SEMILINEAR ELLIPTIC EQUATIONS IN THIN DOMAINS WITH REACTION TERMS CONCENTRATING ON BOUNDARY

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Abstract. In this paper we analyze the behavior of a family of steady state solutions of a semilinear reaction-diffusion equation with homogeneous Neumann boundary condition, posed in a two-dimensional thin domain whit reaction terms concentrated in a narrow oscillating neighborhood of the boundary. We assume that the domain, and therefore, the oscillating boundary neighborhood, degenerates into an interval as a small parameter ϵ goes to zero. Our main result is that this family of solutions converges to the solution of a one-dimensional limit equation capturing the geometry and oscillatory behavior of the open sets where the problem is established.

1. Introduction

In this work we are interested in analyzing the asymptotic behavior of a family of steady state solutions of a semilinear reaction-diffusion equation with homogeneous Neumann boundary conditions on a thin domain $R^\epsilon \subset \mathbb{R}^2$, with reaction terms concentrated in a very narrow oscillating neighborhood θ_ϵ of the boundary. Roughly speaking, we deal with a nonlinear elliptic problem posed in an open region of $\mathbb{R}^2$ which degenerates into a line segment as a positive parameter ϵ goes to zero. The reaction terms of the equation occur only in an extremely thin region close to the border, which can also present oscillatory structure. In Figure 1 we illustrate the thin domain R^ϵ as well as the narrow oscillating neighborhood θ_ϵ where some reactions of the model take place.

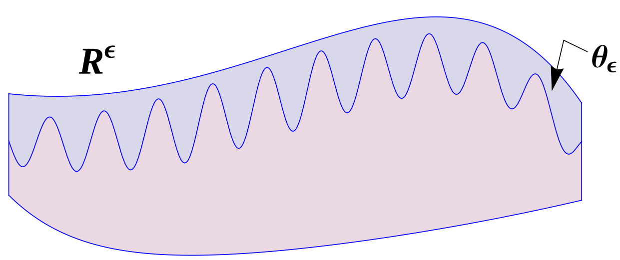

Figure 1. The thin domain R^ϵ and the oscillating ϵ-strip θ_ϵ.

This type of elliptic boundary value problem models diffusion and interactions among agents which can be cells, amount of chemicals or biological organisms, and which are located in an extremely thin region in a small neighborhood of the border, where reactions take place. It is worth noting that our model includes the possibility that the narrow strip presents oscillating behavior, modeling complex regions of interactions. Potential applications of our results include fields like lubrication, nanotechnology, fluid-structure interaction mechanism in vascular dynamics and management and control of aquatic ecological systems, where one can find localized concentrations in connection with boundary complexity in thin channels. For instance, we

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mention [1, 2, 3, 4, 5, 6, 7] where theoretical and practical aspects of mathematical modeling and applications in these fields are investigated.

Using an appropriate functional setting we show that our singular problem defined in a two-dimensional open set can be approximated by a one-dimensional boundary value problem. In this problem, the variable profile of the thin domain and the oscillatory behavior of the narrow strip where the reactions occur are captured by a one-dimensional *limit equation* displaying a *reaction limit term.* This limit equation is not singular and provides an option to approximate the original problem when the parameter ϵ is close to zero (i.e., for very thin domains and boundary strips). Moreover, it preserves some important features of the original system, giving conditions to access the qualitative behavior of the modeled problem. We will also provide some numerical evidence of the convergence to the limit equation.

There are several works in the literature dealing with partial differential equations in thin domains. Let us first cite the pioneering works [11, 12], as well as the subsequent papers [13, 14, 15], where the authors investigate the asymptotic behavior of dynamical systems given by a class of semilinear parabolic equations posed on a thin domain in $\mathbb{R}^n$, $n \geq 2$. We also mention [16] where the author has studied asymptotic approximations of the solutions of a p-Laplacian problem defined in a thin region and [8], which consider a linear elliptic problem in perforated thin domains with rapidly varying thickness. See also [9, 10] which consider nonlinear monotone problems in a multidomain with a highly oscillating boundary. Recently, we also have studied many classes of oscillating thin domains for elliptic and parabolic equations with Neumann boundary conditions, discussing limit problems and convergence properties [17, 18, 19, 20, 21, 22, 23].

On the other hand, there are many works dealing with singular elliptic and parabolic problems featuring potential and reactions terms concentrated in a small neighborhood of a portion of the boundary of fixed bounded domains. For instance, we mention [24, 25, 26]. In [27, 28, 29] we also have studied problems allowing narrow strips with oscillatory border. Our goal here is to introduce a model combining these both singular situations in a more general framework. For this, we adapt methods and techniques discussed and developed in [11, 24, 27] as well as in [30], in order to pass to the limit obtaining the asymptotic behavior of our model as $\epsilon \to 0$.

2. Assumptions, notations and main result

We analyze the family of solutions defined by the following nonlinear elliptic problem

$$\begin{cases} -\Delta u^\epsilon + u^\epsilon = f(u^\epsilon) + \dfrac{1}{\epsilon^\alpha} \chi_{\theta_\epsilon}\, g(u^\epsilon) \quad \text{ in } R^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial \nu^\epsilon} = 0 \quad \text{ on } \partial R^\epsilon \end{cases}. \tag{2.1}$$

The domain R^ϵ is an ordinary thin domain given by

$$R^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 \in (0,1), \quad -\epsilon\, b(x_1) < x_2 < \epsilon\, G(x_1)\} \tag{2.2}$$

where G and $b : (0,1) \mapsto \mathbb{R}^+$ are positive smooth functions, uniformly bounded, with $0 < G_0 \leq G(x) \leq G_1$ and $0 < b_0 \leq b(x) \leq b_1$ for all $x \in (0,1)$ and for fixed positive constants G_0, G_1, b_0, b_1. The vector $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and $\frac{\partial}{\partial \nu^\epsilon}$ is the outside normal derivative. Note that the functions b and G, independent of ϵ, define the lower and upper boundary of the thin domain respectively. [1]

Nonlinearities f and $g : \mathbb{R} \mapsto \mathbb{R}$ are supposed to be $\mathcal{C}^2$-functions, and $\chi_{\theta_\epsilon} : \mathbb{R}^2 \mapsto \mathbb{R}$ is the characteristic function of the narrow strip θ_ϵ defined by

$$\theta_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 \in (0,1), \quad \epsilon\,(G(x_1) - \epsilon^\alpha\, H_\epsilon(x_1)) < x_2 < \epsilon\, G(x_1)\}, \tag{2.3}$$

where $\alpha > 0$ is a parameter, $H_\epsilon : (0,1) \mapsto \mathbb{R}^+$ is a smooth non negative function satisfying $0 \leq H_\epsilon(x) \leq G_0 + b_0$ for all $x \in (0,1)$ and $\epsilon > 0$. H_ϵ also can oscillate when ϵ goes to zero, and so, we express it as

$$H_\epsilon(x) = H(x, x/\epsilon^\beta), \quad \beta > 0, \tag{2.4}$$

[1]We could suppose G and b piecewise C^1-functions with respect to the first variable as in [19]. For simplicity we have assumed more regularity here.

where $H : (0,1) \times \mathbb{R} \mapsto \mathbb{R}$ is a non negative function, continuous in x_1 uniformly with respect to second variable x_2, (that is, for each $\eta > 0$, there exists $\delta > 0$ such that $|H(x_1, x_2) - H({x_1}', x_2)| \leq \eta$ for all x_1, ${x_1}' \in [0,1]$, $|x_1 - {x_1}'| < \delta$, and $x_2 \in \mathbb{R}$). We still assume that H is $l(x_1)$-periodic in x_2 for each $x_1 \in (0,1)$: $H(x_1, x_2 + l(x_1)) = H(x_1, x_2)$, for all x_2, with the period function l uniformly positive and bounded, $0 < l_0 \leq l(x_1) \leq l_1$ for all $x_1 \in (0,1)$.

