The p-Laplacian in thin channels with locally periodic roughness and different scales

Jean Carlos Nakasato* [1] and Marcone Corrêa Pereira† [2]

[1] Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, Zagreb, Croatia
[2] Depto. Matemática Aplicada, IME, Universidade de São Paulo, Rua do Matão 1010, São Paulo - SP, Brazil

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Abstract

In this work we analyze the asymptotic behavior of the solutions of the p-Laplacian equation with homogeneous Neumann boundary conditions posed in bounded thin domains as

$$R^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in (0,1) \text{ and } 0 < y < \varepsilon G\left(x, x/\varepsilon^\alpha\right)\right\}$$

for some $\alpha > 0$. We take a smooth function $G : (0,1) \times \mathbb{R} \mapsto \mathbb{R}$, L-periodic in the second variable, which allows us to consider locally periodic oscillations at the upper boundary. The thin domain situation is established passing to the limit in the solutions as the positive parameter ε goes to zero and we determine the limit regime for three case: $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$.

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1 Introduction

Partial Differential Equations on thin domains (domains in which the size in some directions is much larger than the size in others) appear naturally in biological systems and industrial applications [13, 14, 24]. In most of the applications, the boundary of those domains is not perfectly flat and one can see irregularities. Then, the influence of such boundary distortions might not be neglected because its effect on the effective equation of the considered system, even far from the rough boundary, can be meaningful [1, 9, 11]. This motivates researchers to employ different homogenization techniques and try to determine the effective flow behavior on a lower-dimensional domain which captures the influence of the geometry, roughness and thickness of the perturbed domain on the solutions of such singular boundary value problems. The obtained equations are then suitable for numerical simulations and provide rigorous justification of various natural phenomenon seen in such complex systems.

A simple manner to consider such irregularities is to stody domains of type

$$Q^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in (0,1) \text{ and } 0 < y < \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right\} \quad \text{for } \varepsilon > 0,$$

where g is a positive, bounded and periodic function satisfying some regularity hypothesis and $\varepsilon > 0$ is a small parameter which goes to zero. Thereby, in the limit $\varepsilon \to 0$, the open set Q^ε degenerates to the

*e-mail: jean.carlos.nakasato@math.hr. Partially supported by CNPq 141675/2015-2 (Brazil), INCT/CAPES 88887.507830/2020-00 (Brazil) and Croatian Science Foundation under the project AsAn (IP-2018-01-2735).

†e-mail: marcone@ime.usp.br. Partially supported by CNPq 308950/2020-8, FAPESP 2020/04813-0 and 2020/14075-6 (Brazil), PID2019-103860GB-I00 MICINN (Spain) and MultiFM IP-2019-04-1140 (Croatia).

unit interval presenting oscillatory behavior on the upper boundary (see for instance [1, 22, 21, 5, 3] where similar approach are performed).

The periodic rough boundary considered above is certainly a first step, but usually not enough, since most of the irregularities present in real applications are not periodic. In this work we are interested in the following family of rough thin domains

$$R^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in (0,1) \text{ and } 0 < y < \varepsilon G_\varepsilon(x)\right\} \quad \text{for } \varepsilon > 0 \tag{1.1}$$

where

$$G_\varepsilon(x) = G\left(x, \frac{x}{\varepsilon^\alpha}\right)$$

for some parameter $\alpha > 0$ with function G satisfying the conditions given by hypothesis **(H)** set in Section 2. This kind of domain perturbation is called in the literature locally periodic thin domain and it is illustrated in figure 1 below.

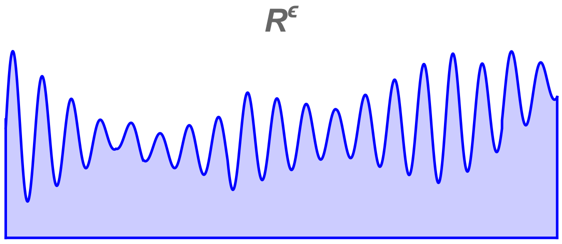

Figure 1: A locally periodic thin channel with rough boundary.

As an example, one can consider $G_\varepsilon(x) = a(x)+b(x)g(x/\varepsilon^\alpha)$ where $a, b : (0,1) \to \mathbb{R}$ are C^1-piecewise positive functions and $g : \mathbb{R} \to \mathbb{R}$ is a L-periodic function of class C^1 also positive. This includes the case where a, b are positive constants recovering the perturbed regions discussed for instance in [3, 5]. Notice that in the case in which $\alpha = 0$, we also recover the open sets considered in [13] where evolution equations on thin domains without roughness were studied. We observe that the hypothesis **(H)** considered here is as general as possible for our framework.

In a unified way, we treat the three cases of roughness that can be modeled by the parameter $\alpha > 0$. We analyse our boundary value problem for $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$, which represents weak, resonant and strong harshness on the upper boundary respectively. In each case, we have a different effective equation featuring the roughness induced effects on the perturbed model for small values of the parameter ε.

Several references treat issues related to the effect of thickness and rough boundaries on the feature of the solutions of partial differential equations. Indeed, thin structures with oscillating boundaries appear in many fields of science: fluid dynamics (lubrication), solid mechanics (thin rods, plates or shells) or even physiology (blood circulation). Therefore, analyzing the asymptotic behavior of models set on thin structures understanding how the geometry and the roughness affect the problem is of considerable current interest in applied science. In these directions, let us mention [7, 10, 15, 21] and references therein.

In this paper, we are interested in analyzing the asymptotic behavior of the solutions of a p-Laplacian equation given by

$$\begin{cases} -\Delta_p u_\varepsilon + |u_\varepsilon|^{p-2} u_\varepsilon = f^\varepsilon \text{ in } R^\varepsilon \\ |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \eta_\varepsilon = 0 \text{ on } \partial R^\varepsilon \end{cases} \tag{1.2}$$

where η_ε is the unit outward normal vector to the boundary ∂R^ε, $1 < p < \infty$ with $p^{-1} + p'^{-1} = 1$, and

$$\Delta_p \cdot = \text{div } \left(|\nabla \cdot|^{p-2} \nabla \cdot\right)$$

denotes the p-Laplacian differential operator. We also assume $f^\varepsilon \in L^{p'}(R^\varepsilon)$ is uniformly bounded.

Such quasilinear equations play an important role in applications, given the fact that many models cannot be described by linear equations. In this sense, considering the p-Laplacian equation becomes natural. Moreover, the p-Laplacian is strongly related to non-Newtonian fluids, which arise in many applications related to polymer processing, hydrology, food processing, turbulent filtration, glaceology (see e.g. [6, 25, 16, 17]). Here, differently from many works [11, 12], we deal also with the case $1 < p < 2$, which is the most relevant range of p in applications (e.g. [6]) and, of course, the case $p \geq 2$.

We improve the results from [3] (where the Laplacian operator in locally periodic thin domains were considered) dealing with the p-Laplacian equation for any $p \in (1, \infty)$. Moreover, we are improving our previous results from [2] where the purely periodic case in bidimensional thin regions were studied. It is worth noticing that the techniques developed in [2, 3] cannot be directly applied in this case. On the one hand, the results concerning the unfolding operator obtained in [4] do not guarantee strong convergence in L^p for the unfolding operator applied on solutions of quasilinear operators. On the other hand, the analysis performed in [3] just works on L^2-spaces. Our goal here is to overcome this situation. We discretize the oscillating region passing to the limit using uniform estimates on two parameters: one associated to the roughness, and other given by the variable profile of the thin domain. In this way, a continuous dependence property for the solutions with respect to the function G in L^p-norms is crucial and it is obtained in Theorem 4.1. We point out that these techniques also work for the dimension reduction from 3-dimensional thin sets to two-dimensional ones. The main change is in the limit problem. In 3D, we somehow lose the explicit p-Laplacian form, as in the unidimensional limit, but, clearly, the monotonicity of this limit operator is preserved (it will be done in a forthcoming work).

Notice that our work also goes a step further from [23] where the p-Laplacian operator is studied in standard thin domains. Let us emphasize that the standard thin domains were previously introduced and rigorously studied in the paper [13] of J. Hale and G. Raugel where the continuity of the family of attractors set by a semilinear parabolic equation in thin domains was considered.

According to [1] and references therein, it is expected that the sequence u_ε will converge to a function of just one variable $x \in (0, 1)$ satisfying a one-dimensional equation of the same type. Combining boundary perturbation techniques [3, 4, 5] and monotone operator analysis [17], we identify the effective limit model of (2.1) at $\varepsilon = 0$.

The paper is organized as follows. In Section 2 we state the main result of the paper. In Section 3, we introduce some notations and state some basic results which will be needed in the sequel. In Section 4, we prove the continuous dependence of the solutions in L^p-spaces with respect to the function G uniformly in the parameter $\varepsilon > 0$ improving [3, Theorem 4.1] from L^2 to L^p-spaces. In Section 5, we perform the asymptotic analysis of (1.2) in piecewise periodic thin domains (that is, in thin domains set by functions G which are piecewise constants in the first variable x, and L-periodic in the second one). See Figure 2 below which illustrates piecewise periodic open sets.

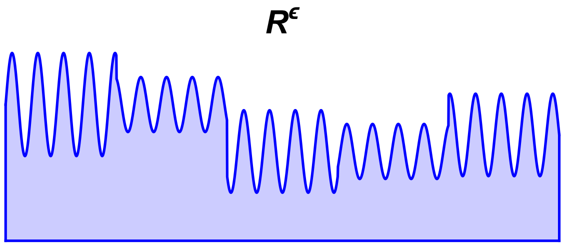

Figure 2: A piecewise periodic thin domain.

Next, we provide in Section 6 the proof of the main result of the paper (namely Theorem 2.1) as a consequence of the analysis performed in the previous sections. Finally, we discuss in Section 7, the convergence of the resolvent and semigroup associated to the equation (1.2) under the additional assumption $p \geq 2$. As we will see, it is obtained combining the classical result [8, Theorem 4.2] and our main result Theorem 2.1. Furthermore, we include an Appendix where the dependence of the auxiliary solution v on admissible functions G is analysed.

2 Hypothesis on function G and the main result

First, recall that the variational formulation of (1.2) is given by

$$\int_{R^\varepsilon} \left\{ |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \varphi + |u_\varepsilon|^{p-2} u_\varepsilon \varphi \right\} dxdy = \int_{R^\varepsilon} f^\varepsilon \varphi \, dxdy, \qquad \varphi \in W^{1,p}(R^\varepsilon). \tag{2.1}$$

Moreover, existence and uniqueness of the solutions are guaranteed by Minty-Browder's Theorem setting a family of solutions u_ε.

Next, we state the main hypothesis on function G setting the main conditions on our rough thin domain R^ε introduced in (1.1).

(H)

Let $G : (0,1) \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying that there exist a finite number of points

$$0 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$$

such that $G : (\xi_{i-1}, \xi_i) \times \mathbb{R} \to (0, \infty)$ is C^1 and such that G, $\partial_x G$ and $\partial_y G$ are uniformly bounded in $(\xi_{i-1}, \xi_i) \times \mathbb{R}$ getting limits when we approach ξ_{i-1} and ξ_i. Further, we assume there exist two constants G_0 and G_1 such that

$$0 < G_0 \leq G(x, y) \leq G_1, \quad \forall (x, y) \in (0,1) \times \mathbb{R},$$

and a real number $L > 0$ such that $G(x, y + L) = G(x, y)$ for all $(x, y) \in (0,1) \times \mathbb{R}$.[a]

[a] $G(x, \cdot)$ is a L-periodic function for each $x \in (0,1)$.

As we will see, the homogenized limit equation is a one-dimensional p-Laplacian equation with variable coefficients $q(x)$ and $r(x)$. It assumes the following form

$$\begin{cases} -\left(q(x)|u'|^{p-2}u'\right)' + r(x)|u|^{p-2}u = \hat{f} \quad \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

where the homogenized coefficients are given by

$$q(x) = \begin{cases} \dfrac{1}{L} \displaystyle\int_{Y^*(x)} |\nabla v|^{p-2} \partial_{y_1} v \, dy_1 dy_2, & \text{if} \quad \alpha = 1, \\ \dfrac{1}{\langle 1/G^{p'-1}(x, \cdot) \rangle_{(0,L)}^{p-1}}, & \text{if} \quad \alpha < 1, \\ G_0(x) = \min\limits_{y \in \mathbb{R}} G(x, y), & \text{if} \quad \alpha > 1, \end{cases} \tag{2.2}$$

and

$$r(x) = \frac{|Y^*(x)|}{L} = \langle G(x, \cdot) \rangle_{(0,L)} \, .$$

We emphasize here the dependence of the function $q(x)$ with respect to the parameter $\alpha > 0$ and variable $x \in (0,1)$ which generalizes our previous work [2]. The function $\hat{f}$ is the weak limit of $\hat{f}^\varepsilon$ in $L^{p'}(0,1)$ with $\hat{f}^\varepsilon$ defined by the family of known forcing terms $f^\varepsilon \in L^{p'}(R^\varepsilon)$ in the following way

$$\hat{f}^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon G(x, x/\varepsilon^\alpha)} f^\varepsilon(x, y) dy.$$

$|Y^*(x)|$ denotes the Lebesgue measure of the representative cell

$$Y^*(x) = \{(y_1, y_2) : 0 < y_1 < L, 0 < y_2 < G(x, y_1)\}$$

which also depends on variable $x \in (0,1)$. The function v used to set the homogenized coefficient $q(x)$ in (2.2) is is the unique solution of the problem

$$\begin{aligned} &\int_{Y^*(x)} |\nabla v|^{p-2} \nabla v \nabla \varphi \, dy_1 dy_2 = 0 \quad \forall \varphi \in W^{1,p}_{\#}(Y^*(x)), \qquad \langle \varphi \rangle_{Y^*(x)} = 0, \\ &\qquad (v - y_1) \in W^{1,p}_{\#}(Y^*(x)) \qquad \text{with} \qquad \langle (v-y_1) \rangle_{Y^*(x)} = 0 \end{aligned} \tag{2.3}$$

where

$$W^{1,p}_{\#}(Y^*(x)) = \{\varphi \in W^{1,p}(Y^*(x)) \, : \, \varphi|_{\partial_{left} Y^*(x)} = \varphi|_{\partial_{right} Y^*(x)}\}$$

is the space of periodic functions on the horizontal variable y_1, and $\langle \varphi \rangle_{\mathcal{O}}$ denotes the average of any function $\varphi \in L^1_{loc}(\mathbb{R}^M)$ on measurable sets $\mathcal{O} \subset \mathbb{R}^M$.

