilon) = \lambda^\epsilon \psi^\epsilon.$$

It follows from (5.7) that

$$\lambda^\epsilon \to -1, \quad \text{as } \epsilon \to 0,$$

and then, there exist $\epsilon_0 > 0$ and $K > 0$ such that

$$|\epsilon^{N_2} \beta_1^\epsilon| \geq K \quad \text{for all } \epsilon \in (0, \epsilon_0].$$

In this way, we obtain (5.3), proving Theorem 1.4.

6. RESCALING THE KERNEL. THE DIRICHLET CASE.

Now let us introduce an extra parameter δ that controls the size of the support of the kernel. As we did for the Neumann case, we consider here the following problem:

(6.1) $$f(x) = \frac{C(\epsilon)}{\delta^{N+2}} \int_{\mathbb{R}^N} J\left(\frac{x_1 - y_1}{\delta}, \epsilon\frac{(x_2 - y_2)}{\delta} \right) (u^{\epsilon,\delta}(y) - u^{\epsilon,\delta}(x))dy, \qquad x \in \Omega,$$

with

$$u^{\epsilon,\delta}(x) \equiv 0, \qquad x \in \mathbb{R}^N \setminus \Omega,$$

and $C(\epsilon)$ given by

$$C(\epsilon) = \left(\frac{1}{2} \int_{\mathbb{R}^N} J(x_1, \epsilon x_2)\, x_{1_1}^2\, dx \right)^{-1}$$

where x_{1_1} is the first coordinate of $x_1 \in \Omega_1$.

Here J is supposed to satisfy assumption (**H**).

Existence and uniqueness of the solutions $u^{\epsilon,\delta}$ of (6.1) are guaranteed in Section 5 for any $f \in L^2(\Omega)$, ϵ and $\delta > 0$. Thus, we can proceed as in Section 4 to analyze the behavior of $u^{\epsilon,\delta}$ as ϵ and δ goes to zero.

First, let us argue as in (5.6) rewriting (6.1) as

$$
\begin{aligned}
f(x) &= \frac{C(\epsilon)}{\delta^{N+2}} \left[\int_\Omega J_\epsilon \left(\frac{x-y}{\delta} \right) u^{\epsilon,\delta}(y)\, dy - u^{\epsilon,\delta}(x) \left(\int_{\mathbb{R}^N} J_\epsilon \left(\frac{x-y}{\delta} \right) dy \right) \right] \\
&= \frac{C(\epsilon)}{\delta^{N+2}} \left[\int_\Omega J_\epsilon \left(\frac{x-y}{\delta} \right) u^{\epsilon,\delta}(y)\, dy - \frac{\delta^N}{\epsilon^{N_2}} u^{\epsilon,\delta}(x) \right] \\
&= \frac{\epsilon^{N_2}}{\delta^{N+2}} \int_\Omega J_\epsilon \left(\frac{x-y}{\delta} \right) w^{\epsilon,\delta}(y)\, dy - \frac{1}{\delta^2} w^{\epsilon,\delta}(x), \qquad x \in \Omega,
\end{aligned}
$$

where

$$
w^{\epsilon,\delta} = \frac{C(\epsilon)}{\epsilon^{N_2}} u^{\epsilon,\delta}
$$

and

$$
J_\epsilon(x) = J(x_1, \epsilon x_2), \quad (x_1, x_2) \in \Omega_1 \times \Omega_2.
$$

Then, we obtain

$$
T_{\epsilon,\delta}(w^{\epsilon,\delta}) = f \quad \text{in } L^2(\Omega)
$$

for all ϵ and δ where $T_{\epsilon,\delta} : L^2(\Omega) \mapsto L^2(\Omega)$ is given by

$$
T_{\epsilon,\delta}(w)(x) = \frac{\epsilon^{N_2}}{\delta^{N+2}} \int_\Omega J_\epsilon \left(\frac{x-y}{\delta} \right) w(y)\, dy - \frac{1}{\delta^2} w(x).
$$

Observe that the bounded linear operator $T_{\epsilon,\delta}$ converges strongly to $-1/\delta^2\, Id$ as $\epsilon \to 0$ for any $\delta > 0$ fixed since

$$
\left\| \int_\Omega J_\epsilon \left(\frac{x-y}{\delta} \right) w(y)\, dy \right\|_{L^2(\Omega)} \le \|J\|_\infty |\Omega| \|w\|_{L^2(\Omega)}.
$$

Remember that Id is the identity operator in the space of bounded linear operator in $L^2(\Omega)$. Thus, for ϵ small enough and any $\delta > 0$, $T_{\epsilon,\delta}$ is an invertible operator, with

$$
T_{\epsilon,\delta}^{-1} \to -\delta^2\, Id \quad \text{as } \epsilon \to 0.
$$

Therefore, we can pass to the limit in $w^{\epsilon,\delta}$ getting

$$
w^{\epsilon,\delta} = T_{\epsilon,\delta}^{-1} f \to -\delta^2 f \quad \text{in } L^2(\Omega).
$$

Hence, we can conclude that

(6.2)
$$
\lim_{\delta \to 0} \left(\lim_{\epsilon \to 0} w^{\epsilon,\delta} \right) = 0 \quad \text{in } L^2(\Omega).
$$

Moreover, since

$$
w^{\epsilon,\delta} = \frac{C(\epsilon)}{\epsilon^{N_2}} u^{\epsilon,\delta} = \left(\frac{1}{2} \int_{\mathbb{R}^N} J(x) x_{1_1}^2\, dx \right)^{-1} u^{\epsilon,\delta}
$$

with $0 < \int_{\mathbb{R}^N} J(x) x_{1_1}^2\, dx < \infty$ independent on ϵ and δ, we obtain from (6.2) that

$$
\lim_{\delta \to 0} \left(\lim_{\epsilon \to 0} u^{\epsilon,\delta} \right) = 0 \quad \text{in } L^2(\Omega)
$$

proving Theorem 1.6.

Finally, we observe that the proof of Theorem 1.5 is analogous to the one of Theorem 1.2 using [7] instead of [6]. We leave the details to the reader.

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