Clearly the open set θ_ϵ is a neighborhood for the upper boundary of R^ϵ whose thickness and oscillatory behavior depend on the positive parameters α and β respectively. Actually, α and β represent the thickness and oscillating order when ϵ goes to zero. Moreover, if H only depends on the first variable x_1, then the function H_ϵ is independent of ϵ and the narrow strip θ_ϵ does not possess oscillatory behavior.

Here we proceed as in [24, 29]. We combine the characteristic function χ_ϵ, the positive parameter ϵ and a fixed value of α to set concentration of reactions on the small region $\theta_\epsilon \subset R^\epsilon$ through the term

$$\frac{1}{\epsilon^\alpha} \chi_{\theta_\epsilon} \in L^\infty(R^\epsilon).$$

Furthermore, since $R^\epsilon \subset (0,1) \times (-\epsilon\, b_1, \epsilon\, G_1)$ is thin and degenerates into the unit interval as ϵ goes to zero, it is reasonable to expect that the family of solutions u^ϵ converges to a solution of a one-dimensional equation of the same type with homogeneous Neumann boundary condition, capturing the variable profile of the thin domain R^ϵ as well as the oscillatory behavior of the narrow strip θ_ϵ. Indeed, we show that under these conditions, the limit problem for (2.1) is the following one

$$\begin{cases} -\dfrac{1}{p(x)}(p(x)u_x)_x + u = f(u) + \dfrac{\mu(x)}{p(x)} g(u) \quad \text{ in } (0,1) \\ u_x(0) = u_x(1) = 0 \end{cases} \tag{2.5}$$

where p and $\mu : (0,1) \mapsto (0,\infty)$ are smooth functions given by

$$\begin{aligned} p(x) &= G(x) + b(x), \\ \mu(x) &= \frac{1}{l(x)} \int_0^{l(x)} H(x,y)\, dy. \end{aligned} \tag{2.6}$$

The positive function p is associated with the geometry of the thin domain and is defined by the functions b and G. On the other hand, the non negative coefficient $\mu \in L^\infty(0,1)$ is related to the oscillating strip θ_ϵ given by H_ϵ. As mentioned, we get a limit problem that captures the variable profile of the thin channel R^ϵ and the oscillating behavior of the narrow strip θ_ϵ combining results previously obtained in [11, 27]. Applying techniques from [11] we compute the coefficient p established by the variable profile of the thin channel, and using *concentrated integrals* discussed in [24, 26, 27], we obtain the function $\mu(x)$, which is the mean value of $H(x,\cdot)$ for each $x \in (0,1)$.

Notice that μ captures the oscillatory behavior and the geometry of the narrow strip where the reactions are concentrated. If H does not depend on the second variable y, then the narrow neighborhood does not have oscillatory behavior, and so, $\mu(x) = H(x)$ in $(0,1)$. Further, if we take $H \equiv 0$, then problem (2.1) does not present concentration of reaction terms in any region and is reduced to the problem considered in [11, 12].

In order to study problem (2.1) in the thin domain R^ϵ, we take a convenient change of variables considering the following equivalent problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f(u^\epsilon) + \dfrac{1}{\epsilon^\alpha}\chi_{o_\epsilon}\, g(u^\epsilon) \quad \text{ in } \Omega \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1 + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2 = 0 \quad \text{ on } \partial\Omega \end{cases} \tag{2.7}$$

where the function $\chi_{o_\epsilon} : \mathbb{R}^2 \mapsto \mathbb{R}$ is the characteristic function of the narrow strip o_ϵ given by

$$o_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \in (0,1), \quad (G(x_1) - \epsilon^\alpha\, H_\epsilon(x_1)) < x_2 < G(x_1)\}. \tag{2.8}$$

The vector $N = (N_1, N_2)$ is the outward unit normal to $\partial\Omega$ and $\Omega \subset \mathbb{R}^2$ is given by

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_1 \in (0,1), \quad -b(x_1) < x_2 < G(x_1)\}. \tag{2.9}$$

The equivalence between problems (2.1) and (2.7) is obtained by changing the scale of the domain R^ϵ and the narrow strip θ_ϵ through the transformation $(x_1, x_2) \to (x_1, \epsilon^{-1} x_2)$ which consists in stretching the x_2-direction by a factor of ϵ^{-1} (see [11] for more details). The factor ϵ^{-2} in front of the derivative in the x_2-direction establishes a very fast diffusion in this direction. In some sense, we have rescaled the neighborhood θ_ϵ into the strip $o_\epsilon \subset \Omega$ and substituted the thin domain R^ϵ for a domain Ω independent on ϵ, at a cost of introducing a very strong diffusion mechanism in the x_2-direction.

Due to the presence of this strong diffusion mechanism it is expected that solutions of (2.7) will become more and more homogeneous in the x_2-direction when ϵ decreases, such that the limit solution will not depend on x_2 and therefore the limit problem will be one dimensional. This is in full agreement with the intuitive idea that an equation in a thin domain should approach one in a line segment.

Now we are in position to state our main result:

Theorem 2.1. *Let u^ϵ be a family of solutions of problem* (2.7) *satisfying $\|u^\epsilon\|_{L^\infty(\Omega)} \leq R$ for some positive constant R independent of ϵ. Then:*

(i) *There exists a subsequence, still defined by u^ϵ, and a function $u \in H^1(\Omega)$, $\|u\|_{L^\infty(\Omega)} \leq R$, depending only on the first variable, that is, $u(x_1, x_2) = u(x_1)$, solution of the problem* (2.5)*, such that*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \;\; \textit{as } \epsilon \to 0.$$

(ii) *Moreover, if the solution u of* (2.5) *belonging to the ball of radius R in $L^\infty(\Omega)$ is hyperbolic, then we also have that there exists a sequence u^ϵ of solutions of problem* (2.7) *satisfying*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \;\; \textit{as } \epsilon \to 0.$$

Remark 2.2. *We say that a solution u of a boundary value problem is hyperbolic if $\lambda = 0$ is not an eigenvalue of the linearized problem around u. For instance, if u satisfies equation* (2.5) *and is hyperbolic, then $\lambda = 0$ is not an eigenvalue of the eigenvalue problem*

$$\begin{cases} -\dfrac{1}{p(x)}(p(x)v_x)_x + v = f_u(u)v + \dfrac{\mu(x)}{p(x)} g_u(u)v + \lambda v \qquad \textit{in } (0,1) \\ \qquad\qquad v_x(0) = v_x(1) = 0 \end{cases}.$$

Remark 2.3. *Since we are concerned with solutions which are uniformly bounded in $L^\infty(\Omega)$, we may assume nonlinearities f and g of class $\mathcal{C}^2$ with bounded derivatives. Indeed, we may perform a cut-off in f and g outside the region $|u| \leq R$ without modifying any of these solutions.*

Remark 2.4. *Let us denote $\mathcal{E}_\epsilon = \{u^\epsilon \in H^1(\Omega) \, : \, u^\epsilon$ is a solution of* (2.7)$\}$ *for each $\epsilon > 0$. If $\|u^\epsilon\|_{H^1(\Omega)} \leq R$ for all $u^\epsilon \in \mathcal{E}_\epsilon$, then assertions (i) and (ii) at Theorem 2.1 respectively mean* upper *and* lower *semicontinuity of the equilibria set of the parabolic problem associated with* (2.7) *at $\epsilon = 0$.*

Finally we mention that we hope to employ the results obtained here in order to investigate the asymptotic behavior of the attractors of the Dynamical System generated by the parabolic equation associated with the semilinear elliptic problem (2.1).