It is worth noticing that problem (2.3) is well posed for each $x \in (0,1)$, due to Minty-Browder's Theorem, and then, the coefficient $q(x)$ is also well defined. Further, $q(x)$ is a positive function setting a well posed homogenized equation. Indeed, since v is the solution of (2.3), there exists $\psi \in W^{1,p}_{\#}(Y^*(x))$ with $\langle \psi \rangle_{Y^*(x)} = 0$ for each $x \in (0,1)$ such that $v = y_1 + \psi$ and then

$$\begin{aligned} 0 < \int_{Y^*(x)} |\nabla v|^p \, dy_1 dy_2 &= \int_{Y^*(x)} |\nabla v|^{p-2} \nabla v \nabla (y_1 + \psi) \, dy_1 dy_2 \\ &= \int_{Y^*(x)} |\nabla v|^{p-2} \partial_{y_1} v \, dy_1 dy_2 = L \, q(x). \end{aligned}$$

The main result of the paper is the following:

Theorem 2.1. *Let u_ε be the solution of* (1.2) *with $f^\varepsilon \in L^{p'}(R^\varepsilon)$ uniformly bounded. Suppose that*

$$\hat{f}^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon G(x, x/\varepsilon^\alpha)} f^\varepsilon(x,y) dy$$

satisfies $\hat{f}^\varepsilon \rightharpoonup \hat{f}$ weakly in $L^{p'}(0,1)$.

Let $u \in W^{1,p}(0,1)$ be the unique solution of the homogenized equation

$$\int_0^1 \left\{ q(x) |u'|^{p-2} u' \varphi' + r(x) \, |u|^{p-2} u \varphi \right\} dx = \int_0^1 \hat{f} \varphi dx, \quad \forall \varphi \in W^{1,p}(0,1),$$

where the homogenized coefficients $q(x)$ and $r(x)$ depend on the parameter $\alpha > 0$ and are given by the expression (2.2).

Then,

$$\frac{L}{|Y^*(x)| \varepsilon} \int_0^{\varepsilon G(x, x/\varepsilon^\alpha)} u_\varepsilon(x,y) dy \rightharpoonup u \text{ weakly in } L^p(0,1),$$

and

$$\varepsilon^{-1/p} ||u_\varepsilon - u||_{L^p(R^\varepsilon)} \to 0, \qquad \text{as } \varepsilon \to 0.$$

As mentioned before, we are improving the results from [3] where the Laplacian operator in locally periodic thin domains were considered. We recover them taking $p = 2$ in Theorem 2.1. Moreover, we also have improved our previous results from [2] where the purely periodic case in bidimensional thin regions were studied to the p-Laplacian operator where constant homogenized coefficients are obtained. Here, since we are considering locally periodic thin domains, variable homogenized coefficients can be produced. The main step in the proof is to pass to the limit in the solutions with the representative cell depending on vairable $x \in (0,1)$ assuming different orders of roughness (different values for the parameter $\alpha > 0$). To do that, we discretize the oscillating thin region passing to the limit using uniform estimates on two parameters: one associated to the roughness, and other given by the variable profile of the thin domain. In this way, a continuous dependence property for the solutions with respect to the function G in L^p-norms is crucial and it is shown in Theorem 4.1 below.

3 Basic Facts and the unfolding operator

In this section, we introduce some basic facts, definitions and results concerning to the unfolding method making some straightforward adaptations to our propose. First, let us just recall some basic properties to the p-Laplacian which can be found for instance in [17].

Proposition 3.1. *Let $x, y \in \mathbb{R}^n$.*

- *If $p \geq 2$, then*
$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y \rangle \geq c_p |x-y|^p.$$

- *If $1 < p < 2$, then*
$$\begin{aligned} \langle |x|^{p-2}x - |y|^{p-2}y, x-y \rangle &\geq c_p |x-y|^2 (|x|+|y|)^{p-2} \\ &\geq c_p |x-y|^2 (1+|x|+|y|)^{p-2}. \end{aligned}$$

Corollary 3.1.1. *Let $a_p : \mathbb{R}^n \to \mathbb{R}^n$ such that $a_p(s) = |s|^{p-2}s$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then, a_p is the inverse of $a_{p'}$. Moreover,*

- *If $1 < p' < 2$ (i.e, $p \geq 2$), then*
$$\left| |u|^{p'-2}u - |v|^{p'-2}v \right| \leq c|u-v|^{p'-1}.$$

- *If $p' \geq 2$ (i.e, $1 < p < 2$), then*
$$\begin{aligned} \left| |u|^{p'-2}u - |v|^{p'-2}v \right| &\leq c|u-v|(|u|+|v|)^{p'-2} \\ &\leq c|u-v|(1+|u|+|v|)^{p'-2}. \end{aligned}$$

Proposition 3.2. *Let $x, y \in \mathbb{R}^n$ and $p \geq 1$. Then.*
$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y-x)$$

Moreover,
$$\begin{aligned} &|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y-x) + c_p|y-x|^p \ \text{if } p \geq 2, \\ &|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y-x) + c_p|x-y|^2(1+|x|+|y|)^{p-2} \ \text{if } 1 < p < 2. \end{aligned}$$

From now on, we use the following rescaled norms
$$\begin{aligned} |||\varphi|||_{L^p(R^\varepsilon)} &= \varepsilon^{-1/p} \, ||\varphi||_{L^p(R^\varepsilon)} \, \forall \varphi \in L^p(R^\varepsilon), 1 \leq p < \infty, \\ |||\varphi|||_{W^{1,p}(R^\varepsilon)} &= \varepsilon^{-1/p} \, ||\varphi||_{W^{1,p}(R^\varepsilon)} \, \forall \varphi \in W^{1,p}(R^\varepsilon), 1 \leq p < \infty. \end{aligned}$$

For completeness we may denote $|||\varphi|||_{L^\infty(R^\varepsilon)} = ||\varphi||_{L^\infty(R^\varepsilon)}$.

Next, we get the following uniform bound for the solutions of (1.2):

Proposition 3.3. *Consider the variational formulation of our problem:*
$$\int_{R^\varepsilon} \left\{ |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \varphi + |u_\varepsilon|^{p-2} u_\varepsilon \varphi \right\} dxdy = \int_{R^\varepsilon} f^\varepsilon \varphi dxdy, \ \varphi \in W^{1,p}(R^\varepsilon), \tag{3.1}$$

where f^ε satisfies
$$|||f^\varepsilon|||_{L^{p'}(R^\varepsilon)} \leq c$$

for some positive constant c independent of $\varepsilon > 0$. Then,
$$\begin{gathered} |||u_\varepsilon|||_{L^p(R^\varepsilon)} \leq c, \qquad |||u_\varepsilon|||_{W^{1,p}(R^\varepsilon)} \leq c, \\ \left|\left|\left| |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \right|\right|\right|_{L^{p'}(R^\varepsilon)} \leq c. \end{gathered}$$

Proof. Take $\varphi = u_\varepsilon$ in (3.1). Then,

$$||u_\varepsilon||^p_{W^{1,p}(R^\varepsilon)} = \int_{R^\varepsilon} \{|\nabla u_\varepsilon|^p + |u_\varepsilon|^p\} \, dxdy \leq ||f^\varepsilon||_{L^{p'}(R^\varepsilon)} \, ||u_\varepsilon||_{L^p(R^\varepsilon)} \, .$$

Hence,

$$|||u_\varepsilon|||_{W^{1,p}(R^\varepsilon)} \leq c.$$

Therefore, the sequence u_ε and $|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$, are respectively bounded in $L^p(R^\varepsilon)$ and $(L^{p'}(R^\varepsilon))^2$ under the norm $|||\cdot|||$. ☐

3.1 Unfolding operator

Here, we present the unfolding operators for thin domains in the purely and locally periodic settings. We rewrite it to our context in order to simplify our proofs. They were first introduced in [4, 5] where details and proofs can be found.

3.1.1 The purely periodic unfolding

Let $G_i : \mathbb{R} \to \mathbb{R}$ be a L-periodic function, lower semicontinuous satisfying $0 < g_{0,i} \leq G_i(x) \leq g_{1,i}$ with $g_{0,i} = \min_{x\in\mathbb{R}} G_i(x)$ and $g_{1,i} = \sup_{x\in\mathbb{R}} G_i(x)$ for any $i = 1, ..., N$. Now consider the thin region

$$R^\varepsilon_i = \{(x, y) \in \mathbb{R} : \xi_{i-1} < x < \xi_i, 0 < y < \varepsilon G_i(x/\varepsilon)\} \, .$$

The basic cell associated to R^ε_i is

$$Y^*_i = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < G_i(y_1)\}.$$

By

$$\langle \varphi \rangle_{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \varphi(x) \, dx,$$

we denote the average of $\varphi \in L^1_{loc}(\mathbb{R}^2)$ for any open measurable set $\mathcal{O} \subset \mathbb{R}^2$. We also set functional spaces which are defined by periodic functions in the variable $y_1 \in (0, L)$. Namely

$$L^p_\#(Y^*_i) = \{\varphi \in L^p(Y^*_i) \, : \, \varphi(y_1, y_2) \text{ is } L\text{-periodic in } y_1 \, \},$$
$$L^p_\# \left((0, 1) \times Y^*_i\right) = \{\varphi \in L^p((0, 1) \times Y^*_i) \, : \, \varphi(x, y_1, y_2) \text{ is } L\text{-periodic in } y_1 \, \},$$
$$W^{1,p}_\#(Y^*_i) = \{\varphi \in W^{1,p}(Y^*_i) \, : \, \varphi|_{\partial_{left} Y^*_i} = \varphi|_{\partial_{right} Y^*_i}\}.$$

For each $\varepsilon > 0$ and any $x \in (\xi_{i-1}, \xi_i)$, there exists an integer denoted by $\left[\frac{x}{\varepsilon}\right]_L$ such that

$$x = \varepsilon \left[\frac{x}{\varepsilon}\right]_L L + \varepsilon \left\{\frac{x}{\varepsilon}\right\}_L \text{ where } \left\{\frac{x}{\varepsilon}\right\}_L \in [0, L).$$

We still set

$$I^i_\varepsilon = \text{ Int } \left(\bigcup_{k=1}^{N^i_\varepsilon} [kL\varepsilon + \xi_{i-1}, (k + 1)L\varepsilon + \xi_{i-1}] \right)$$

where N^i_ε is largest integer such that $\varepsilon L(N^i_\varepsilon + 1) + \xi_{i-1} \leq \xi_i$, as well

$$\Lambda^i_\varepsilon = (\xi_{i-1}, \xi_i) \backslash I^i_\varepsilon = [\varepsilon L(N^i_\varepsilon + 1) + \xi_{i-1}, \xi_i),$$
$$R^\varepsilon_{0i} = \left\{(x, y) \in \mathbb{R}^2 : x \in I^i_\varepsilon, 0 < y < \varepsilon G_i \left(\frac{x}{\varepsilon}\right)\right\},$$
$$R^\varepsilon_{1i} = \left\{(x, y) \in \mathbb{R}^2 : x \in \Lambda^i_\varepsilon, 0 < y < \varepsilon G_i \left(\frac{x}{\varepsilon}\right)\right\}.$$

Now we can introduce the unfolding operator. In the sequel, we point out its main properties.

Definition 3.4. *Let φ be a Lebesgue-measurable function in R_i^ε. The unfolding operator $\mathcal{T}_\varepsilon^i$ acting on φ is defined as the following function in $(\xi_{i-1}, \xi_i) \times Y_i^*$*

$$\mathcal{T}_\varepsilon^i \varphi(x, y_1, y_2) = \begin{cases} \varphi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_L L + \varepsilon y_1, \varepsilon y_2\right), \text{ for } (x, y_1, y_2) \in I_\varepsilon^i \times Y_i^*, \\ 0, \text{ for } (x, y_1, y_2) \in \Lambda_\varepsilon^i \times Y_i^*. \end{cases}$$

Proposition 3.5. *The unfolding operator satifies the following properties:*

1. *$\mathcal{T}_\varepsilon^i$ is linear;*
2. *$\mathcal{T}_\varepsilon^i(\varphi\psi) = \mathcal{T}_\varepsilon^i(\varphi)\mathcal{T}_\varepsilon(\psi)$, for all φ, ψ Lebesgue mesurable in R_i^ε;*
3. *$\forall \varphi \in L^p(R_i^\varepsilon)$, $1 \le p \le \infty$,*
$$\mathcal{T}_\varepsilon^i(\varphi)\left(x, \left\{\frac{x}{\varepsilon}\right\}_L, \frac{y}{\varepsilon}\right) = \varphi(x, y),$$
for $(x, y) \in R_{0i}^\varepsilon$.
4. *Let (φ_ε) be a sequence in $L^p(R_i^\varepsilon)$, $1 < p \le \infty$ with the norm $||\varphi_\varepsilon||_{L^p(R_i^\varepsilon)}$ uniformly bounded. Then,*
$$\frac{1}{\varepsilon} \int_{R_{1i}^\varepsilon} |\varphi_\varepsilon| dx dy \to 0.$$
5. *Let (φ_ε) be a sequence in $L^p(\xi_{i-1}, \xi_i)$, $1 \le p < \infty$, such that*
$$\varphi_\varepsilon \to \varphi \text{ strongly in } L^p(\xi_{i-1}, \xi_i).$$
Then,
$$\mathcal{T}_\varepsilon^i \varphi_\varepsilon \to \varphi \text{ strongly in } L^p\left((\xi_{i-1}, \xi_i) \times Y_i^*\right).$$

Proof. See [5, Proposition 2.5]. □

The above result sets several basic and somehow immediate properties of the unfolding operator. Property 5 will be essential to pass to the limit when dealing with solutions of differential equations since it allow us to transform any integral over the thin sets depending on the parameter ε and function G_i into an integral over the fixed set $(\xi_{i-1}, \xi_i) \times Y_i^*$.

3.1.2 Locally Periodic Unfolding

Next we set the locally periodic unfolding operator discussing some properties that will be needed in the sequel.

Definition 3.6. *We define the locally periodic unfolding operator T_ε^{lp} acting on a measurable function φ, as the function $T_\varepsilon^{lp}\varphi$ defined in $(0, 1) \times (0, L) \times (0, G_1)$ by expression*

$$T_\varepsilon^{lp}\varphi(x, y_1, y_2) = \widetilde{\varphi}\left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha}\right] L + \varepsilon^\alpha y_1, \varepsilon y_2\right) \text{ for } (x, y_1, y_2) \in (0, 1) \times (0, L) \times (0, G_1),$$

where $\widetilde{\cdot}$ denotes the extension by zero to the whole space.

As in classical periodic homogenization, we have the unfolding operator reflecting two scales. The macroscopic one, denoted by x which gives the position in the interval $(0, 1)$, and the microscopic scale given by (y_1, y_2) which sets the position in the cell $(0, L) \times (0, G_1)$. However, due to the locally periodic oscillations of the domain R^ε, the definition given here differs from the usual ones. In this case, we do not have a fixed cell that describes the domain R^ε which makes the extesion by zero needed.