3. Basic facts and technical results

In this section we set some basic and technical results that will be needed in the proof of the main result. We initially introduce some notation, stating basic results and writing our problem in a more abstract setting. Next we discuss how concentrating integrals converge to boundary integrals, adapting results from [24, 26, 27].

Throughout this paper we denote $H^1_\epsilon(U)$ the Hilbert space given by $H^1(U)$ with the equivalent norm

$$\|w\|^2_{H^1_\epsilon(U)} = \|w\|^2_{L^2(U)} + \Big\| \frac{\partial w}{\partial x_1} \Big\|^2_{L^2(U)} + \frac{1}{\epsilon^2} \Big\| \frac{\partial w}{\partial x_2} \Big\|^2_{L^2(U)} \tag{3.1}$$

defined by the inner product

$$(\phi,\varphi)_{H^1_\epsilon(U)} = \int_U \left\{ \frac{\partial \phi}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial \phi}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + \phi\,\varphi \right\} dx_1 dx_2$$

where U is an arbitrary open set of $\mathbb{R}^2$. It is worth noting that this space is a suitable one to deal with thin domain problems due to the strong diffusion mechanism in front of the second derivative of the equation.

Remark 3.1. *It follows from* (3.1) *that if a sequence* $u^\epsilon \in H^1_\epsilon(\Omega)$ *sastifies* $\|u^\epsilon\|_{H^1_\epsilon(\Omega)} \le K$ *for some positive constant* K *independent of* ϵ*, then*

$$\left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega)} \le \epsilon K, \quad \forall \epsilon > 0,$$

and so,

$$\left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega)} \to 0, \ \text{ as } \epsilon \to 0.$$

Lemma 3.2. *If* H_ϵ *is defined as in (2.4), then*

$$H_\epsilon(\cdot) \to \mu(\cdot) = \frac{1}{l(\cdot)} \int_0^{l(\cdot)} H(\cdot, s)\, ds, \quad w^* - L^\infty(0,1).$$

Proof. See [27, Lemma 2.3]. □

3.1. **Abstract settings and existence of solutions.** In order to write problems (2.5) and (2.7) in an appropriated abstract form, we first consider the linear operator $A_\epsilon : D(A_\epsilon) \subset L^2(\Omega) \mapsto L^2(\Omega)$ defined by

$$A_\epsilon u^\epsilon = -\frac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \frac{1}{\epsilon^2}\frac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon,$$
$$D(A_\epsilon) = \{u^\epsilon \in H^2(\Omega) \,:\, \frac{\partial u^\epsilon}{\partial x_1} N_1 + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2} N_2 = 0 \text{ on } \partial\Omega\}.$$

It is known from [11] that A_ϵ is a self-adjoint positive linear operator with compact resolvent. Denoting $E^0_\epsilon = L^2(\Omega)$ and $E^1_\epsilon = D(A_\epsilon)$, we can consider the scale of Hilbert spaces $\{(E^s_\epsilon, A_{\epsilon,s}),\ s \in \mathbb{R}\}$ constructed by complex interpolation [31]. Since we are in a Hilbert settings we have that this scale coincides with the standard fractional power spaces of A_ϵ where the negative exponents are given by $E^{-s}_\epsilon = (E^s_\epsilon)'$, $s > 0$. Moreover we know that $E^s_\epsilon \hookrightarrow H^{2s}(\Omega)$. Hence, if we consider the realizations of operator A_ϵ in this scale, we get $A_{\epsilon,-1/2} \in \mathcal{L}(E^{1/2}_\epsilon, E^{-1/2}_\epsilon)$ given by

$$\left\langle A_{\epsilon,-1/2} u, v \right\rangle = \int_\Omega \left\{ \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} + u\,v \right\} dx_1 dx_2, \quad \text{ for } v \in E^{1/2}_\epsilon.$$

It follows from [32, Section 2.9] and [33, Theorem 5.2] that $E^{1/2}_\epsilon = H^1_\epsilon(\Omega)$ where $H^1_\epsilon(\Omega)$ has been defined in (3.1). With some abuse of notation we identify all different realizations writing them as A_ϵ.

Thus, we can write problem (2.7) as $A_\epsilon u = F_\epsilon(u)$, $\epsilon > 0$, with $F_\epsilon : H^1_\epsilon(\Omega) \mapsto E^{-s}_\epsilon$ for $1/4 < s < 1/2$,

$$\begin{gathered} F_\epsilon = \hat{F} + \hat{F}_\epsilon, \\ \left\langle \hat{F}(u), v \right\rangle = \int_\Omega f(u)\, v\, d\xi, \quad \left\langle \hat{F}_\epsilon(u), v \right\rangle = \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u)\, v\, d\xi, \end{gathered} \tag{3.2}$$

for $u \in H^1_\epsilon(\Omega)$ and $v \in E^s_\epsilon$. Therefore, we have that u^ϵ is a solution of (2.7), if and only if $u^\epsilon \in H^1_\epsilon(\Omega)$ satisfies $u^\epsilon = A^{-1}_\epsilon F_\epsilon(u^\epsilon)$. Thus u^ϵ must be a fixed point of the nonlinear map

$$A^{-1}_\epsilon \circ F_\epsilon : H^1_\epsilon(\Omega) \mapsto H^1_\epsilon(\Omega). \tag{3.3}$$

The existence of solutions of problem (2.7) can be derived from Schaefer's Fixed Point Theorem [34, Section 9.2.2], in the case that f and g are bounded with bounded derivatives. We then have that F_ϵ is continuous (see Remarks 2.3 and 3.3). Hence, since A_ϵ has compact resolvent, $A_\epsilon^{-1} \circ F_\epsilon$ is a continuous and compact mapping. It remains to prove that

$$\mathcal{O}_\epsilon = \{\varphi \in H^1_\epsilon(\Omega) \,:\, \varphi = A_\epsilon^{-1} F_\epsilon(\varphi)\}$$

is a bounded set in order to conclude the existence. But, for any $\varphi \in \mathcal{O}_\epsilon$, it follows from Lemma 3.4 (shown in the next section) that

$$\begin{aligned} \|\varphi\|^2_{H^1_\epsilon(\Omega)} &= \langle A_\epsilon \varphi, \varphi\rangle \;=\; \langle F_\epsilon(\varphi), \varphi\rangle \\ &\le |\Omega|^{1/2} \, \|f\|_\infty \, \|\varphi\|_{L^2(\Omega)} + C\, \|g\|_\infty \, \|\varphi\|_{H^1(\Omega)} \\ &\le \Big(|\Omega|^{1/2} \, \|f\|_\infty + C\, \|g\|_\infty \Big) \|\varphi\|_{H^1_\epsilon(\Omega)}. \end{aligned}$$

Therefore, there exists $K > 0$ such that $\|\varphi\|_{H^1_\epsilon(\Omega)} \le K$ for all $\varphi \in \mathcal{O}_\epsilon$, concluding the proof.