Theorem 3.7. *Let $\varphi_\varepsilon \in W^{1,p}(R^\varepsilon)$ for $1 < p < \infty$ such that $|||\varphi_\varepsilon|||_{W^{1,p}(R^\varepsilon)}$ is uniformly bounded. Then, there exists $\varphi \in W^{1,p}(0, 1)$ such that, up to subsequences,*

$$T_\varepsilon^{lp}\varphi_\varepsilon \rightharpoonup \varphi \chi_{(0,1)\times Y^*(x)},$$

weakly in $L^p\left((0, 1) \times (0, L) \times (0, G_1)\right)$ where $\chi_{(0,1)\times Y^*(x)}$ is the characteristic function of the set

$$\{(x, y) \in \mathbb{R}^2 \, : \, x \in (0, 1) \text{ and } y \in Y^*(x)\}.$$

Proof. See [4, Theorem 3.14]. $\square$

Remark 3.1. *We point out that the convergence above cannot be improved because of the definition of locally periodic unfolding operator.*

Proposition 3.8. *1. Let $\varphi \in L^1(R^\varepsilon)$. Then,*

$$\frac{1}{L}\int_{(0,1)\times(0,L)\times(0,G_1)} T_\varepsilon^{lp}\varphi(x,y_1,y_2)dxdy_1dy_2 = \frac{1}{\varepsilon}\int_{R^\varepsilon}\varphi(x,y)dxdy.$$

2. Let $\varphi \in L^p(0,1)$. Then,

$$T_\varepsilon^{lp}\varphi \to \chi_{(0,1)\times Y^*(x)}\varphi \text{ strongly in } L^p\left((0,1)\times(0,L)\times(0,G_1)\right).$$

Proof. See [4]. $\square$

Proposition 3.9. *Let $\varphi_\varepsilon \in L^p(R^\varepsilon)$ such that*

$$T_\varepsilon^{lp}\varphi_\varepsilon \rightharpoonup \chi_{(0,1)\times Y^*(x)}\varphi \text{ weakly in } L^p\left((0,1)\times(0,L)\times(0,G_1)\right),$$

where $\varphi(x,y_1,y_2)=\varphi(x)$. Then,

$$\frac{L}{\varepsilon}\int_0^{\varepsilon G_\varepsilon(\cdot)}\varphi_\varepsilon(\cdot,y)dy \rightharpoonup |Y^*(\cdot)|\varphi \text{ weakly in } L^p(0,1).$$

Proof. Notice that

$$\frac{1}{L}\int_{(0,1)\times(0,L)\times(0,G_1)} T_\varepsilon^{lp}\varphi_\varepsilon T_\varepsilon^{lp}\psi(x)dxdy_1dy_2 \to \frac{1}{L}\int_{(0,1)\times(0,L)\times(0,G_1)}\varphi(x)\psi(x)\chi_{(0,1)\times Y^*(x)}dxdy_1dy_2,$$

for all $\psi \in L^{p'}(0,1)$. By the Proposition 3.8, we have

$$\begin{aligned}\frac{1}{L}\int_{(0,1)\times(0,L)\times(0,G_1)} T_\varepsilon^{lp}\varphi_\varepsilon T_\varepsilon^{lp}\psi(x)dxdy_1dy_2 &= \frac{1}{\varepsilon}\int_{R^\varepsilon}\varphi_\varepsilon(x,y)\psi(x)dxdy\\ &= \int_0^1\left(\frac{1}{\varepsilon}\int_0^{\varepsilon G_\varepsilon(x)}\varphi_\varepsilon(x,y)dy\right)\psi(x)dx\end{aligned}$$

and

$$\frac{1}{L}\int_{(0,1)\times(0,L)\times(0,G_1)}\varphi(x)\psi(x)\chi_{(0,1)\times Y^*(x)}dxdy_1dy_2 = \frac{1}{L}\int_0^1|Y^*(x)|\varphi(x)\psi(x)dx,$$

for all $\psi \in L^{p'}(0,1)$. Thus,

$$\frac{1}{\varepsilon}\int_0^{\varepsilon G_\varepsilon(x)}\varphi_\varepsilon(x,y)dy \rightharpoonup \frac{1}{L}|Y^*(x)|\varphi(x)$$

weakly in $L^p(0,1)$. $\square$

4 A domain dependence result

In this section we analyze how the solutions of (1.2) depends on the function G_ε. Let us take

$$G_\varepsilon(x) = G\left(x, \frac{x}{\varepsilon^\alpha}\right) \quad \text{and} \quad \hat{G}_\varepsilon(x) = \hat{G}\left(x, \frac{x}{\varepsilon^\alpha}\right)$$

satisfying hypothesis (H) and considering the associated thin domains R^ε and $\hat{R}^\varepsilon$ by

$$\begin{aligned}R^\varepsilon &= \left\{(x,y)\in\mathbb{R}^2 : x\in(0,1), 0<y<\varepsilon G_\varepsilon(x)\right\} \quad \text{and}\\ \hat{R}^\varepsilon &= \left\{(x,y)\in\mathbb{R}^2 : x\in(0,1), 0<y<\varepsilon \hat{G}_\varepsilon(x)\right\}.\end{aligned}$$

Now, let u_ε and $\hat{u}_\varepsilon$ be the solutions of (1.2) for the domains R^ε and $\hat{R}^\varepsilon$ respectively with $f^\varepsilon \in L^{p'}(\mathbb{R}^2)$. We have the following result.

Theorem 4.1. *Let G_ε and $\hat{G}_\varepsilon$ be piecewise C^1 functions satisfying (H) with*

$$\|G_\varepsilon - \hat{G}_\varepsilon\|_{L^\infty(0,1)} \le \delta.$$

Assume also $f^\varepsilon \in L^{p'}(\mathbb{R}^2)$ satisfying $\|f^\varepsilon\|_{L^p(\mathbb{R}^2)} \le 1$.

Then, there exists a positive real function $\rho : [0,\infty) \mapsto [0,\infty)$ such that

$$|||u_\varepsilon - \hat{u}_\varepsilon|||^p_{W^{1,p}(R^\varepsilon \cap \hat{R}^\varepsilon)} + |||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon \setminus \hat{R}^\varepsilon)} + |||\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon \setminus R^\varepsilon)} \le \rho(\delta), \tag{4.1}$$

with $\rho(\delta) \to 0$ as $\delta \to 0$ uniformly for all $\varepsilon > 0$.

Remark 4.1. *The important part of this result is that the function $\rho(\delta)$ does not depend on ε. As we will see, it only depends on the positive constants G_0 and G_1.*

In order to prove Theorem 4.1, we use the fact that u_ε and $\hat{u}_\varepsilon$ are minimizers of the the functionals

$$\begin{aligned} V_\varepsilon(\varphi) &= \frac{1}{p\,\varepsilon}\int_{R^\varepsilon} \left(|\nabla\varphi|^p + |\varphi|^p\right) dxdy - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon \varphi dxdy \\ \hat{V}_\varepsilon(\hat{\varphi}) &= \frac{1}{p\,\varepsilon}\int_{\hat{R}^\varepsilon} \left(|\nabla\hat{\varphi}|^p + |\hat{\varphi}|^p\right) dxdy - \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon \hat{\varphi} dxdy \end{aligned} \tag{4.2}$$

that is,

$$V_\varepsilon(u_\varepsilon) = \min_{\varphi \in W^{1,p}(R^\varepsilon)} V_\varepsilon(\varphi) \quad \text{and} \quad \hat{V}_\varepsilon(\hat{u}_\varepsilon) = \min_{\hat{\varphi} \in W^{1,p}(\hat{R}^\varepsilon)} \hat{V}_\varepsilon(\hat{\varphi}).$$

We will need to evaluate the minimizers plugging them into different functionals. For this, we set the following operators introduced in [3]:

$$\begin{aligned} P_{1+\eta} &: W^{1,p}(U) \mapsto W^{1,p}\left(U(1+\eta)\right) \\ \left(P_{1+\eta}\varphi\right)(x,y) &= \varphi\left(x, \frac{y}{1+\eta}\right), \quad (x,y) \in U(1+\eta), \end{aligned} \tag{4.3}$$

where

$$U(1+\eta) = \left\{(x,(1+\eta)y) \in \mathbb{R}^2 : (x,y) \in U\right\} \tag{4.4}$$

and $U \subset \mathbb{R}^2$ is an arbitrary open set. We also consider the following norm in $W^{1,p}(U)$

$$||w||^p_{W^{1,p}_{1+\eta}(U)} = \frac{1}{1+\eta}\left[||w||^p_{L^p(U)} + ||K_{1+\eta}\nabla w||^p_{L^p(U)}\right] \tag{4.5}$$

where

$$K_{1+\eta} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\eta \end{pmatrix}.$$

We can easily see that

$$||w||^p_{W^{1,p}(U)} = ||P_{1+\eta}w||^p_{W^{1,p}_{1+\eta}(U(1+\eta))} \tag{4.6}$$

and

$$\frac{1}{1+\eta}||w||^p_{W^{1,p}(U)} \le ||w||_{W^{1,p}_{1+\eta}(U)} \le (1+\eta)||w||^p_{W^{1,p}(U)} \quad \text{as } \eta \ge 0.$$

Also, we need the following result about the behavior of the solutions near the oscillating boundary.

Lemma 4.2. *Let u_ε be the solution of problem (1.2) and let $P_{1+\eta}$ be the operator given by (4.3). Then, there exists a positive function $\rho = \rho(p,\eta)$ satisfying $\rho(p,\eta) \to 0$ as $\eta \to 0$, such that*

$$|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon \setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + |||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon)} + |||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \le \rho(p,\eta),$$

for $1 < p < \infty$.

Proof. Since $\eta > 0$, we have that $R^\varepsilon\left(\frac{1}{1+\eta}\right) \subset R^\varepsilon$. Then,

$$\begin{aligned}
V(u_\varepsilon) &= \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + \frac{1}{p}|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}_{1+\eta}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&\geq \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + \frac{1}{p(1+\eta)}|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy.
\end{aligned} \tag{4.7}$$

Now, let us first assume $p \geq 2$. We use the notations of Corollary 3.1.1 to simplify proofs. By Proposition 3.2, (4.2) and (2.1) for $\varphi = P_{1+\eta}u_\varepsilon - u_\varepsilon$, we get

$$\begin{aligned}
|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} &\geq |||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} + \frac{p}{\varepsilon}\int_{R^\varepsilon} [a_p(\nabla u_\varepsilon)\nabla\left(P_{1+\eta}u_\varepsilon - u_\varepsilon\right) \\
&\quad + a_p(u_\varepsilon)\left(P_{1+\eta}u_\varepsilon - u_\varepsilon\right)]\, dxdy + c_p|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \\
&= pV(u_\varepsilon) + \frac{p}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy + \frac{p}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon\left(P_{1+\eta}u_\varepsilon - u_\varepsilon\right)dxdy + c_p|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \\
&= pV(u_\varepsilon) + \frac{p}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon P_{1+\eta}u_\varepsilon dxdy + c_p|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)}.
\end{aligned} \tag{4.8}$$

Putting together (4.7) and (4.8), we obtain

$$\begin{aligned}
V(u_\varepsilon) &\geq \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + \frac{1}{p(1+\eta)}|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&\geq \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + \frac{1}{1+\eta}V(u_\varepsilon) \\
&\quad + \frac{1}{\varepsilon(1+\eta)}\int_{R^\varepsilon} f^\varepsilon P_{1+\eta}u_\varepsilon dxdy + \frac{c_p}{1+\eta}|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy.
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{\eta}{1+\eta}V(u_\varepsilon) &\geq \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} \\
&\quad + \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon\left[\frac{P_{1+\eta}u_\varepsilon}{(1+\eta)} - u_\varepsilon\right]dxdy + \frac{c_p}{1+\eta}|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)}
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + \frac{c_p}{1+\eta}|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \\
&\quad \leq \frac{\eta}{1+\eta}V(u_\varepsilon) + \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon\left[u_\varepsilon - \frac{P_{1+\eta}u_\varepsilon}{(1+\eta)}\right]dxdy.
\end{aligned} \tag{4.9}$$

Now, let us analyze the integral:

$$\frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon\left[u_\varepsilon - \frac{P_{1+\eta}u_\varepsilon}{(1+\eta)}\right]dxdy.$$

To do this, notice that

$$u_\varepsilon(x,y) - (P_{1+\eta}u_\varepsilon)(x,y) = u_\varepsilon(x,y) - u_\varepsilon\left(x, \frac{y}{1+\eta}\right) = \int_{\frac{y}{1+\eta}}^{y} \partial_y u_\varepsilon(x,s)ds,$$

which implies

$$|u_\varepsilon(x,y) - (P_{1+\eta}u_\varepsilon)(x,y)| \leq \left[\int_{\frac{y}{1+\eta}}^{y} |\partial_y u_\varepsilon(x,s)|^p ds\right]^{1/p}\left(\frac{\eta y}{(1+\eta)}\right)^{1/p'}$$

putting the power p, multiplying by $1/\varepsilon$, integrating between 0 and $\varepsilon G_\varepsilon(x)$ and using that $(y/(1+\eta), y) \subset (\varepsilon G_\varepsilon(x))$, we get

$$\frac{1}{\varepsilon}\int_0^{\varepsilon G_\varepsilon(x)} |u_\varepsilon(x,y) - (P_{1+\eta}u_\varepsilon)(x,y)|^p \, dy \le \left[\frac{1}{\varepsilon}\int_0^{\varepsilon G_\varepsilon(x)} |\partial_y u_\varepsilon(x,s)|^p ds\right] \left(\frac{\eta}{1+\eta}\right)^{p-1} \frac{(\varepsilon G_\varepsilon(x))^p}{p}.$$

Thus, we have

$$|||u_\varepsilon - P_{1+\eta}u_\varepsilon|||_{L^p(R^\varepsilon)} \le |||\partial_y u_\varepsilon|||_{L^p(R^\varepsilon)} \left(\frac{\eta}{1+\eta}\right)^{1/p'} \frac{G_1}{p^{1/p}},$$

for $\varepsilon < 1$. Consequently, we get

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon \left[u_\varepsilon - \frac{P_{1+\eta}u_\varepsilon}{(1+\eta)}\right] dxdy\right| \\
&\quad \le \frac{\eta}{\varepsilon(1+\eta)} \int_{R^\varepsilon} |f^\varepsilon u_\varepsilon| dxdy + \frac{1}{\varepsilon(1+\eta)} \int_{R^\varepsilon} |f^\varepsilon u_\varepsilon - f^\varepsilon P_{1+\eta}u_\varepsilon| \, dxdy \\
&\quad \le \frac{\eta}{1+\eta} |||f^\varepsilon|||_{L^{p'}(R^\varepsilon)} |||u_\varepsilon|||_{L^p(R^\varepsilon)} + |||f^\varepsilon|||_{L^{p'}(R^\varepsilon)} |||\partial_y u_\varepsilon|||_{L^p(R^\varepsilon)} \frac{\eta^{1/p'}}{(1+\eta)^{1+1/p'}} \frac{G_1}{p^{1/p}}.
\end{aligned} \tag{4.10}$$

Hence, due Proposition 3.3, (4.9) and (4.10), one gets

$$\begin{aligned}
&\frac{1}{p} |||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon \setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + c_p |||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \\
&\quad \le \frac{\eta}{1+\eta} c + \frac{\eta}{1+\eta} c + \frac{\eta^{1/p'}}{(1+\eta)^{1+1/p'}} c \\
&\quad \le c\eta + c\eta^{1/p'}.
\end{aligned} \tag{4.11}$$