Similarly, we can consider the self-adjoint positive linear operator $A_0 : D(A_0) \subset \hat{L}^2(0,1) \mapsto \hat{L}^2(0,1)$

$$A_0 u = -\frac{1}{p}\,(p\, u_x)_x + u,$$
$$D(A_0) = \{u \in H^2(0,1) \,:\, u_x(0) = u_x(1) = 0\},$$

where $\hat{L}^2(0,1)$ is the space $L^2(0,1)$ with the convenient inner product

$$(u,v) = \int_0^1 u(x)\, v(x)\, p(x)\, dx,$$

and $p : (0,1) \mapsto (0,\infty)$ is the positive function given by (2.6). We can denote $E_0^0 = \hat{L}^2(0,1)$ and $E_0^1 = D(A_0)$ considering the scale of Hilbert spaces $\{(E_0^s, A_{0,s}),\ s \in \mathbb{R}\}$ by complex interpolation, and also extending this scale to spaces of negative exponents by taking $E_0^{-s} = (E_0^s)'$ for $s > 0$. Integrating by parts we see that

$$\langle A_0 u, v\rangle = \int_0^1 \{u_x(x)\, v_x(x) + u(x)\, v(x)\}\, p(x)\, dx, \qquad \text{for } u, v \in H^1(0,1),$$

and so, we get that solutions of the limiting problem (2.5) can be expressed as fixed points of the map

$$A_0^{-1} \circ F_0 : \hat{H}^1(0,1) \mapsto \hat{H}^1(0,1), \tag{3.4}$$

where $F_0 : \hat{H}^1(0,1) \mapsto E_0^{-s}$ with $0 \le s < 1/2$ is defined by

$$\langle F_0(u), v\rangle = \int_0^1 \left\{ f(u(x)) + \frac{\mu(x)}{p(x)}\, g(u(x)) \right\} v(x)\, p(x)\, dx, \qquad \text{for } v \in E_0^s. \tag{3.5}$$

The smooth function $\mu : (0,1) \mapsto [0,\infty)$ is given by (2.6), while the Hilbert space $\hat{H}^1(0,1)$ is $H^1(0,1)$ with the Lebesgue measure re-scaled by p, which coincides with $E_0^{1/2}$.

The existence of solutions to problem (2.5) in the case f, g and their derivatives are bounded also follows from Schaefer's Fixed Point Theorem (similarly to problem (2.7)). We observe that some solutions to (2.5) may also be obtained as limits of solutions from problem (2.7) by Theorem 2.1.

Remark 3.3. *Finally we notice that functions F_ϵ and F_0 defined respectively in* (3.2) *and* (3.5) *are Fréchet differentiable. The proof can be taken from* [28, Lemma 3.7] *and* [30, Lemma 5.3].

3.2. **Concentrating integrals.**

Lemma 3.4. *Suppose that $v \in H^s(\Omega)$ with $1/2 < s \le 1$ and $s - 1 \ge -1/q$. Then, for small ϵ_0, there exists a constant $C > 0$ independent of ϵ and v, such that for any $0 < \epsilon \le \epsilon_0$, we have*

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} |v(\xi)|^q \; d\xi \le C \, \|v\|^q_{H^s(\Omega)}.$$

Proof. First we observe that

$$\frac{1}{\epsilon^\alpha}\int_{o_\epsilon}|v(\xi)|^q\,d\xi = \frac{1}{\epsilon^\alpha}\int_0^1\int_0^{\epsilon^\alpha H_\epsilon(x_1)}|v(x_1,G(x_1)-x_2)|^q\,dx_2dx_1.$$

Next, using [27, Lemma 2.1], we have that there exists ϵ_0 and $C>0$ independent of ϵ and $w = v\circ\tau$ such that

$$\frac{1}{\epsilon^\alpha}\int_{o_\epsilon}|v(\xi)|^q\;d\xi \le C\,\|w\|^q_{H^s(\tau^{-1}(\Omega))},\quad \forall\epsilon\in(0,\epsilon_0).$$

Here we are taking $\tau:\mathbb{R}^2\mapsto\mathbb{R}^2$ given by $\tau(x_1,x_2)=(x_1,G(x_1)-x_2)$. Thus, we conclude the proof using that the norms $\|w\|_{H^s(\tau^{-1}(\Omega))}$ and $\|v\|_{H^s(\Omega)}$ are equivalents [24, Section 2].

□

The following result follows from Lemma 3.4.

Lemma 3.5. *Assume ϕ^ϵ, $\varphi^\epsilon\in H^1_\epsilon(\Omega)$ uniformly bounded in $\epsilon>0$, and ϕ, $\varphi\in H^1(0,1)$ satisfying $\phi^\epsilon\rightharpoonup\phi$ and $\varphi^\epsilon\rightharpoonup\varphi$, $w-H^1(\Omega)$. Then,*

$$\lim_{\epsilon\to0}\frac{1}{\epsilon^\alpha}\int_{o_\epsilon}\phi^\epsilon\,\varphi^\epsilon\,d\xi = \int_0^1\mu\,\phi\,\varphi\,dS, \tag{3.6}$$

where $\mu\in L^\infty(0,1)$ is given by (2.6).

Proof. We have that

$$\frac{1}{\epsilon^\alpha}\int_{o_\epsilon}\phi^\epsilon\,\varphi^\epsilon\,d\xi = \frac{1}{\epsilon^\alpha}\int_0^1\int_0^{\epsilon^\alpha H_\epsilon(x_1)}\phi^\epsilon(x_1,G(x_1)-x_2)\,\varphi^\epsilon(x_1,G(x_1)-x_2)\,dx_2dx_1.$$

Hence, rescaling x_2 by $\epsilon^\alpha H_\epsilon(x_1)x_2$ we obtain

$$\frac{1}{\epsilon^\alpha}\int_{o_\epsilon}\phi^\epsilon\,\varphi^\epsilon\,d\xi = \int_0^1\int_0^1\phi^\epsilon(x_1,G(x_1)-\epsilon^\alpha H_\epsilon(x_1)x_2)\,\varphi^\epsilon(x_1,G(x_1)-\epsilon^\alpha H_\epsilon(x_1)x_2)\,H_\epsilon(x_1)\,dx_2dx_1.$$

Then, adding and subtracting appropriate terms, we get

$$\begin{aligned}\frac{1}{\epsilon^\alpha}\int_{o_\epsilon}\phi^\epsilon\,\varphi^\epsilon\,d\xi &= \int_0^1\int_0^1(\phi^\epsilon(x_1,G(x_1)-\epsilon^\alpha H_\epsilon(x_1)x_2)-\phi(x_1))\,\varphi^\epsilon(x_1,G(x_1)-\epsilon^\alpha H_\epsilon(x_1)x_2)\,H_\epsilon(x_1)\,dx_2dx_1\\ &+\int_0^1\int_0^1(\varphi^\epsilon(x_1,G(x_1)-\epsilon^\alpha H_\epsilon(x_1)x_2)-\varphi(x_1))\,\phi(x_1)\,H_\epsilon(x_1)\,dx_2dx_1\\ &+\int_0^1\int_0^1\phi(x_1)\,\varphi(x_1)\,H_\epsilon(x_1)\,dx_2dx_1 = I_1+I_2+I_3,\end{aligned}$$

where I_i represent in an obvious way the first, second and third integrals.