On the other hand, we have

$$\begin{aligned}
V(u_\varepsilon) &= \frac{1}{p} |||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{p} |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}_{1+\eta}(R^\varepsilon(1+\eta))} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{p} |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}_{1+\eta}(R^\varepsilon(1+\eta)\setminus R^\varepsilon)} + \frac{1}{p} |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}_{1+\eta}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&\ge \frac{1}{p(1+\eta)} \left[|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon))} + |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)}\right] - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy.
\end{aligned}$$

Hence, due to (4.8), we get

$$\begin{aligned}
V(u_\varepsilon) &\ge \frac{1}{p(1+\eta)} \left[|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon))} + |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)}\right] - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&\ge \frac{1}{p(1+\eta)} |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon))} + \frac{1}{(1+\eta)} V(u_\varepsilon) + \frac{1}{(1+\eta)\varepsilon}\int_{R^\varepsilon} f^\varepsilon P_{1+\eta}u_\varepsilon dxdy \\
&\qquad + c_p |||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy,
\end{aligned}$$

and then,

$$\begin{aligned}
&\frac{1}{p(1+\eta)} |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon))} + c_p |||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \\
&\quad \le \frac{\eta}{1+\eta} V(u_\varepsilon) + \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon \left(u_\varepsilon - \frac{P_{1+\eta}u_\varepsilon}{(1+\eta)}\right) dxdy.
\end{aligned}$$

Thus, due Proposition 3.3 and (4.10), we get for $p > 2$ that

$$\begin{aligned}
&\frac{1}{p} |||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon)} + c_p |||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \\
&\quad \le c\eta + c\eta^{1/p'}.
\end{aligned} \tag{4.12}$$

Notice that to the case $p > 2$, we have mainly estimated the term $|x-y|^p$. Now, for the case $1 < p < 2$, we have to estimate $(1+|x|+|y|)^{p-2}|x-y|^2$ in view of Propositions 3.1 and 3.2. Indeed, we can argue as in (4.11) and (4.12), to get, for $1 < p < 2$ that

$$\begin{aligned}\frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} &+ \frac{c_p}{\varepsilon}\int_{R^\varepsilon} |\nabla P_{1+\eta}u_\varepsilon - \nabla u_\varepsilon|^2 \left(1+|\nabla P_{1+\eta}u_\varepsilon| + |\nabla u_\varepsilon|\right)^{p-2} dxdy \\ &+ \frac{c_p}{\varepsilon}\int_{R^\varepsilon} |P_{1+\eta}u_\varepsilon - u_\varepsilon|^2 \left(1+|P_{1+\eta}u_\varepsilon| + |u_\varepsilon|\right)^{p-2} dxdy \le c\eta + c\eta^{1/p'}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\frac{1}{p}|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon(1+\eta)\setminus R^\varepsilon)} &+ \frac{c_p}{\varepsilon}\int_{R^\varepsilon} |\nabla P_{1+\eta}u_\varepsilon - \nabla u_\varepsilon|^2 \left(1+|\nabla P_{1+\eta}u_\varepsilon| + |\nabla u_\varepsilon|\right)^{p-2} dxdy \\ &+ \frac{c_p}{\varepsilon}\int_{R^\varepsilon} |P_{1+\eta}u_\varepsilon - u_\varepsilon|^2 \left(1+|P_{1+\eta}u_\varepsilon| + |u_\varepsilon|\right)^{p-2} dxdy \le c\eta + c\eta^{p-1}.\end{aligned}$$

Now, notice that

$$\begin{aligned}|||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)}d \le & \left(\frac{1}{\varepsilon}\int_{R^\varepsilon} |\nabla P_{1+\eta}u_\varepsilon - \nabla u_\varepsilon|^2 \left(1+|\nabla P_{1+\eta}u_\varepsilon| + |\nabla u_\varepsilon|\right)^{p-2} dxdy\right)^{p/2} \\ & \cdot \left[\frac{1}{\varepsilon}\int_{R^\varepsilon} \left(1+|\nabla P_{1+\eta}u_\varepsilon| + |\nabla u_\varepsilon|\right)^{p} dxdy\right]^{(2-p)/2} \\ & + \left(\frac{1}{\varepsilon}\int_{R^\varepsilon} |P_{1+\eta}u_\varepsilon - u_\varepsilon|^2 \left(1+|P_{1+\eta}u_\varepsilon| + |u_\varepsilon|\right)^{p-2} dxdy\right)^{p/2} \\ & \cdot \left[\frac{1}{\varepsilon}\int_{R^\varepsilon} \left(1+|P_{1+\eta}u_\varepsilon| + |u_\varepsilon|\right)^{p} dxdy\right]^{(2-p)/2}.\end{aligned}$$

Finally, putting together the last inequality and (4.13), we also obtain

$$\frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} + |||P_{1+\eta}u_\varepsilon - u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \le c\eta + c\eta^{1/p'} + \left[c\eta + c\eta^{1/p'}\right]^{p/2}$$

for $1 < p < 2$ finishing the proof. □

Now, we are in condition to show Theorem 4.1.

Proof of Theorem 4.1. Taking $\eta = \delta/G_0$, we get under condition $\|G_\varepsilon - \hat{G}_\varepsilon\| \le \delta$ that

$$R^\varepsilon\left(\frac{1}{1+\eta}\right) \subset \hat{R}^\varepsilon \subset R^\varepsilon(1+\eta) \quad \text{and} \quad \hat{R}^\varepsilon\left(\frac{1}{1+\eta}\right) \subset R^\varepsilon \subset \hat{R}^\varepsilon(1+\eta). \tag{4.14}$$

Applying Lemma 4.2, we get

$$\begin{aligned}|||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon\setminus\hat{R}^\varepsilon)} \le |||u_\varepsilon|||^p_{W^{1,p}\left(R^\varepsilon\setminus R^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} \le c\rho(\eta) \quad \text{and} \\ |||u_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon\setminus R^\varepsilon)} \le |||u_\varepsilon|||^p_{W^{1,p}\left(\hat{R}^\varepsilon\setminus \hat{R}^\varepsilon\left(\frac{1}{1+\eta}\right)\right)} \le c\rho(\eta).\end{aligned} \tag{4.15}$$

Now, let us focus to the first term of (4.1). We have

$$\begin{aligned}V_\varepsilon(u_\varepsilon) &\le V_\varepsilon\left(\left(P_{1+\eta}\hat{u}_\varepsilon\right)|_{R^\varepsilon}\right) \\ &= \frac{1}{p}|||\left(P_{1+\eta}\hat{u}_\varepsilon\right)|_{R^\varepsilon}|||_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon \left(P_{1+\eta}\hat{u}_\varepsilon\right)|_{R^\varepsilon} dxdy \\ &\le \frac{1}{p}|||P_{1+\eta}\hat{u}_\varepsilon|||_{W^{1,p}(\hat{R}^\varepsilon(1+\eta))} - \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy + \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon\setminus R^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy.\end{aligned} \tag{4.16}$$

But from the definition of $P_{1+\eta}$ (see (4.3)) and a change of variables, we get

$$|||P_{1+\eta}\hat{u}_\varepsilon|||^p_{W^{1,p}\left(\hat{R}^\varepsilon(1+\eta)\right)} \le (1+\eta)|||\hat{u}_\varepsilon|||^p_{W^{1,p}\left(\hat{R}^\varepsilon\right)}. \tag{4.17}$$

From Lemma 4.2 we get

$$\frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon\left(P_{1+\eta}\hat{u}_\varepsilon-\hat{u}_\varepsilon\right)dxdy \le |||f^\varepsilon|||_{L^{p'}(\hat{R}^\varepsilon)}|||P_{1+\eta}\hat{u}_\varepsilon-\hat{u}_\varepsilon|||_{L^p(\hat{R}^\varepsilon)} \le c\rho(\eta)^{1/p}. \tag{4.18}$$

Also, by (4.14), (4.15) and Lemma 4.2, we obtain

$$\frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon\setminus R^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy \le |||f^\varepsilon|||_{L^{p'}(\hat{R}^\varepsilon)}|||P_{1+\eta}\hat{u}_\varepsilon|||_{L^p(\hat{R}^\varepsilon\setminus R^\varepsilon)} \le c\rho(\eta)^{1/p}. \tag{4.19}$$

Hence, using (4.2), (4.16), (4.17), Proposition 3.3, (4.18), (4.19), we get

$$\begin{aligned}
V_\varepsilon(u_\varepsilon) &\le \frac{(1+\eta)}{p}|||\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)} - \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy + \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon\setminus R^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy \\
&= (1+\eta)\hat{V}_\varepsilon(\hat{u}_\varepsilon) + \frac{(1+\eta)}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon \hat{u}_\varepsilon dxdy - \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy + \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon\setminus R^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy \\
&= (1+\eta)\hat{V}_\varepsilon(\hat{u}_\varepsilon) + \frac{\eta}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon \hat{u}_\varepsilon dxdy + \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon (\hat{u}_\varepsilon - P_{1+\eta}\hat{u}_\varepsilon) dxdy + \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon\setminus R^\varepsilon} f^\varepsilon P_{1+\eta}\hat{u}_\varepsilon dxdy \\
&\le (1+\eta)\hat{V}_\varepsilon(\hat{u}_\varepsilon) + \eta|||f^\varepsilon|||_{L^{p'}(\hat{R}^\varepsilon)}|||\hat{u}_\varepsilon|||_{L^p(\hat{R}^\varepsilon)} + c\rho(\eta)^{1/p} \\
&= (1+\eta)\hat{V}_\varepsilon(\hat{u}_\varepsilon) + \bar{\rho}(\eta),
\end{aligned} \tag{4.20}$$

where $\bar{\rho}$ denotes a function such that $\bar{\rho}(\eta)\to 0$ as $\eta\to 0$.

On the other hand, by (4.2), (4.5), (4.6), (4.14) and Proposition 3.2, we get, for $p\ge 2$,

$$\begin{aligned}
V_\varepsilon(u_\varepsilon) &= \frac{1}{p}|||u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{p}|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}_{1+\eta}(R^\varepsilon(1+\eta))} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&\ge \frac{1}{p(1+\eta)}|||P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)} - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&\ge \frac{1}{p(1+\eta)}\Big[|||\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)} + \frac{p}{\varepsilon}\int_{\hat{R}^\varepsilon}\big(a_p(\nabla\hat{u}_\varepsilon)\nabla(P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon) \\
&\quad + a_p(\hat{u}_\varepsilon)(P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon)\big)dxdy + c_p|||P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)}\Big] - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{p(1+\eta)}\Big[p\hat{V}(\hat{u}_\varepsilon) + \frac{p}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon\hat{u}_\varepsilon dxdy + \frac{p}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon(P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon)dxdy \\
&\quad + c_p|||P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)}\Big] - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy \\
&= \frac{1}{(1+\eta)}\hat{V}(\hat{u}_\varepsilon) + \frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon\frac{1}{1+\eta}P_{1+\eta}u_\varepsilon dxdy - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy + \frac{c_p}{p(1+\eta)}|||P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)}.
\end{aligned} \tag{4.21}$$

Now, due (4.10), a Hölder's inequality and Lemma 4.2, we obtain

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon} f^\varepsilon\frac{1}{1+\eta}P_{1+\eta}u_\varepsilon dxdy - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy\right| \\
&\le \left|\frac{1}{\varepsilon}\int_{\hat{R}^\varepsilon\setminus R^\varepsilon} f^\varepsilon P_{1+\eta}u_\varepsilon dxdy\right| + \left|\frac{1}{\varepsilon}\int_{R^\varepsilon\setminus\hat{R}^\varepsilon} f^\varepsilon P_{1+\eta}u_\varepsilon dxdy\right| \\
&\quad + \left|\frac{1}{(1+\eta)\varepsilon}\int_{R^\varepsilon} f^\varepsilon P_{1+\eta}u_\varepsilon dxdy - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon u_\varepsilon dxdy\right| \le c\rho(\delta)^{1/p}.
\end{aligned} \tag{4.22}$$

First, one can put together (4.20) and (4.21), and then use (4.22) to lead us to

$$\frac{c_p}{p(1+\eta)}|||P_{1+\eta}u_\varepsilon-\hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)} \le \frac{\eta^2+2\eta}{1+\eta}\hat{V}_\varepsilon(\hat{u}_\varepsilon) + \rho(\delta)^{1/p} + \bar{\rho}(\delta),$$

which implies that

$$|||P_{1+\eta}u_\varepsilon - \hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)} \leq \hat{\rho}(\delta), \tag{4.23}$$

for $p \geq 2$, where $\hat{\rho}(\eta)$ is a nonnegative function that tends to zero as $\eta \to 0$.

From Lemma 4.2, we have $|||u_\varepsilon - P_{1+\eta}u_\varepsilon|||^p_{W^{1,p}(R^\varepsilon)} \leq c\rho(\delta)$. It follows from (4.23) that

$$|||u_\varepsilon - \hat{u}_\varepsilon|||^p_{W^{1,p}(R^\varepsilon \cap \hat{R}^\varepsilon)} \leq \tilde{\rho}(\delta),$$

for $p \geq 2$, where $\tilde{\rho}(\eta)$ is a nonnegative function that tends to zero as $\eta \to 0$.

For $1 < p < 2$, we can perform analogous argument to obtain

$$\frac{c_p}{p(1+\eta)} \left[\frac{1}{\varepsilon} \int_{R^\varepsilon} |\nabla P_{1+\eta}u_\varepsilon - \nabla u_\varepsilon|^2 \left(1 + |\nabla P_{1+\eta}u_\varepsilon| + |\nabla u_\varepsilon|\right)^{p-2} dxdy \right.$$
$$\left. + \frac{1}{\varepsilon} \int_{R^\varepsilon} |P_{1+\eta}u_\varepsilon - u_\varepsilon|^2 \left(1 + |P_{1+\eta}u_\varepsilon| + |u_\varepsilon|\right)^{p-2} dxdy \right] \leq \frac{\eta}{1+\eta} \hat{V}_\varepsilon(\hat{u}_\varepsilon) + \rho(\delta)^{1/p}$$

which gives us

$$|||u_\varepsilon - \hat{u}_\varepsilon|||^p_{W^{1,p}(R^\varepsilon \cap \hat{R}^\varepsilon)} \leq \tilde{\rho}(\delta),$$

where $\tilde{\rho}(\eta)$ is a nonnegative function which tends to zero as $\eta \to 0$. □

Remark 4.2. *It follows from* (4.23) *that there exists* $\rho : [0, \infty) \mapsto [0, \infty)$ *such that*

$$|||P_{1+\delta/G_0}u_\varepsilon - \hat{u}_\varepsilon|||^p_{W^{1,p}(\hat{R}^\varepsilon)} \leq \rho(\delta)$$

with $\rho(\delta) \to 0$ *as* $\delta \to 0$ *uniformly in* ε *and any piecewise* C^1 *functions* G_ε *and* $\hat{G}_\varepsilon$ *uniformly bounded with* $\|G_\varepsilon - \hat{G}_\varepsilon\|_{L^\infty(0,1)} \leq \delta$ *and* $f^\varepsilon \in L^{p'}(\mathbb{R}^2)$ *satisfying* $\|f^\varepsilon\|_{L^{p'}(\mathbb{R}^2)} \leq 1$.

5 The piecewise periodic case

Now, we analyze the limit of $\{u_\varepsilon\}_{\varepsilon>0}$ assuming the upper boundary of R^ε is piecewise periodic.

More precisely, we assume G satisfies (H) being independent on the first variable in each interval (ξ_{i-1}, ξ_i). We suppose that G satisfies

$$G(x, y) = G_i(y) \text{ in } x \in I_i = (\xi_{i-1}, \xi_i) \text{ for any } y \in \mathbb{R} \tag{5.1}$$

with $G_i(y+L) = G_i(y)$ for all $y \in \mathbb{R}$. Moreover, we assume the function $G_i(\cdot)$ is C^1 for all $i = 1, \dots, N$ and there exist $0 < G_0 < G_1$ such that $\min_{y\in\mathbb{R}} G_i(y) = G_i^0 \leq G_i(\cdot) \leq G_1$ for all $i = 1, \dots, N$.

Notice that the domain R^ε can now be rewritten as

$$R^\varepsilon = \left(\bigcup_{i=1}^N R_i^\varepsilon \right) \cup \left(\bigcup_{i=1}^{N-1} \{(\xi_i, y) : 0 < y < \varepsilon \min\{G_{i-1}(\xi_i/\varepsilon), G_i(\xi_i/\varepsilon)\}\} \right) \tag{5.2}$$

with

$$R_i^\varepsilon = \{(x, y) \in \mathbb{R} : \xi_{i-1} < x < \xi_i, 0 < y < \varepsilon G_i(x/\varepsilon)\}.$$

See Figure 2 which illustrates this piecewise periodic thin domain.

Before proving the main result of this section, let us recall an important result proved, for instance, in [21]. It is concerned to the purely periodic thin domain situation.

Proposition 5.1. *Assume* G *satisfies the condition* (5.1) *and let* u_ε *be the solution of* (1.2) *with* f^ε *satisfying* $|||f^\varepsilon|||_{L^{p'}(R_i^\varepsilon)} \leq c$ *for some* $c > 0$ *independent of* $\varepsilon > 0$. *Suppose that*

$$\hat{f}^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon G_i(x/\varepsilon^\alpha)} f^\varepsilon(x, y)dy \rightharpoonup \hat{f} \quad \textit{weakly in } L^{p'}(\xi_{i-1}, \xi_i).$$

If $\alpha = 1$, then there exists $(u^i, u_1^i) \in W^{1,p}(\xi_{i-1}, \xi_i) \times L^p((0,1); W^{1,p}_{\#}(Y_i^*))$ such that

$$\begin{cases} \mathcal{T}_\varepsilon^i u_\varepsilon \to u^i \text{ strongly in } L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)), \\ \mathcal{T}_\varepsilon^i(\partial_x u_\varepsilon) \rightharpoonup \partial_x u^i + \partial_{y_1} u_1^i(x, y_1, y_2) \text{ weakly in } L^p\left((\xi_{i-1}, \xi_i) \times Y_i^*\right), \\ \mathcal{T}_\varepsilon^i(\partial_y u_\varepsilon) \rightharpoonup \partial_{y_2} u_1^i(x, y_1, y_2) \text{ weakly in } L^p\left((\xi_{i-1}, \xi_i) \times Y_i^*\right), \end{cases}$$

with

$$\partial_x u^i(x) \nabla_y v^i(y_1, y_2) = (\partial_x u^i(x), 0) + \nabla_y u_1^i(x, y_1, y_2)$$

where $\nabla_y \cdot = (\partial_{y_1} \cdot, \partial_{y_2} \cdot)$ and v^i is the solution of the auxiliar problem

$$\int_{Y_i^*} \left|\nabla v^i\right|^{p-2} \nabla v^i \nabla \varphi dy_1 dy_2 = 0, \quad \forall \varphi \in W^{1,p}_{\#,0}(Y_i^*),$$
$$(v^i - y_1) \in W^{1,p}_{\#,0}(Y_i^*),$$

where $W^{1,p}_{\#,0}(Y_i^*)$ denotes the subspace of $W^{1,p}_{\#}(Y_i^*)$ of functions with zero average.

If $\alpha < 1$, then there exists $(u^i, u_1^i) \in W^{1,p}(\xi_{i-1}, \xi_i) \times L^p\left((\xi_{i-1}, \xi_i); W^{1,p}_{\#}(Y_i^*)\right)$ with $\partial_{y_2} u_1 = 0$ such that

$$\mathcal{T}_\varepsilon^i u_\varepsilon \to u^i \text{ strongly in } L^p\left((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)\right)$$
$$\mathcal{T}_\varepsilon^i \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1^i \text{ weakly in } L^p\left((\xi_{i-1}, \xi_i) \times Y_i^*\right)$$

and

$$\partial_{y_1} u_1^i(x, y_1) = \partial_x u^i \left(\frac{1}{G_i^{p'-1}(y_1) \left\langle 1/G_i^{p'-1} \right\rangle_{(0,L)}} - 1 \right).$$

If $\alpha > 1$, then there exists an unique $u^i \in W^{1,p}(\xi_{i-1}, \xi_i)$ such that

$$\mathcal{T}_\varepsilon^i u_\varepsilon \to u^i \quad \text{strongly in} \quad L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)),$$
$$\Pi_\varepsilon u_\varepsilon^+ \to u^i \quad \text{strongly in} \quad W^{1,p}(R_{i-}),$$
$$\mathcal{T}_\varepsilon^i(|\nabla u_\varepsilon^+|^{p-2} \partial_x u_\varepsilon^+) \rightharpoonup 0 \qquad \text{weakly in} \quad L^p\left((\xi_{i-1}, \xi_i) \times Y_{i+}^*\right),$$

where

$$R_{i-} = \left\{(x, y) \in \mathbb{R}^2 : x \in (\xi_{i-1}, \xi_i), 0 < y < G_0\right\},$$
$$Y_{i+}^* = \left\{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L,\, G_0 < y_2 < G_i(y_1)\right\},$$

and the scaling operator $\Pi_\varepsilon : L^p((\xi_{i-1}, \xi_i) \times (0, \varepsilon G_0)) \to L^p(R_{i-})$ is defined by

$$\Pi_\varepsilon(\varphi)(x, y) = \varphi(x, \varepsilon y) \; \forall (x, y) \in R_{i-}.$$

Also, we denote by $u_\varepsilon^+ = u_\varepsilon|_{R_i^\varepsilon \setminus \{(\xi_{i-1}, \xi_i) \times (0, \varepsilon G_0)\}}$ and $u_\varepsilon^- = u_\varepsilon|_{(\xi_{i-1}, \xi_i) \times (0, \varepsilon G_0)}$.

Proof. It follows from [21, Theorems 3.1, 4.1 and 5.3]. □

Remark 5.1. *We point out that the results in [21] are proved in the unit interval. Here, we just rewrite it to (ξ_{i-1}, ξ_i). The limit problems are stated in the next result.*

Now, we are in condition to show the following result.

Theorem 5.2. *Suppose G satisfies the assumption (5.1) and let u_ε be the solution of problem (1.2) with $f^\varepsilon \in L^{p'}(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^{p'}(R^\varepsilon)} \le c$, for some $c > 0$ independent of $\varepsilon > 0$. Suppose the function*

$$\hat{f}^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon G\left(x, \frac{x}{\varepsilon^\alpha}\right)} f(x, y) dy$$

satisfies $\hat{f}^\varepsilon \rightharpoonup \hat{f}$ *weakly in* $L^{p'}(0,1)$.

Then, if $\alpha = 1$, *there exist* $u \in W^{1,p}(0,1)$ *and* $u_1^i \in L^p((\xi_{i-1},\xi_i); W^{1,p}_\#(Y_i^*))$ *such that*

$$\begin{cases} \mathcal{T}_\varepsilon^i u_\varepsilon \to u \text{ strongly in } L^p((\xi_{i-1},\xi_i); W^{1,p}(Y_i^*)), \\ \mathcal{T}_\varepsilon^i(\partial_x u_\varepsilon) \rightharpoonup \partial_x u + \partial_{y_1} u_1^i(x,y_1,y_2) \text{ weakly in } L^p\left((\xi_{i-1},\xi_i); W^{1,p}(Y_i^*)\right), \\ \mathcal{T}_\varepsilon^i(\partial_y u_\varepsilon) \rightharpoonup \partial_{y_2} u_1^i(x,y_1,y_2) \text{ weakly in } L^p\left((\xi_{i-1},\xi_i); W^{1,p}(Y_i^*)\right) \end{cases}$$

and u *is the unique solution of the problem*

$$\int_0^1 \left\{ q(x)|u'|^{p-2}u'\varphi' + r(x)\,|u|^{p-2}u\varphi \right\} dx = \int_0^1 \hat{f}\varphi dx, \quad \varphi \in W^{1,p}(0,1), \tag{5.4}$$

where $q,\ r : (0,1) \to \mathbb{R}$ *are piecewise constant functions such that*

$$q(x) = q_i \text{ and } r(x) = r_i \quad \text{for } x \in (\xi_{i-1},\xi_i)$$

with the homogenized constants r_i *and* q_i *given by*

$$q_i = \frac{1}{L}\int_{Y_i^*} |\nabla v^i|^{p-2}\partial_{y_1} v^i \, dy_1 dy_2 \quad \text{and} \quad r_i = \frac{|Y_i^*|}{L} \tag{5.5}$$

where Y_i^* *is the basic cell associated to* R_i^ε

$$Y_i^* = \{(y_1,y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < G_i(y_1)\}$$

and v^i *is the solution of the auxiliary problem*

$$\int_{Y_i^*} \left|\nabla v^i\right|^{p-2} \nabla v^i \nabla \psi dy_1 dy_2 = 0, \quad \forall \psi \in W^{1,p}_\#(Y_i^*), \quad \langle \psi \rangle_{Y_i^*} = 0$$

$$(v^i - y_1) \in W^{1,p}_\#(Y_i^*), \quad \langle v - y_1 \rangle_{Y_i^*} = 0.$$

If $\alpha < 1$, *then there exists* $(u, u_1^i) \in W^{1,p}(0,1) \times L^p\left((\xi_{i-1},\xi_i); W^{1,p}_\#(Y_i^*)\right)$ *with* $\partial_{y_2} u_1^i = 0$ *such that*

$$\mathcal{T}_\varepsilon^i u_\varepsilon \to u \text{ strongly in } L^p\left((\xi_{i-1},\xi_i); W^{1,p}(Y_i^*)\right),$$
$$\mathcal{T}_\varepsilon^i \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1^i \text{ weakly in } L^p\left((\xi_{i-1},\xi_i) \times Y_i^*\right).$$

Also, u *is the unique solution of the problem* (5.4) *with*

$$q(x) = q_i \quad \text{and} \quad r(x) = r_i \quad \text{for } x \in (\xi_{i-1},\xi_i) \tag{5.7}$$

where

$$q_i = \frac{1}{L\left\langle 1/G_i^{p'-1}\right\rangle_{(0,L)}^{p-1}} \quad \text{and} \quad r_i = \frac{|Y_i^*|}{L}.$$

If $\alpha > 1$, *then there exists a unique* $u \in W^{1,p}(0,1)$ *such that*

$$\mathcal{T}_\varepsilon^i u_\varepsilon \to u \quad \text{strongly in} \quad L^p((\xi_{i-1},\xi_i); W^{1,p}(Y_i^*)),$$
$$\Pi_\varepsilon u_\varepsilon^+ \to u \quad \text{strongly in} \quad W^{1,p}(R_{i-}),$$
$$\mathcal{T}_\varepsilon^i(|\nabla u_\varepsilon^+|^{p-2}\partial_x u_\varepsilon^+) \rightharpoonup 0 \qquad \text{weakly in} \quad L^p\left((\xi_{i-1},\xi_i) \times Y_{i+}^*\right).$$

Furthermore, u *is the unique solution of the problem* (5.4) *with*

$$G_0(x) = G_i^0 \quad \text{for } x \in (\xi_{i-1},\xi_i) \quad \text{and} \quad r(x) = \frac{|Y_i^*|}{L}.$$

Proof. By (5.2), we can rewrite (2.1) taking into account the partition $\{\xi_i\}_{i=1}^N$ getting

$$\sum_{i=1}^{N}\int_{R_i^\varepsilon}\left\{|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi+|u_\varepsilon|^{p-2}u_\varepsilon\varphi\right\}dxdy=\int_{R^\varepsilon}f^\varepsilon\varphi\,dxdy,\qquad \varphi\in W^{1,p}(R^\varepsilon). \tag{5.8}$$

Hence, we obtain from (5.8) (with test functions $\varphi(x,y)=\varphi(x)\in W^{1,p}(0,1)$) and Proposition 3.5 that

$$\sum_{i=1}^{N}\int_{R_i^\varepsilon}\Bigg[\int_{(\xi_{i-1},\xi_i)\times Y_i^*}\mathcal{T}_\varepsilon^i\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)\mathcal{T}_\varepsilon^i\nabla\varphi dxdy_1dy_2+\frac{L}{\varepsilon}\int_{R_{1i}^\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi dxdy$$
$$+\int_{(\xi_{i-1},\xi_i)\times Y_i^*}\mathcal{T}_\varepsilon^i\left(|u_\varepsilon|^{p-2}u_\varepsilon\right)\mathcal{T}_\varepsilon^i\varphi dxdy_1dy_2+\frac{L}{\varepsilon}\int_{R_{1i}^\varepsilon}|u_\varepsilon|^{p-2}u_\varepsilon\varphi dxdy\Bigg]=\frac{L}{\varepsilon}\int_{R^\varepsilon}f^\varepsilon\varphi dxdy.$$

By Proposition 5.1, we can pass to the limit in each subinterval (ξ_{i-1},ξ_i). If we assume $\alpha\le 1$, we obtain

$$\sum_{i=1}^{N}\int_{\xi_{i-1}}^{\xi_i}\int_{Y_i^*}\left[(|\nabla_y v^i|^{p-2}\partial_{y_1}v^i)|\partial_x u^i|^{p-2}\partial_x u^i\partial_x\varphi+\left|u^i\right|^{p-2}u^i\varphi\right]dxdY=L\int_0^1\hat f\varphi dx$$

which is equivalent to

$$\sum_{i=1}^{N}\int_{\xi_{i-1}}^{\xi_i}\left\{\left[\int_{Y_i^*}|\nabla_y v^i|^{p-2}\partial_{y_1}v^i dY\right]|\partial_x u^i|^{p-2}\partial_x u^i\partial_x\varphi+|Y_i^*|\left|u^i\right|^{p-2}u^i\varphi\right\}dx=L\int_0^1\hat f\varphi dx \tag{5.9}$$

for all $\varphi\in W^{1,p}(0,1)$.