First we investigate integral I_1. We come back to the open set o_ϵ and use Lemma 3.4 to get

$$\begin{aligned}|I_1| &= \left|\int_0^1\int_{G(x_1)-\epsilon^\alpha H_\epsilon(x_1)}^{G(x_1)}(\phi^\epsilon(x_1,x_2)-\phi(x_1))\;\varphi^\epsilon(x_1,x_2)\,\epsilon^{-\alpha}\,dx_2dx_1\right|\\ &\le \left(\epsilon^{-\alpha/2}\|\phi^\epsilon-\phi\|_{L^2(o_\epsilon)}\right)\left(\epsilon^{-\alpha/2}\|\varphi^\epsilon\|_{L^2(o_\epsilon)}\right)\\ &\le C\|\phi^\epsilon-\phi\|_{H^s(\Omega)}\|\varphi^\epsilon\|_{H^1(\Omega)},\end{aligned} \tag{3.7}$$

for some C independent of ϵ and $1\ge s>1/2$. The sequence φ^ϵ is uniformly bounded in $H^1_\epsilon(\Omega)$, and then, uniformly bounded in $H^1(\Omega)$ for all $\epsilon\in(0,1)$. Also, $\phi^\epsilon\rightharpoonup\phi$, $w-H^1(\Omega)$, implies $\phi^\epsilon\rightharpoonup\phi$, $s-H^s(\Omega)$, for any $1>s\ge0$. Thus, due to (3.7), we obtain $I_1\to0$ as $\epsilon\to0$.

Analogously we can show that $I_2\to0$ as $\epsilon\to0$. Finally applying Lemma 3.2 in I_3 we obtain

$$\int_0^1\int_0^1\phi(x_1)\,\varphi(x_1)\,H_\epsilon(x_1)\,dx_2dx_1\to\int_0^1\mu\,\phi(x_1)\,\varphi(x_1)\,dx_1$$

where $\mu\in L^\infty(0,1)$ is defined in (2.6) completing the proof. □

Now let us study the convergence of the concentrated integrals given by nonlinear terms.

Lemma 3.6. *Assume u^ϵ, $\varphi^\epsilon \in H^1_\epsilon(\Omega)$ for each $\epsilon > 0$, and u, $\varphi \in H^1(0,1)$ satisfying $u^\epsilon \rightharpoonup u$ and $\varphi^\epsilon \rightharpoonup \varphi$, $w - H^1(\Omega)$. Then,*

$$\int_\Omega f(u^\epsilon)\, \varphi^\epsilon \, dx_1 dx_2 \to \int_0^1 p\, f(u)\, \varphi \, dx, \tag{3.8}$$

$$\frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)\, \varphi^\epsilon \, dx_1 dx_2 \to \int_0^1 \mu\, g(u)\, \varphi \, dx, \tag{3.9}$$

as $\epsilon \to 0$, where μ and p are given by (2.6). Consequently $\langle F_\epsilon(u^\epsilon), \varphi^\epsilon \rangle \to \langle F_0(u), \varphi \rangle$ as $\epsilon \to 0$.

Proof. Since f has bounded derivative, it is easy to see that $f(u^\epsilon) \to f(u)$ in $L^2(\Omega)$, and so, we have

$$\int_\Omega f(u^\epsilon)\, \varphi^\epsilon \, dx_1 dx_2 \to \int_0^1 \left(\int_{-b(x_1)}^{G(x_1)} dx_2 \right) f(u)\, \varphi \, dx_1 = \int_0^1 p\, f(u)\, \varphi \, dx,$$

proving the first convergence (3.8).

Next let us evaluate

$$\begin{aligned} \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)\, \varphi^\epsilon \, dx_1 dx_2 - \int_0^1 \mu\, g(u)\, \varphi \, dx &= \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)\, (\varphi^\epsilon - \varphi)\, dx_1 dx_2 + \int_0^1 g(u)\, (H_\epsilon(x) - \mu(x))\, \varphi \, dx \\ &+ \int_0^1 H_\epsilon(x)\, g'(\theta)\, (u^\epsilon(x_1, 0) - u(x))\, \varphi \, dx + \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g'(\theta)\, (u^\epsilon(x_1, x_2) - u^\epsilon(x_1, 0))\, \varphi \, dx_1 dx_2 \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where I_i represents the integrals in an obvious way with $u(x_1) \le \theta(x_1, x_2) \le u^\epsilon(x_1, x_2)$ in Ω. We will show that $I_i \to 0$ as $\epsilon \to 0$ for each $i = 1, 2, 3, 4$ getting the second convergence (3.9).

From Lemma 3.2 we obtain $I_2 \to 0$. Due to Lemma 3.4, we have $|I_1| \le C\, \|g\|_\infty\, \|\varphi^\epsilon - \varphi\|_{H^s(\Omega)}$ for $1/2 < s \le 1$. Hence, since $\varphi^\epsilon \rightharpoonup \varphi$, $w - H^1(\Omega)$ implies $\varphi^\epsilon \to \varphi$, $s - H^s(\Omega)$, for all $0 \le s < 1$, we get $I_1 \to 0$.

Now observe that, for any $1/2 < s \le 1$, there exists K depending only on s and Ω such that

$$|I_3| \le H_1 \|g'\|_\infty \|\varphi\|_{L^2(\Omega)} \|u^\epsilon(\cdot, 0) - u(\cdot)\|_{L^2(0,1)} \le K\, H_1 \|g'\|_\infty \|\varphi\|_{L^2(\Omega)}\, \|u^\epsilon - u\|_{H^s(\Omega)}.$$

Thus, using $u^\epsilon \to u$, $s - H^s(\Omega)$ for all $0 \le s < 1$, we also obtain $I_3 \to 0$ as $\epsilon \to 0$. Finally, since

$$\begin{aligned} |I_4| &\le \frac{\|g'\|_\infty}{\epsilon^\alpha} \int_{o_\epsilon} |\varphi(x_1)| \left(\int_0^{x_2} \left| \frac{\partial u^\epsilon}{\partial x_2}(x_1, s) \right| ds \right) dx_2 dx_1 \\ &\le \frac{\|g'\|_\infty}{\epsilon^\alpha} \int_0^1 \left(\int_{-b(x_1)}^{G(x_1)} |\varphi(x_1)| \left| \frac{\partial u^\epsilon}{\partial x_2}(x_1, s) \right| ds \right) \left(\int_{G(x_1) - \epsilon^\alpha H_\epsilon(x_1)}^{G(x_1)} dx_2 \right) dx_1 \\ &\le \|g'\|_\infty\, H_1 \|\varphi\|_{L^2(\Omega)} \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega)} \end{aligned}$$

we conclude the proof by Remark 3.1 once $\left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega)} \to 0$ as ϵ goes to zero. ∎

4. Continuity of the equilibria set

In this section we provide a proof of the main result of the paper, namely Theorem 2.1. In order to do so, we break its two assertions, concerning the upper and lower semicontinuity of the equilibria set $\mathcal{E}_\epsilon$ at $\epsilon = 0$, into Proposition 4.1 and 4.3 respectively.