For $\alpha<1$, Proposition 5.1 guarantees

$$\sum_{i=1}^{N}\int_{\xi_{i-1}}^{\xi_i}\int_{Y_i^*}\left|\frac{\partial_x u^i}{G_i^{p'-1}(y_1)\left\langle 1/G_i^{p'-1}\right\rangle_{(0,L)}}\right|^{p-2}\left(\frac{\partial_x u^i}{G_i^{p'-1}(y_1)\left\langle 1/G_i^{p'-1}\right\rangle_{(0,L)}}\right)\partial_x\varphi\,dx$$
$$+\sum_{i=1}^{N}\int_{\xi_{i-1}}^{\xi_i}\int_{Y_i^*}\left|u^i\right|^{p-2}u^i\varphi dx=L\int_0^1\hat f\varphi dx. \tag{5.10}$$

Since $(p'-1)(p-1)=1$, (5.10) can be rewritten as

$$\sum_{i=1}^{N}\int_{\xi_{i-1}}^{\xi_i}\left[\int_0^L\frac{1}{G_i(y_1)\left\langle 1/G_i^{p'-1}\right\rangle_{(0,L)}^{p-1}}G_i(y_1)dy_1\right]|\partial_x u^i|^{p-2}\partial_x u^i\partial_x\varphi\,dx$$
$$+\sum_{i=1}^{N}\int_{\xi_{i-1}}^{\xi_i}|Y_i^*|\left|u^i\right|^{p-2}u^i\varphi dx=L\int_0^1\hat f\varphi dx. \tag{5.11}$$

Hence, for any $\alpha\le 1$, it follows from (5.5), (5.7), (5.9) and (5.11) that

$$\int_0^1\left[q(x)|\partial_x u|^{p-2}\partial_x u\partial_x\varphi+r(x)|u|^{p-2}u\varphi\right]dx=\int_0^1\hat f\varphi dx,\qquad\forall\varphi\in W^{1,p}(0,1), \tag{5.12}$$

with

$$u(x)=u^i(x)\text{ a.e. in }(\xi_{i-1},\xi_i)$$

where the functions u^i are given by Proposition 5.1. Notice that $q_i>0$ for each i. Indeed, by (5.6), we can take $(v^i-y_1)\in W^{1,p}_{\#,0}(Y_i^*)$ as a test function in such way that

$$q_i=\frac{1}{L}\int_{Y_i^*}|\nabla v^i|^{p-2}\nabla v^i\left((1,0)+\nabla v^i-(1,0)\right)dy_1dy_2=\frac{1}{L}\int_{Y_i^*}|\nabla v^i|^p dy_1dy_2>0.$$

Consequently, we obtain from the Minty-Browder's Theorem that the problem (5.12) has a unique solution in $W^{1,p}(0,1)$, and then, we can conclude that $u \in W^{1,p}(0,1)$ proving the theorem for $\alpha \leq 1$.

Now, let us assume $\alpha > 1$. Then, from (5.8) and Proposition 3.5, we obtain that

$$\sum_{i=1}^{N} \left[\frac{1}{L} \int_{(\xi_{i-1},\xi_i)\times Y^*_{i+}} \mathcal{T}^i_\varepsilon \left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \right) \mathcal{T}^i_\varepsilon \nabla \varphi_i dx dy_1 dy_2 + \frac{1}{\varepsilon} \int_{R^\varepsilon_{1i+}} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \varphi_i dx dy \right.$$
$$+ \int_{R_{i-}} |\Pi_\varepsilon \nabla u_\varepsilon|^{p-2} \Pi_\varepsilon \nabla u_\varepsilon \Pi_\varepsilon \nabla \varphi \, dx dy$$
$$\left. + \frac{1}{L} \int_{(\xi_{i-1},\xi_i)\times Y^*_i} \mathcal{T}^i_\varepsilon \left(|u_\varepsilon|^{p-2} u_\varepsilon \right) \mathcal{T}^i_\varepsilon \varphi_i dx dy_1 dy_2 + \frac{1}{\varepsilon} \int_{R^\varepsilon_{1i}} |u_\varepsilon|^{p-2} u_\varepsilon \varphi_i dx dy \right] = \frac{1}{\varepsilon} \int_{R^\varepsilon} f^\varepsilon \varphi_i dx dy$$

where Π_ε is the scaling operator introduced in Proposition 5.1. Hence, by Proposition 5.1, we can pass to the limit taking test functions $\varphi(x,y) = \varphi(x) \in W^{1,p}(0,1)$. We obtain

$$\sum_{i=1}^{N} \left[\int_{R_{i-}} |\partial_x u|^{p-2} \partial_x u \partial_x \varphi dx + \frac{1}{L} \int_{(\xi_{i-1},\xi_i)\times Y^*_i} |u|^{p-2} u \varphi dx \right] = \int_0^1 \hat{f} \varphi dx$$

for all $\varphi \in W^{1,p}(0,1)$ with

$$u(x) = u^i(x) \text{ a.e. in } (\xi_{i-1}, \xi_i)$$

where the functions u^i are given by Proposition 5.1. Thus,

$$\int_0^1 \left[G_0(x) |\partial_x u|^{p-2} \partial_x u \partial_x \varphi + r(x) |u|^{p-2} u \varphi \right] dx = \int_0^1 \hat{f} \varphi dx, \quad \forall \varphi \in W^{1,p}(0,1). \tag{5.13}$$

As $G_0 > 0$, it follows from Minty-Browder's Theorem that (5.13) is well posed. Hence, we get that $u \in W^{1,p}(0,1)$ is the unique solution concluding the proof of the theorem. □

6 The locally periodic case

In this section, we provide the proof of our main result, Theorem 2.1.

Proof of Theorem 2.1. Using Proposition 3.3 and Theorem 3.7, there is $u_0 \in W^{1,p}(0,1)$ such that, up to subsequences,

$$T^{lp}_\varepsilon u_\varepsilon \rightharpoonup \chi u_0 \text{ weakly in } L^p\left((0,1)\times(0,L)\times(0,G_1)\right), \tag{6.1}$$

where χ is the characteristic function of $(0,1) \times Y^*(x)$.

We show that u_0 satisfies the Neumann problem (5.4). To do this, we use a kind of discretization argument on the oscillating thin domains. We first proceed as in [3, Theorem 2.3] fixing a parameter $\delta > 0$ in order to set a piecewise periodic function $G^\delta(x,y)$ satisfying (5.1) and $0 \leq G^\delta(x,y) - G(x,y) \leq \delta$ in $(0,1)\times\mathbb{R}$.

Let us construct this function. Recall that G is uniformly C^1 in each of the domains $(\xi_{i-1},\xi_i)\times\mathbb{R}$. Also, it is periodic in the second variable. In particular, for $\delta > 0$ small enough and for a fixed $z \in (\xi_{i-1},\xi_i)$ we have that there exists a small interval $(z-\eta, z+\eta)$ with η depending only on δ such that $|G(x,y) - G(z,y)| + |\partial_y G(x,y) - \partial_y G(z,y)| < \delta/2$ for all $x \in (z-\eta, z+\eta) \cap (\xi_{i-1},\xi_i)$ and for all $y \in \mathbb{R}$. This allows us to select a finite number of points: $\xi_{i-1} = \xi^1_{i-1} < \xi^2_{i-1} < \cdots < \xi^r_{i-1} = \xi_i$ with $\xi^r_{i-1} - \xi^{r-1}_{i-1} < \eta$ in such way that $G^\delta(x,y) = G(\xi^r_{i-1}, y) + \delta/2$ defined for $x \in (\xi^r_{i-1}, \xi^{r+1}_{i-1})$ and $y \in \mathbb{R}$ satisfies $|\partial_y G^\delta(x,y) - \partial_y G(z,y)| \leq \delta$ in $(\xi^r_{i-1}, \xi^{r+1}_{i-1})\times\mathbb{R}$. Notice that this construction can be done for all $i = 1, \ldots, N$. In particular, if we rename all the constructed points ξ^k_i by $0 = z_0 < z_1 < \cdots < z_m = 1$, for some $m = m(\delta)$, we get that $G^\delta(x,y) = G^\delta_i(y)$ for $(x,y) \in (z_{i-1}, z_i)\times\mathbb{R}$ and $i = 1, \ldots, m$ is a piecewise C^1-function which is L-periodic in the second variable y.

Finally, we set $G^\delta_\varepsilon(x) = G^\delta(x, x/\varepsilon^\alpha)$, for any $\alpha > 0$, considering the following domains

$$R^{\varepsilon,\delta} = \{(x,y) : x \in (0,1), 0 < y < \varepsilon G^\delta_\varepsilon(x)\}.$$

In such domains, if we assume $\alpha = 1$, we obtain from Theorem 5.2 that, for each $\delta > 0$ fixed, there exist $u^\delta \in W^{1,p}(0,1)$ and $u_1^{i,\delta} \in L^p((\xi_{i-1}, \xi_i); W^{1,p}_\#(Y_i^*))$ such that the solutions $u_{\varepsilon,\delta}$ of (1.2) in $R^{\varepsilon,\delta}$ satisfy

$$\begin{cases} \mathcal{T}_\varepsilon^\delta u_{\varepsilon,\delta} \to u^\delta \text{ strongly in } L^p((z_{i-1}, z_i); W^{1,p}(Y_i^*)), \\ \mathcal{T}_\varepsilon^\delta (\partial_x u_{\varepsilon,\delta}) \rightharpoonup \partial_x u^\delta + \partial_{y_1} u_1^{i,\delta}(x, y_1, y_2) \text{ weakly in } L^p\left((z_{i-1}, z_i); W^{1,p}(Y_i^*)\right), \\ \mathcal{T}_\varepsilon^\delta (\partial_y u_{\varepsilon,\delta}) \rightharpoonup \partial_{y_2} u_1^{i,\delta}(x, y_1, y_2) \text{ weakly in } L^p\left((z_{i-1}, z_i); W^{1,p}(Y_i^*)\right), \\ \mathcal{T}_\varepsilon^\delta(|\nabla u_{\varepsilon,\delta}|^{p-2} \nabla u_{\varepsilon,\delta}) \rightharpoonup q^\delta a_p(\partial_x u^\delta) \text{ weakly in } L^p\left((z_{i-1}, z_i) \times Y_i^*\right)^2. \end{cases} \tag{6.2}$$

On the other side, if we assume $\alpha < 1$, we get that, for each $\delta > 0$ fixed, there exist $u^\delta \in W^{1,p}(0,1)$ and $u_1^{i,\delta} \in L^p((\xi_{i-1}, \xi_i); W^{1,p}_\#(Y_i^*))$ with $\partial_{y_2} u_1^{i,\delta} = 0$ in such way that the solutions $u_{\varepsilon,\delta}$ of (1.2) in $R^{\varepsilon,\delta}$ satisfy

$$\begin{cases} \mathcal{T}_\varepsilon^\delta u_{\varepsilon,\delta} \to u^\delta \text{ strongly in } L^p((z_{i-1}, z_i); W^{1,p}(Y_i^*)), \\ \mathcal{T}_\varepsilon^\delta (\partial_x u_{\varepsilon,\delta}) \rightharpoonup \partial_x u^\delta + \partial_{y_1} u_1^{i,\delta}(x, y_1, y_2) \text{ weakly in } L^p\left((z_{i-1}, z_i); W^{1,p}(Y_i^*)\right), \\ \mathcal{T}_\varepsilon^\delta(|\nabla u_{\varepsilon,\delta}|^{p-2} \nabla u_{\varepsilon,\delta}) \rightharpoonup q^\delta a_p(\partial_x u^\delta) \text{ weakly in } L^p\left((z_{i-1}, z_i) \times Y_i^*\right)^2. \end{cases}$$

Finally, if we take $\alpha > 1$, we have that

$$\begin{gathered} \mathcal{T}_\varepsilon^\delta u_{\varepsilon,\delta} \rightharpoonup u^\delta \quad \text{weakly in} \quad L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)), \\ \Pi_\varepsilon^\delta u_{\varepsilon,\delta}^+ \to u^\delta \quad \text{strongly in} \quad W^{1,p}(R_{i-}), \\ \mathcal{T}_\varepsilon^\delta(|\nabla u_{\varepsilon,\delta}^+|^{p-2} \partial_x u_{\varepsilon,\delta}^+) \rightharpoonup 0 \qquad \text{weakly in} \quad L^p\left((\xi_{i-1}, \xi_i) \times Y_{i+}^*\right). \end{gathered}$$

Furthermore, we have that u^δ is the unique solution of the Neumann problem

$$\int_0^1 \left\{ q^\delta(x) |(u^\delta)'|^{p-2} (u^\delta)' \varphi' + r^\delta(x)\, |u^\delta|^{p-2} u^\delta \varphi \right\} dx = \int_0^1 \hat{f} \varphi dx, \quad \forall \varphi \in W^{1,p}(0,1), \tag{6.3}$$

with

$$\begin{gathered} q^\delta(x) = \frac{1}{L} \sum_{i=1}^{N-1} \chi_{I_i}(x) \begin{cases} \displaystyle\int_{Y_i^*} |\nabla v^i|^{p-2} \partial_{y_1} v^i \, dy_1 dy_2 & \text{if} \quad \alpha = 1, \\ \dfrac{1}{\left\langle 1/G_i^{p'-1} \right\rangle_{(0,L)}^{p-1}} & \text{if} \quad \alpha < 1, \end{cases} \\ q^\delta(x) = \sum_{i=1}^{N-1} \chi_{I_i}(x) G_i^0 \quad \text{and} \quad r^\delta(x) = \sum_{i=1}^{N-1} \chi_{I_i}(x) \frac{|Y_i^*|}{L} \quad \text{if} \quad \alpha > 1. \end{gathered} \tag{6.4}$$

χ_{I_i} is the characteristic function of (ξ_{i-1}, ξ_i) and v^i is the solution of (5.6) in Y_i^* which is given by

$$Y_i^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < G_i(y_1)\}.$$

Now, we pass to the limit in (6.3) as $\delta \to 0$. From Lemmas A.1 and A.2, we have the uniform convergence of q^δ and r^δ to q and r where

$$q(x) = \begin{cases} \dfrac{1}{L} \displaystyle\int_{Y^*(x)} |\nabla v|^{p-2} \partial_{y_1} v \, dy_1 dy_2 & \text{if} \quad \alpha = 1, \\ \dfrac{1}{L} \dfrac{1}{\langle 1/G(x, \cdot)^{p'-1} \rangle_{(0,L)}^{p-1}} & \text{if} \quad \alpha < 1, \\ G_0(x) = \min_{y \in \mathbb{R}} G(x, y) & \text{if} \quad \alpha > 1 \end{cases} \qquad \text{and} \qquad r(x) = \frac{|Y^*(x)|}{L}. \tag{6.5}$$

Notice that $q(x) > 0$. Furthermore, we have that the solutions $u^\delta \in W^{1,p}(0,1)$ of (6.3) are uniformly bounded in δ. Thus, there exists $u^* \in W^{1,p}(0,1)$ such that $u^\delta \rightharpoonup u^*$ weakly in $W^{1,p}(0,1)$ and strongly in $L^p(0,1)$. Indeed, we have the strong convergence

$$u^\delta \to u^* \text{ in } W^{1,p}(0,1). \tag{6.6}$$

To prove this, we set the following norm

$$\| \cdot \|_{L^p_\delta(0,1)}^p = \int_0^1 q^\delta |\cdot|^p dx.$$

By Proposition 3.1 and equation (6.3), we get for $\varphi = u^\delta - u^*$ and $p \geq 2$ that

$$
\begin{aligned}
\|(u^\delta)' - (u^*)'\|^p_{L^p_\delta(0,1)} &\leq c \int_0^1 q^\delta \left[a_p \left((u^\delta)' \right) - a_p \left((u^*)' \right) \right] \left[(u^\delta)' - (u^*)' \right] dx \\
&= c \int_0^1 (\hat{f} - a_p(u^\delta))(u^\delta - u^*) dx - c \int_0^1 q^\delta a_p \left((u^*)' \right) \left[(u^\delta)' - (u^*)' \right] dx \\
&\to 0.
\end{aligned}
$$

Hence, using the equivalence of norms, we get

$$
\|(u^\delta)' - (u^*)'\|_{L^p(0,1)} \leq \|(u^\delta)' - (u^*)'\|_{L^p_\delta(0,1)} \to 0,
$$

as $\delta \to 0$, which implies (6.6). Thus, we have that $u^* \in W^{1,p}(0,1)$ satisfies

$$
\int_0^1 \left\{ q(x) |(u^*)'|^{p-2} (u^*)' \varphi' + r(x) \, |u^*|^{p-2} u^* \varphi \right\} dx = \int_0^1 \hat{f} \varphi dx, \tag{6.7}
$$

for all $\varphi \in W^{1,p}(0,1)$ and $p \geq 2$. For $1 < p < 2$, one can show using similar arguments.

Now, let us see that $u^* = u_0$ in $(0,1)$ where u_0 is given by (6.1). Let η be a positive small number and let $\varphi \in C_0^\infty(0,1)$. Notice that

$$
\begin{aligned}
\int_0^1 (u_0 - u^*) \varphi dx = &\int_0^1 \left(u_0 - \frac{L}{|Y^*(x)| \varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} u_\varepsilon(x,y) dy \right) \varphi(x) dx \\
&+ \int_0^1 \left(\frac{L}{|Y^*(x)| \varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} u_\varepsilon(x,y) - P_{1+\delta/G_0} u_{\varepsilon,\delta}(x,y) dy \right) \varphi(x) dx \\
&+ \int_0^1 \left(\frac{L}{|Y^*(x)| \varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} P_{1+\delta/G_0} u_{\varepsilon,\delta}(x,y) - u^\delta(x) dy \right) \varphi(x) dx \\
&+ \int_0^1 \left(\frac{L}{|Y^*(x)| \varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} u^\delta(x) - u^*(x) dy \right) \varphi(x) dx,
\end{aligned} \tag{6.8}
$$

where $P_{1+\delta/G_0}$ is the operator defined in (4.3).

Now, due to definition (4.3), notation (4.4) and an appropriated change of variables, we get

$$
\begin{aligned}
\int_0^1 \left(\frac{L}{\varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} P_{1+\delta/G_0} u_{\varepsilon,\delta}(x,y) - u^\delta(x) dy \right) \varphi(x) dx &\leq c |||P_{1+\delta/G_0} u_{\varepsilon,\delta} - u^\delta|||_{L^p(R^\varepsilon)} \\
&\leq c |||P_{1+\delta/G_0} u_{\varepsilon,\delta} - u^\delta|||_{L^p(R^{\varepsilon,\delta}(1+\delta))} = c |||u_{\varepsilon,\delta} - u^\delta|||_{L^p(R^{\varepsilon,\delta})}
\end{aligned}
$$

for some $c > 0$ independent of δ and $\varepsilon > 0$. Thus, we can rewrite (6.8) as

$$
\begin{aligned}
\left| \int_0^1 (u_0 - u^*) \varphi dx \right| \leq &\left| \int_0^1 \left(u_0 - \frac{L}{\varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} u_\varepsilon(x,y) dy \right) \varphi(x) dx \right| \\
&+ c |||u_\varepsilon - P_{1+\delta/G_0} u_{\varepsilon,\delta}|||_{L^p(R^\varepsilon)} + c |||u_{\varepsilon,\delta} - u^\delta|||_{L^p(R^{\varepsilon,\delta})} + c \|u^\delta - u^*\|_{L^p(0,1)}.
\end{aligned}
$$

From (6.2) and Remark 4.2, we can take $\delta > 0$ small enough such that

$$
|||u_\varepsilon - P_{1+\delta/G_0} u_{\varepsilon,\delta}|||_{L^p(R^\varepsilon)} \leq \eta \quad \text{and} \quad |||u_{\varepsilon,\delta} - u^\delta|||_{L^p(R^{\varepsilon,\delta})} \leq \eta \tag{6.9}
$$

uniformly in $\varepsilon > 0$. Also, from (6.6), we can choose $\varepsilon_1 > 0$ such that $|||u^* - u^\delta|||_{L^p(0,1)} \leq \eta$ for $0 < \varepsilon < \varepsilon_1$.

Moreover, from (6.1) and Proposition 3.9, we have

$$
\int_0^1 \left(u_0 - \frac{L}{|Y^*(x)| \varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} u_\varepsilon(x,y) dy \right) \varphi(x) dx \to 0, \quad \text{as} \quad \varepsilon \to 0.
$$

Therefore, there exists $\varepsilon_2 > 0$ such that

$$\left| \int_0^1 \left(u_0 - \frac{L}{|Y^*(x)|\varepsilon} \int_0^{\varepsilon G_\varepsilon(x)} u_\varepsilon(x,y)dy \right) \varphi(x)dx \right| \le \eta$$

whenever $0 < \varepsilon < \varepsilon_2$. Hence, setting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ we get

$$\left| \int_0^1 (u_0 - u^*)\varphi dx \right| \le 4\eta.$$

Since φ and η are arbitrary, we conclude that $u^* = u_0$.

Finally, let us see that the convergence

$$|||u_\varepsilon - u_0|||_{L^p(R^\varepsilon)} \to 0 \tag{6.10}$$

holds. Notice that

$$|||u_\varepsilon - u_0|||_{L^p(R^\varepsilon)} \le |||u_\varepsilon - P_{1+\delta/G_0} u_{\varepsilon,\delta}|||_{L^p(R^\varepsilon)} + |||P_{1+\delta/G_0} u_{\varepsilon,\delta} - u^\delta|||_{L^p(R^\varepsilon)} + |||u^\delta - u_0|||_{L^p(R^\varepsilon)}.$$

Hence, we can argue as in (6.9) getting (6.10) from (6.2), Remark 4.2 and (6.6). And then, we conclude the proof of the theorem. □

7 Convergence of the Resolvent and semigroups

In this Section, we show the convergence of the resolvent and semigroup associated to the p-Laplacian operator given by the equation (1.2) under the additional condition $p \ge 2$. For that, let us first consider the operator $M_\varepsilon : L^p(R^\varepsilon) \mapsto L^p(0,1)$ given by

$$M_\varepsilon f^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon G(x, x/\varepsilon^\alpha)} f^\varepsilon(x,y)dy.$$

Next, let $A_\varepsilon : W^{1,p}(R^\varepsilon) \to (W^{1,p}(R^\varepsilon))'$ and $A_0 : W^{1,p}(0,1) \to (W^{1,p}(0,1))'$ be given by

$$\begin{aligned} \langle A_\varepsilon u, v\rangle_\varepsilon &= \frac{1}{\varepsilon} \int_{R^\varepsilon} \left\{ |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv \right\} dxdy \\ \langle A_0 u, v\rangle_0 &= \int_0^1 \left\{ q(x) |\partial_x u|^{p-2} \partial_x u \partial_x v + r(x) |u|^{p-2} uv \right\} dx. \end{aligned} \tag{7.1}$$

We consider the L^2-realization of A_ε and A_0, that is,

$$\begin{gathered} D(A_{\varepsilon,2}) = \left\{ u \in W^{1,p}(R^\varepsilon) : A_\varepsilon u \in L^2(R^\varepsilon) \right\}), \\ A_{\varepsilon,2} u = A_\varepsilon u, \quad \forall u \in D(A_{\varepsilon,2}), \quad \text{ and} \\ D(A_{0,2}) = \left\{ u \in W^{1,p}(0,1) : A_0 u \in L^2(0,1) \right\}), \\ A_{0,2} u = A_0 u, \quad \forall u \in D(A_{0,2}). \end{gathered}$$

Then, for any $p \ge 2$, $\lambda > 0$ and forcing terms $f^\varepsilon \in L^2(R^\varepsilon)$, we can consider the following problems

$$(I + \lambda A_\varepsilon) u_\varepsilon = f^\varepsilon \tag{7.2}$$

and

$$(I + \lambda A_0) u = \hat{f} \tag{7.3}$$

which are well posed (existence and uniqueness of solutions) by the Minty-Browder's Theorem. Notice that here, we are using the dual products $\langle\cdot,\cdot\rangle_\varepsilon$ and $\langle\cdot,\cdot\rangle_0$ from $W^{1,p}(R^\varepsilon)$ and $W^{1,p}(0,1)$ respectively to set the equations (7.2) and (7.3).

Hence, with the additional conditions $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded and $M_\varepsilon f^\varepsilon \rightharpoonup \hat{f}$ weakly in $L^2(0,1)$, it follows from Theorem 2.1 that the family of solutions defined by (7.2) converges to the solution of (7.3) as $\varepsilon \to 0$. Consequently, we obtain the convergence of the resolvent operators defined by the equation (1.2). In fact, we have for any $\lambda > 0$ that

$$|||(I+\lambda A_\varepsilon)^{-1}f^\varepsilon - (I+\lambda A_0)^{-1}\hat{f}|||_{L^2(R^\varepsilon)} = |||u_\varepsilon - u|||_{L^2(R^\varepsilon)} \to 0 \quad \text{ as } \varepsilon \to 0.$$

In the next, let us obtain the convergence of the semigroup associated to the equations (7.2) and (7.3). As we will see, it is a consequence of [8, Theorem 4.2, p. 120]. First, let us write the resolvent operators convergence in appropriate spaces. For this purpose, we use the unfolding operator. We have

$$\begin{aligned}\langle A_\varepsilon u, w\rangle_\varepsilon &= \frac{1}{L}\int_W |T_\varepsilon^{lp}\nabla u|^{p-2}T_\varepsilon^{lp}\nabla u T_\varepsilon^{lp}\nabla w + |T_\varepsilon^{lp}u|^{p-2}T_\varepsilon^{lp}uT_\varepsilon^{lp}v dxdY \\ &= \langle T_\varepsilon^{lp}A_\varepsilon u, T_\varepsilon^{lp}w\rangle = \langle B_\varepsilon u, w\rangle,\end{aligned} \tag{7.4}$$

where $W = (0,1)\times(0,L)\times(0,G_1)$ and $\langle\cdot,\cdot\rangle$ is the dual product in $W^{1,p}(W)$. Next,

$$\begin{aligned}\langle A_0 u, w\rangle_0 &= \frac{1}{L}\int_W \left[|\partial_x u\widetilde{\nabla_y v}|^{p-2}\partial_x u\widetilde{\nabla_y v}\partial_x w + \chi_{Y^*}|u|^{p-2}uw\right] dxdY \\ &= \langle B_0 u, w\rangle.\end{aligned} \tag{7.5}$$

Notice that

$$\overline{D(B_0)} \subset \overline{D(B_\varepsilon)}, \quad \forall \varepsilon > 0.$$

It remains to observe that

$$(I+\lambda B_\varepsilon)^{-1}f \to (I+\lambda B_0)^{-1}f \quad \forall f \in \overline{D(B_0)},$$

wich holds due to Theorem 2.1.

Therefore, thanks to Neveu-Trotter-Kato Theorem, the semigroup $S_\varepsilon(t)$ associated to $-B_\varepsilon$ satisfies

$$S_\varepsilon(t)f \to S(t)f, \quad \forall f \in \overline{D(B_0)},$$

where $S(t)$ is the semigroup associated to $-B_0$. We have the following theorem.

Theorem 7.1. *Assume $p \geq 2$ and consider the operators A_ε and B_ε defined respectively by (7.1) and (7.4). Then,*

(a) For any $f^\varepsilon \in L^2(R^\varepsilon)$ with $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded and $M_\varepsilon f^\varepsilon \rightharpoonup \hat{f}$ weakly in $L^2(0,1)$, we have

$$|||(I+\lambda A_\varepsilon)^{-1}f^\varepsilon - (I+\lambda A_0)^{-1}\hat{f}|||_{L^2(R^\varepsilon)} \to 0, \quad \text{as } \varepsilon \to 0.$$

(b) The semigroup $S_\varepsilon(t)$ associated to

$$\begin{cases}\partial_t u_\varepsilon + B_\varepsilon u_\varepsilon = f, \\ u_\varepsilon(0,x,y) = u_\varepsilon^0(x,y)\end{cases}$$

satisfies

$$S_\varepsilon(t)f \to S(t)f, \quad \forall f \in \overline{D(B_0)},$$

where $S(t)$ is the semigroup associated

$$\begin{cases}\partial_t u + B_0 u = f, \\ u(0,x) = u^0(x)\end{cases}$$

with B_0 given by (7.5).

A Appendix

In the proof of the main result, we used $q^\delta \to q$ uniformly to obtain (6.7). Recall that q^δ and q are given by (6.4) and (6.5) respectively. Here we prove such convergence. For this sake, let us first set

$$A(M) = \left\{ G \in C^1(\mathbb{R}) : \, G \text{ is } L-\text{periodic, } 0 < G_0 \leq G(\cdot) \leq G_1 \text{ with } |G'(s)| \leq M \right\}. \tag{A.1}$$

Hence, for any $\bar{G} \in A(M)$, we can consider the problem

$$\int_{Y^*_{\bar{G}}} |\nabla \bar{v}|^{p-2} \nabla \bar{v} \nabla \varphi dy_1 dy_2 = 0, \quad \forall \varphi \in W^{1,p}_{\#,0}(Y^*_{\bar{G}}) \tag{A.2}$$

where $W^{1,p}_{\#,0}(Y^*_{\bar{G}})$ is the space of functions $W^{1,p}_{\#}(Y^*_{\bar{G}})$ with zero average,

$$Y^*_{\bar{G}} = \left\{ (y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L, 0 < y_2 < \bar{G}(y_2) \right\}$$

and we are looking for solutions $\bar{v}$ such that $(\bar{v} - y_1) \in W^{1,p}_{\#,0}(Y^*_{\bar{G}})$.

Now, for any $\bar{G}$, $G \in A(M)$, let us consider the following transformation

$$\begin{array}{rccl} L & : & Y^*_G & \mapsto & Y^*_{\bar{G}} \\ & & (z_1, z_2) & \to & (z_1, F(z_1) z_2) = (y_1, y_2) \end{array}$$

where

$$F = \frac{\bar{G}}{G}.$$

The Jacobian matrix for L is

$$JL(z_1, z_2) = \begin{pmatrix} 1 & 0 \\ F'(z_1) z_2 & F(z_1) \end{pmatrix}$$

with $\det(JL) = F$. Also, we can consider

$$\begin{aligned} \mathcal{L} \nabla U &= \begin{pmatrix} 1 & -\dfrac{F'}{F} z_2 \\ 0 & 1/F \end{pmatrix} \nabla U = \left(\partial_{z_1} U - \frac{F'}{F} z_2 \partial_{z_2} U, \frac{1}{F} \partial_{z_2} U \right) \qquad \text{and} \\ \mathcal{B} \nabla U &= \left(\partial_{z_1} U + \frac{F' z_2}{F} \partial_{z_2} U, -\frac{F' z_2}{F} \partial_{z_1} U + \frac{1}{F^2} \left[1 + (z_2 F')^2 \right] \partial_{z_2} U \right). \end{aligned}$$

It is not difficult to see that $\mathcal{B} = \mathcal{L}^T \mathcal{L}$.