We first consider the upper semicontinuity of solutions u^ϵ.

Proposition 4.1. *Let u^ϵ be a family of solutions of problem (2.7) satisfying $\|u^\epsilon\|_{L^\infty(\Omega)} \le R$ for some positive constant R independent of ϵ.*

Then there exist a subsequence, still denoted by u^ϵ, and a function $u \in H^1(\Omega)$, $\|u\|_{L^\infty(\Omega)} \le R$, depending only on the first variable, that is, $u(x_1, x_2) = u(x_1)$, solution of the problem (2.5), such that

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \ \ as \ \epsilon \to 0.$$

Proof. First, we observe that a family of solutions u^ϵ of (2.7) satisfying $\|u^\epsilon\|_{L^\infty(\Omega)} \le R$ is uniformly bounded in $H^1(\Omega)$ with respect to ϵ. We have $u^\epsilon \in \mathcal{E}_\epsilon$, if and only if

$$\int_\Omega \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_\Omega f(u^\epsilon) \varphi \, dx_1 dx_2 + \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)\, \varphi \, dx_1 dx_2, \tag{4.1}$$

for all $\varphi \in H^1(\Omega)$. Hence, taking $\varphi = u^\epsilon$ in (4.1) and using Lemma 3.4 we obtain

$$\Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|^2_{L^2(\Omega)} + \frac{1}{\epsilon^2} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|^2_{L^2(\Omega)} + \|u^\epsilon\|^2_{L^2(\Omega)} \le |\Omega|^{1/2} \sup_{|x| \le R} |f(x)| \, \|u^\epsilon\|_{L^2(\Omega)} + C \sup_{|x| \le R} |g(x)| \, \|u^\epsilon\|_{H^1(\Omega)},$$

for some $C > 0$ independent of ϵ. Thus, since $H^1(\Omega) \subset L^2(\Omega)$ with compact injection and $\|\cdot\|_{H^1(\Omega)} \le \|\cdot\|_{H^1_\epsilon(\Omega)}$ for all $\epsilon \in (0, 1)$, there exists $K(\Omega, f, g, R, C) = K > 0$, also independent of ϵ, such that

$$\|u^\epsilon\|_{L^2(\Omega)}, \ \Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|_{L^2(\Omega)} \ \text{ and } \ \frac{1}{\epsilon} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega)} \le K \quad \forall \epsilon \in (0, 1). \tag{4.2}$$

Consequently, due to (4.2) we can extract a subsequence of solutions, still denoted by u^ϵ, such that as $\epsilon \to 0$

$$u^\epsilon \rightharpoonup u, \ w - H^1(\Omega), \quad \text{ and } \quad \frac{\partial u^\epsilon}{\partial x_2} \to 0, \ s - L^2(\Omega), \tag{4.3}$$

for some $u \in H^1(\Omega)$. Moreover, it follows from (4.3) that $u(x_1, x_2) = u(x_1)$ in Ω, that is, u does not depend on x_2, and so, $u \in H^1(0, 1)$. Indeed, we have $\frac{\partial u}{\partial x_2}(x_1, x_2) = 0$ a.e. Ω, since for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u \, \frac{\partial \varphi}{\partial x_2} \, dx_1 dx_2 = \lim_{\epsilon \to 0} \int_\Omega u^\epsilon \, \frac{\partial \varphi}{\partial x_2} \, dx_1 dx_2 = - \lim_{\epsilon \to 0} \int_\Omega \frac{\partial u^\epsilon}{\partial x_2} \, \varphi \, dx_1 dx_2 = 0. \tag{4.4}$$

Now it is easy to see that u satisfies our limit problem (2.5). Using Lemma 3.6 and (4.3), we can pass to the limit in the variational formulation (4.1) obtaining

$$\int_\Omega \{ u_x \varphi_x + u \, \varphi \} \, dx_1 dx_2 = \int_0^1 p \, f(u) \, \varphi \, dx + \int_0^1 \mu \, g(u) \, \varphi \, dx,$$

whenever $\varphi \in H^1(0, 1)$. Hence, since u and φ do not depend on x_2, we have that

$$\int_0^1 p \, \{ u_x \varphi_x + u \, \varphi \} \, dx = \int_0^1 p \, f(u) \, \varphi \, dx + \int_0^1 \mu \, g(u) \, \varphi \, dx, \tag{4.5}$$

where p and μ are the functions given by (2.6). Note that (4.5) is the variational formulation of (2.5).

Next we prove strong convergence in $H^1(\Omega)$ showing convergence of the H^1-norm. For this, we also use that the norm is lower semicontinuous with respect to the weak convergence, that is,

$$\|u\|_{H^1(\Omega)} \le \liminf_\epsilon \|u^\epsilon\|_{H^1(\Omega)}. \tag{4.6}$$

Indeed, due to (4.1), (4.3), (4.5) and (4.6) we obtain

$$\begin{aligned} \int_0^1 p \, \frac{du}{dx}^2 dx = \int_\Omega |\nabla u|^2 \, dx_1 dx_2 \quad &\le \quad \liminf_{\epsilon \in (0,1)} \int_\Omega |\nabla u^\epsilon|^2 \, dx_1 dx_2 \le \limsup_{\epsilon \in (0,1)} \int_\Omega |\nabla u^\epsilon|^2 \, dx_1 dx_2 \\ &\le \quad \limsup_{\epsilon \in (0,1)} \int_\Omega \left\{ \frac{\partial u^\epsilon}{\partial x_1}^2 + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2}^2 \right\} dx_1 dx_2 \\ &\le \quad - \int_0^1 p \, u^2 \, dx + \int_0^1 \{ p \, f(u) + \mu \, g(u) \} \, u \, dx = \int_0^1 p \, \frac{du}{dx}^2 dx. \end{aligned}$$

Then $\|u^\epsilon\|_{H^1(\Omega)} \to \|u\|_{H^1(\Omega)}$, and we conclude the proof. $\square$

In order to show lower semicontinuity of equilibria set $\mathcal{E}_\epsilon$, we first discuss some properties of maps $A_\epsilon^{-1}F_\epsilon$.

Lemma 4.2. *Let $A_\epsilon^{-1}F_\epsilon$ be the maps defined in* (3.3) *and* (3.4) *for $\epsilon \in [0,1)$.*

i) *$A_\epsilon^{-1}F_\epsilon$ are compact operators for each fixed $\epsilon \geq 0$.*
ii) *$\{A_\epsilon^{-1}F_\epsilon(u^\epsilon)\}_{\epsilon\in[0,1)}$ is a pre-compact family whenever $\|u^\epsilon\|_{H^1_\epsilon(\Omega)}$ is uniformly bounded, that is, there exist a subsequence, still denoted by $A_\epsilon^{-1}F_\epsilon(u^\epsilon)$, and $u \in H^1(0,1)$, such that*

$$\|A_\epsilon^{-1}F_\epsilon(u^\epsilon) - A_0^{-1}F_0(u)\|_{H^1_\epsilon(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0.$$

iii) *If $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$ as $\epsilon \to 0$, then $\|A_\epsilon^{-1}F_\epsilon(u^\epsilon) - A_0^{-1}F_0(u)\|_{H^1_\epsilon(\Omega)} \to 0$.*

Proof. First we observe that i), for each $\epsilon > 0$ fixed, is a consequence of the continuity of $F_\epsilon : H^1_\epsilon(\Omega) \mapsto E_\epsilon^{-s}$, $1/2 > s > 1/4$, and $A_\epsilon^{-1} : E_\epsilon^{-s} \mapsto E_\epsilon^{1-s}$, as well as the compact imbedding of $H^1_\epsilon(\Omega)$ in E_ϵ^{1-s} with $1-s > 1/2$. Arguing in a similar way, we can get i) at $\epsilon = 0$.

Next we prove ii). To do so, let $u^\epsilon \in H^1_\epsilon(\Omega)$ be such that $\|u^\epsilon\|_{H^1_\epsilon(\Omega)} \leq C$. Then,

$$\Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|^2_{L^2(\Omega)} + \frac{1}{\epsilon^2}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|^2_{L^2(\Omega)} + \|u^\epsilon\|^2_{L^2(\Omega)} \leq C^2,$$

and so, we have

$$\|u^\epsilon\|_{L^2(\Omega)}, \ \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega)} \ \text{and} \ \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega)} \leq C \quad \forall \epsilon > 0.$$

Thus, arguing as in (4.3) and (4.4), we can extract a subsequence, still denoted by u^ϵ, such that

$$u^\epsilon \rightharpoonup u, \ w - H^1(\Omega), \qquad \text{and} \qquad \frac{\partial u^\epsilon}{\partial x_2} \to 0, \ s - L^2(\Omega), \tag{4.7}$$

for some $u \in H^1(0,1)$. Now, let us consider $w^\epsilon = A_\epsilon^{-1}F_\epsilon(u^\epsilon)$, which is equivalent to w^ϵ being a weak solution of $A_\epsilon w^\epsilon = F_\epsilon(u^\epsilon)$. Consequently, we obtain

$$\|w^\epsilon\|^2_{H^1_\epsilon(\Omega)} = \int_\Omega f(u^\epsilon) w^\epsilon \, dx_1 dx_2 + \frac{1}{\epsilon^\alpha}\int_{o_\epsilon} g(u^\epsilon)\, w^\epsilon \, dx_1 dx_2.$$

Hence, since we are assuming f and g bounded with bounded derivatives, see Remark 2.3, we have by Lemma 3.4 that w^ϵ is a uniformly bounded sequence in $H^1_\epsilon(\Omega)$ for $\epsilon \in (0,1)$. Then, we can argue as in (4.3) and (4.4) in order to extract a subsequence, still denoted by w^ϵ, such that

$$w^\epsilon \rightharpoonup w, \ w - H^1(\Omega), \qquad \text{and} \qquad \frac{\partial w^\epsilon}{\partial x_2} \to 0, \ s - L^2(\Omega), \tag{4.8}$$

for some $w \in H^1(0,1)$. Thus, due to (4.7), (4.8) and Lemma 3.6, we can pass to the limit in $\langle A_\epsilon w^\epsilon, \varphi\rangle = \langle F_\epsilon(u^\epsilon), \varphi\rangle$ obtaining $\langle A_0 w, \varphi\rangle = \langle F_0(u), \varphi\rangle$ for each $\varphi \in H^1(0,1)$, and so, $w = A_0^{-1}F_0(u)$. Moreover, using Lemma 3.6 again, we also get $\|w^\epsilon\|_{H^1_\epsilon(\Omega)} = \langle F_\epsilon(u^\epsilon), w^\epsilon\rangle \to \langle F_0(u), w\rangle = \langle A_0 w, w\rangle = \|w\|_{\hat{H}^1(0,1)}$ proving ii).

Now we show iii). Assuming $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$, we have $\|u^\epsilon\|_{H^1_\epsilon(\Omega)} \leq C$. Thus, arguing as in the proof of item ii), for any subsequence, we still can extract another subsequence such that $\|A_\epsilon^{-1}F_\epsilon(u^\epsilon) - A_0^{-1}F_0(u)\|_{H^1_\epsilon(\Omega)} \to 0$ as $\epsilon \to 0$, with $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$. Then, since this has been shown for any arbitrary sequence, we obtain a proof for item iii).

□

Finally we show the lower semicontinuity of the equilibria set $\mathcal{E}_\epsilon$ at $\epsilon = 0$. As we will see, it is a direct consequence of Lemma 4.2 and [36, Theorem 3].

Proposition 4.3. *Let u be a hyperbolic solution of problem* (2.5)*. Then there exists a sequence of solutions u^ϵ of problem* (2.7)*, uniformly bounded in $L^\infty(\Omega)$, such that*

$$\|u^\epsilon - u\|_{H^1(\Omega)} \to 0, \ \text{as } \epsilon \to 0.$$

Proof. First we note that u being a hyperbolic solution of (2.5) implies that u is isolated one, that is, there exists $\delta > 0$ such that u is the unique solution of (2.5) in $B(u, \delta)$, the open ball of radius δ centered at $u \in H^1_\epsilon(\Omega)$. Moreover, we have that its *fixed point index*, relatively to map $A_0^{-1} F_0$, satisfies $|\mathrm{ind}(u, A_0^{-1} F_0)| = 1$. We refer to [35] for an appropriated definition of fixed point index. Next, since the family of compact operators $A_\epsilon^{-1} F_\epsilon$ satisfies items ii) and iii) of Lemma 4.2, it follows from [36, Theorem 3] that there exists $\epsilon_0 > 0$ such that the operator $A_\epsilon^{-1} F_\epsilon$ has at least one fixed point $u^\epsilon \in B(u, \delta)$ satisfying $\|u^\epsilon - u\|_{H^1_\epsilon(\Omega)} \to 0$ as $\epsilon \to 0$. Therefore, $\|u^\epsilon - u\|_{H^1(\Omega)} \to 0$ since $\| \cdot \|_{H^1(\Omega)} \leq \| \cdot \|_{H^1_\epsilon(\Omega)}$ whenever $\epsilon \in (0, 1)$.

Now, we show that u^ϵ is uniformly bounded in $L^\infty(\Omega)$. For all $\varphi \in H^1(\Omega)$,

$$\int_\Omega \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_\Omega f(u^\epsilon) \varphi dx_1 dx_2 + \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)\, \varphi\, dx_1 dx_2. \tag{4.9}$$

Let us take $\varphi = U^\epsilon = (u^\epsilon - k)^+$ in (4.9) for some $k > 0$ where ϕ^+ denotes the positive part of a function ϕ. Thus, adding and subtracting k in an appropriated way, we obtain for any $\delta > 0$ that

$$\begin{aligned}
\|U^\epsilon\|^2_{H^1_\epsilon(\Omega)} &= \int_\Omega (f(u^\epsilon) - k)\, U^\epsilon dx_1 dx_2 + \frac{1}{\epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)\, U^\epsilon\, dx_1 dx_2 \\
&\leq \int_\Omega (f(u^\epsilon) - k)\, U^\epsilon dx_1 dx_2 + \frac{1}{\delta\, \epsilon^\alpha} \int_{o_\epsilon} g(u^\epsilon)^2\, dx_1 dx_2 + \frac{\delta}{\epsilon^\alpha} \int_{o_\epsilon} U^{\epsilon 2}\, dx_1 dx_2 \\
&\leq \int_\Omega (f(u^\epsilon) - k)\, U^\epsilon dx_1 dx_2 + \frac{(G_0 + b_0)}{\delta} \|g\|^2_\infty + C\, \delta\, \|U^\epsilon\|^2_{H^1_\epsilon(\Omega)},
\end{aligned}$$

where $C > 0$ is a constant, independent of ϵ, given by Lemma 3.4. Hence, we obtain

$$\|U^\epsilon\|^2_{H^1_\epsilon(\Omega)} (1 - C\, \delta) \leq \int_\Omega (f(u^\epsilon) - k)\, U^\epsilon dx_1 dx_2 + \frac{(G_0 + b_0)}{\delta} \|g\|^2_\infty. \tag{4.10}$$