Then, we can use the change of variables given by L to rewrite (A.2) in the region Y^*_G as

$$\int_{Y^*_G} |\mathcal{L} \nabla \bar{v}|^{p-2} \mathcal{L} \nabla \bar{v} \mathcal{L} \nabla \left(\frac{\varphi}{F} \right) F \, dz_1 dz_2 = 0 \, , \forall \varphi \in W^{1,p}_{\#,0}(Y^*_G). \tag{A.3}$$

Notice that this problem still has unique solution $\bar{v} \in W^{1,p}(Y^*_G)$ with $(\bar{v} - z_1) \in W^{1,p}_{\#,0}(Y^*_G)$ by Minty-Browder's Theorem.

By the coercivity of (A.3), we get

$$\begin{aligned} \|\nabla \bar{v}\|^p_{L^p(Y^*_G)} &\leq \int_{Y^*_G} |\mathcal{L} \nabla \bar{v}|^{p-2} \mathcal{L} \nabla \bar{v} \mathcal{L} \nabla \left(\frac{\bar{v}}{F} \right) F \, dz_1 dz_2 \\ &= - \int_{Y^*_G} |\mathcal{L} \nabla \bar{v}|^{p-2} \mathcal{L} \nabla \bar{v} \mathcal{L} \nabla \left(\frac{z_1}{F} \right) F \, dz_1 dz_2 \\ &\leq c \|\mathcal{L} \nabla \bar{v}\|^{p-1}_{L^p(Y^*_G)} \leq c \|\nabla \bar{v}\|^{p-1}_{L^p(Y^*_G)}, \end{aligned}$$

which means that the solutions are uniformly bounded by a constant independent on $\bar{G}$ and G.

Now, let us compare the solutions of (A.2) for $\bar{G} = G$ and (A.3). We need to analyze

$$\begin{aligned}&\int_{Y_G^*}\left[|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v} - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar{v} - \nabla v)dz_1dz_2\\ &= \int_{Y_G^*}\left[|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v} - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar{v} - (1,0) + (1,0) - \nabla v)dz_1dz_2.\end{aligned} \tag{A.4}$$

Notice that $\mathcal{L}(1,0) = (1,0)$. We will distribute the terms finding estimative for each one.

First, observe that for any test function $\varphi \in W^{1,p}_{\#,0}(Y_G^*)$ in (A.3), we have

$$\int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}\mathcal{L}\nabla\varphi dz_1dz_2 = \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}\varphi\left(\frac{F'}{F},0\right)dz_1dz_2. \tag{A.5}$$

Now, take $\varphi = (\bar{v} - z_1)$ in (A.5). Then,

$$\int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}\mathcal{L}\nabla(\bar{v} - z_1)dz_1dz_2 = \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(\bar{v} - z_1)\left(\frac{F'}{F},0\right)dz_1dz_2 \tag{A.6}$$

On the other side, we can compute

$$\begin{aligned}&\int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}((1,0) - \nabla v)dz_1dz_2\\ &= \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}((1,0) - \nabla v + \mathcal{L}\nabla v - (1,0) + (1,0) - \mathcal{L}\nabla v)dz_1dz_2\\ &= \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(-\nabla v + \mathcal{L}\nabla v)dz_1dz_2 + \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}\mathcal{L}\nabla(z_1 - v)dz_1dz_2\\ &= -\int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(\mathcal{L} - I)\nabla v dz_1dz_2 + \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(z_1 - v)\left(\frac{F'}{F},0\right)dz_1dz_2\end{aligned} \tag{A.7}$$

by (A.5) with $\varphi = (z_1 - v)$.

Next, take $(\bar{v} - z_1) \in W^{1,p}_{\#,0}(Y_G^*)$ as a test function in (A.2). Then,

$$\int_{Y_G^*}|\nabla v|^{p-2}\nabla v(\nabla\bar{v} - (1,0))dz_1dz_2 = 0. \tag{A.8}$$

Finally, due to (A.8), we have

$$\begin{aligned}&\int_{Y_G^*}|\nabla v|^{p-2}\nabla v(\mathcal{L}\nabla\bar{v} - (1,0))dz_1dz_2\\ &= \int_{Y_G^*}|\nabla v|^{p-2}\nabla v(\mathcal{L}\nabla\bar{v} - (1,0))dz_1dz_2 - \int_{Y_G^*}|\nabla v|^{p-2}\nabla v(\nabla\bar{v} - (1,0))dz_1dz_2\\ &= \int_{Y_G^*}|\nabla v|^{p-2}\nabla v(\mathcal{L} - I)\nabla\bar{v}dz_1dz_2.\end{aligned} \tag{A.9}$$

Hence, putting together (A.4), (A.6), (A.7), (A.8) and (A.9), we obtain

$$\begin{aligned}&\int_{Y_G^*}\left[|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v} - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar{v} - \nabla v)dz_1dz_2\\ &= \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(\bar{v} - z_1)\left(\frac{F'}{F},0\right)dz_1dz_2\\ &- \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(\mathcal{L} - I)\nabla v dz_1dz_2 + \int_{Y_G^*}|\mathcal{L}\nabla\bar{v}|^{p-2}\mathcal{L}\nabla\bar{v}(z_1 - v)\left(\frac{F'}{F},0\right)dz_1dz_2\\ &- \int_{Y_G^*}|\nabla v|^{p-2}\nabla v(\mathcal{L} - I)\nabla\bar{v}dz_1dz_2.\end{aligned} \tag{A.10}$$

Now, one can apply Hölder and Poincaré-Wirtinger's inequalities in (A.10) to obtain

$$\begin{aligned}
&\int_{Y_{\bar G}^*} \left[|\mathcal{L}\nabla\bar v|^{p-2}\mathcal{L}\nabla\bar v - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar v - \nabla v)dz_1dz_2 \\
&\le \|\mathcal{L}\nabla\bar v\|^{p-1}_{L^p(Y_{\bar G}^*)}\|\nabla\bar v\|_{L^p(Y_{\bar G}^*)}\left\|\frac{F'}{F}\right\|_{L^\infty} + \|\mathcal{L}\nabla\bar v\|^{p-1}_{L^p(Y_{\bar G}^*)}\|\mathcal{L}-I\|_{L^\infty}\|\nabla v\|_{L^p(Y_{\bar G}^*)} \\
&+\|\mathcal{L}\nabla\bar v\|^{p-1}_{L^p(Y_{\bar G}^*)}\|\nabla v\|_{L^p(Y_{\bar G}^*)}\left\|\frac{F'}{F}\right\|_{L^\infty} + \|\nabla v\|^{p-1}_{L^p(Y_{\bar G}^*)}\|\mathcal{L}-I\|_{L^\infty}\|\nabla\bar v\|_{L^p(Y_{\bar G}^*)}.
\end{aligned} \tag{A.11}$$

Note that

$$\left\|\frac{F'}{F}\right\|_{L^\infty} \le c\|\bar G - G\|_{C^1} \text{ and } \|\mathcal{L}-I\|_{L^\infty} \le c\|\bar G - G\|_{C^1}. \tag{A.12}$$

Also, $\|\nabla v\|_{L^p(Y_{\bar G}^*)}$, $\|\nabla \bar v\|_{L^p(Y_{\bar G}^*)}$, $\|\mathcal{L}\nabla\bar v\|_{L^p(Y_{\bar G}^*)}$ and $\|\mathcal{L}\nabla v\|_{L^p(Y_{\bar G}^*)}$ are uniformly bounded. Thus, by (A.11)

$$\int_{Y_{\bar G}^*} \left[|\mathcal{L}\nabla\bar v|^{p-2}\mathcal{L}\nabla\bar v - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar v - \nabla v)dz_1dz_2 \le c\|\bar G - G\|_{C^1}. \tag{A.13}$$

If $p \ge 2$, we get from Proposition 3.1 and (A.13) that

$$\begin{aligned}
\|\mathcal{L}\nabla\bar v - \nabla v\|^p_{L^p(Y_{\bar G}^*)} &\le c\int_{Y_{\bar G}^*} \left[|\mathcal{L}\nabla\bar v|^{p-2}\mathcal{L}\nabla\bar v - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar v - \nabla v)dz_1dz_2 \\
&\le c\|\bar G - G\|_{C^1}.
\end{aligned}$$

On the other side, if $1 < p < 2$, we get from Hölder's inequality, Proposition 3.1 and (A.13), that

$$\begin{aligned}
\|\mathcal{L}\nabla\bar v - \nabla v\|^p_{L^p(Y_{\bar G}^*)} &\le c\left\{\int_{Y_{\bar G}^*} \left[|\mathcal{L}\nabla\bar v|^{p-2}\mathcal{L}\nabla\bar v - |\nabla v|^{p-2}\nabla v\right](\mathcal{L}\nabla\bar v - \nabla v)dz_1dz_2\right\}^{p/2} \\
&\qquad \left[\int_{Y_{\bar G}^*}(1+|\mathcal{L}\nabla\bar v| + |\nabla v|)^p\right]^{(2-p)/2} \\
&\le c\|\bar G - G\|^{p/2}_{C^1},
\end{aligned}$$

Therefore, for $1 < p < \infty$, we have

$$\|\mathcal{L}\nabla\bar v - \nabla v\|_{L^p(Y_{\bar G}^*)} \le c\|\bar G - G\|^{\alpha}_{C^1} \tag{A.14}$$

where $\alpha = 1/2$ if $1 < p < 2$ and $\alpha = 1/p$ if $p \ge 2$.

Finally, since

$$\|\nabla\bar v - \nabla v\|_{L^p(Y_{\bar G}^*)} \le \|\mathcal{L}\nabla\bar v - \nabla\bar v\|_{L^p(Y_{\bar G}^*)} + \|\mathcal{L}\nabla\bar v - \nabla v\|_{L^p(Y_{\bar G}^*)},$$

we conclude by (A.14) and (A.12) that

$$\|\nabla\bar v - \nabla v\|_{L^p(Y_{\bar G}^*)} \le c\|\bar G - G\|_{C^1} + c\|\bar G - G\|^{\alpha}_{C^1}.$$

We have the following lemma:

Lemma A.1. *Let us consider the family of admissible functions $G \in A(M)$ for some constant $M > 0$ where $A(M)$ is defined by* (A.1).

Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $G, \bar G \in A(M)$ with $\|\bar G - G\| \le \delta$, then

$$\|\nabla\bar v - \nabla v\|_{L^p(Y_{\bar G}^*)} \le c(\varepsilon + \varepsilon^\alpha),$$

where $\alpha = 1/2$ if $1 < p < 2$ and $\alpha = 1/p$ if $p \ge 2$ and c is a constant which depends only on p, G_0, G_1. In particular, we have that

$$|q(\bar G) - q(G)| \le c(\varepsilon + \varepsilon^\alpha),$$

where

$$q(\bar G) = \int_{Y_{\bar G}^*} |\nabla\bar v|^{p-2}\partial_{y_1}\bar v\, dy_1dy_2$$

and $\bar v$ is the solution of (A.2) *in the region $Y_{\bar G}^*$ set by $\bar G$.*

Lemma A.2. *Let us consider the family of admissible functions $G \in A(M)$ for some constant $M > 0$ where $A(M)$ is defined by* (A.1). *Then*

$$q(G) = \frac{1}{\langle 1/G^{p'-1}\rangle_{(0,L)}^{p-1}}$$

for any $G, \bar{G} \in A(M)$ with $g \neq \bar{G}$,

$$|q(G) - q(\bar{G})| \leq c\|G - \bar{G}\|^{p-1}$$

for a constant that depends on p, G_0 and G_1.

Proof. Notice that

$$\begin{aligned} q(G) - q(\bar{G}) &= \frac{1}{\langle 1/G^{p'-1}\rangle_{(0,L)}^{p-1}} - \frac{1}{\langle 1/\bar{G}^{p'-1}\rangle_{(0,L)}^{p-1}} \\ &= \frac{\left\langle 1/\bar{G}^{p'-1}\right\rangle_{(0,L)}^{p-1} - \left\langle 1/G^{p'-1}\right\rangle_{(0,L)}^{p-1}}{\langle 1/G^{p'-1}\rangle_{(0,L)}^{p-1}\langle 1/\bar{G}^{p'-1}\rangle_{(0,L)}^{p-1}}. \end{aligned}$$

Suppose that $1 < p < 2$. Then, due to Corollary 3.1.1, we get

$$\begin{aligned} &\left\langle 1/\bar{G}^{p'-1}\right\rangle_{(0,L)}^{p-1} - \left\langle 1/G^{p'-1}\right\rangle_{(0,L)}^{p-1} \\ &\leq c\left|\left\langle 1/\bar{G}^{p'-1}\right\rangle_{(0,L)} - \left\langle 1/G^{p'-1}\right\rangle_{(0,L)}\right|^{p-1} \\ &\leq \frac{c}{L}\left|\int_0^L \frac{G^{p'-1}(s) - \bar{G}^{p'-1}(s)}{\bar{G}^{p'-1}(s)G^{p'-1}(s)}ds\right|^{p-1} \\ &\leq \frac{c}{L}\left|\int_0^L (1 + |\bar{G}(s)| + |G(s)|)^{p'-2}|\bar{G}(s) - G(s)|ds\right|^{p-1} \\ &\leq C\|G - \bar{G}\|^{p-1}, \end{aligned}$$

where C is a positive constant that depends on p, G_0, G_1.

Now, suppose $p \geq 2$. Then, by Corollary 3.1.1,

$$\begin{aligned} &\left\langle 1/\bar{G}^{p'-1}\right\rangle_{(0,L)}^{p-1} - \left\langle 1/G^{p'-1}\right\rangle_{(0,L)}^{p-1} \\ &\leq c\left(1 + |\left\langle 1/\bar{G}^{p'-1}\right\rangle_{(0,L)}| + |\left\langle 1/G^{p'-1}\right\rangle_{(0,L)}|\right)^{p-2}\left|\left\langle 1/\bar{G}^{p'-1}\right\rangle_{(0,L)} - \left\langle 1/G^{p'-1}\right\rangle_{(0,L)}\right| \\ &\leq c\left|\int_0^L \frac{G^{p'-1}(s) - \bar{G}^{p'-1}(s)}{\bar{G}^{p'-1}(s)G^{p'-1}(s)}ds\right| \\ &\leq c\left|\int_0^L |G(s) - \bar{G}(s)|^{p-1}ds\right| \\ &\leq C\|G - \bar{G}\|^{p-1}. \end{aligned}$$

□

Remark A.1. *We remark that the result of the Lemma above, works in a more general framework, that is, the functions do not need to be in $A(M)$. On the other hand, to perform the discretization of the domain in the locally periodic case, in the previous section, we need the hypothesis of $A(M)$ functions defining the domains.*

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