Since f and g are bounded functions, we can choose δ small enough and k sufficiently large such that

$$(1 - C\, \delta) > 0, \quad \text{and} \quad \int_\Omega (f(u^\epsilon) - k)\, U^\epsilon dx_1 dx_2 + \frac{(G_0 + b_0)}{\delta} \|g\|^2_\infty < 0.$$

We conclude from (4.10) that

$$\|U^\epsilon\|_{H^1_\epsilon(\Omega)} = 0, \quad \text{for any } \epsilon > 0,$$

and therefore, $u^\epsilon \leq k$. Finally, we can proceed in a similar way for $-u^\epsilon$ getting the desired result. ☐

5. Numerical evidences

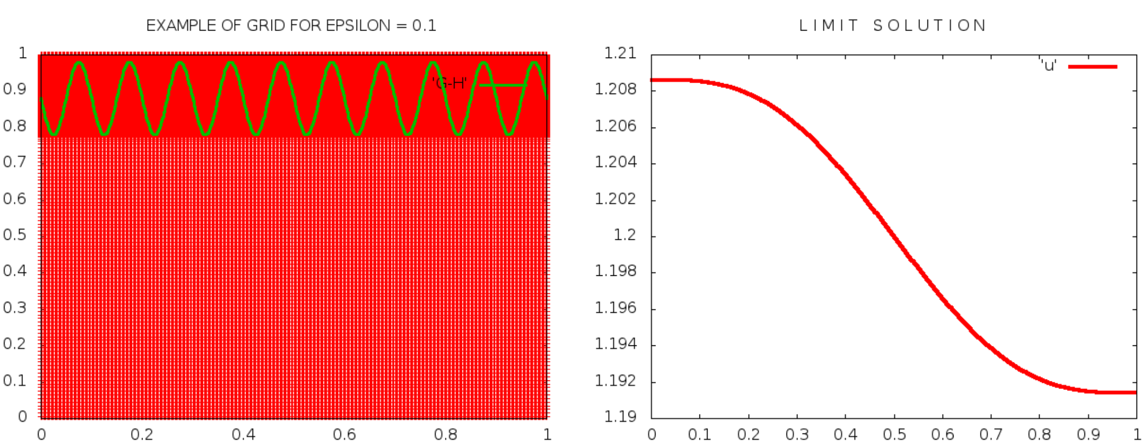

Figure 2. Example of grid with $\epsilon = 0.1$, also showing the narrow strip close to the boundary at $y = 1$. On the right, the limit solution of equation (2.5) is shown.

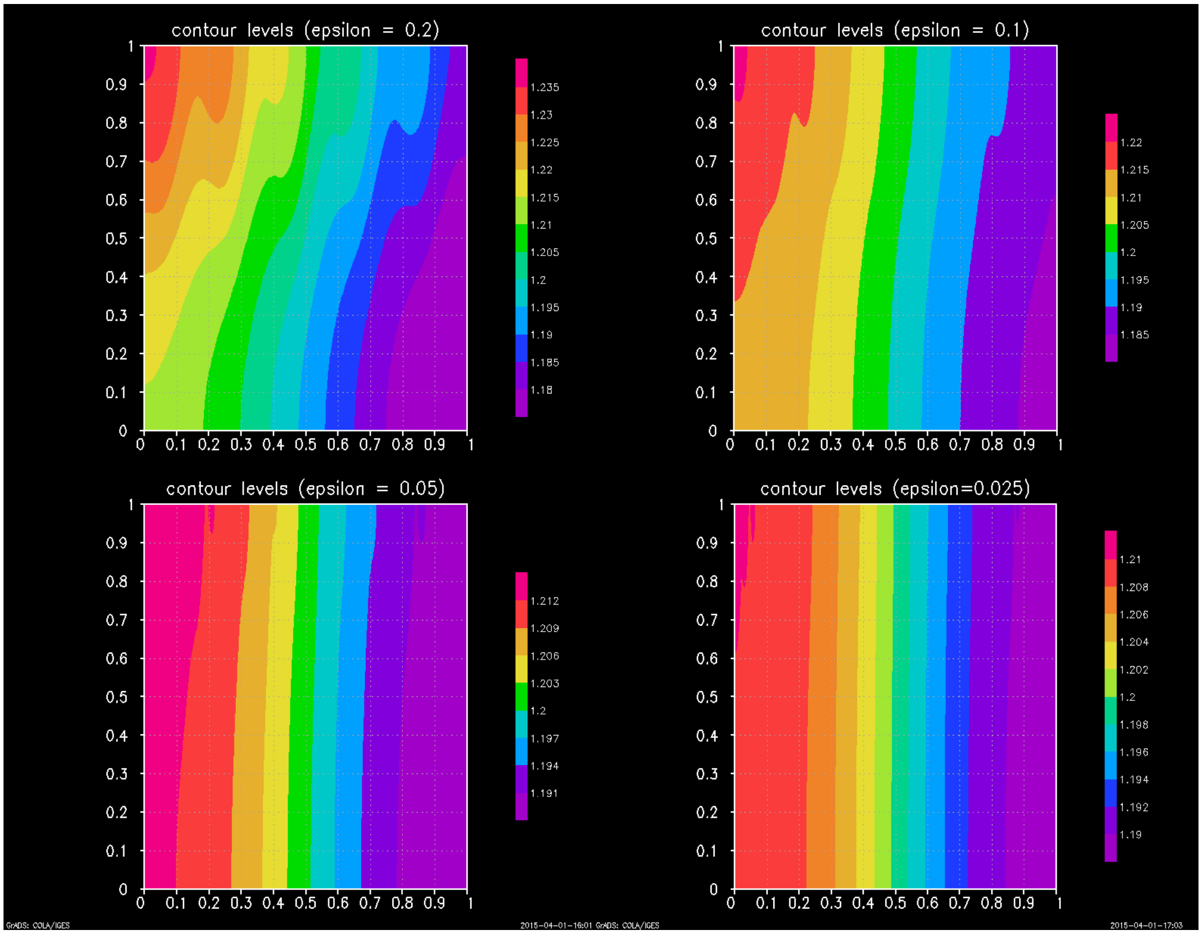

FIGURE 3. Contour levels of solutions for several values of ϵ.

In this section we present numerical results which illustrate the behavior of the solutions when the factor ϵ tends to 0. We consider the linear case with $f(u) = 0$ in equation (2.7). The domain Ω is chosen as the unit square (we use $b(x) = 0$ and $G(x) = 1$). The forcing term g is set as $g(x) = 1 + 0.1\sin(2\pi x)$ and the function H_ϵ defining the narrow strip o_ϵ is chosen as $H_\epsilon(x) = 1.2 + \sin(2\pi x/\epsilon)$. The problem has been discretized through centered finite-differences on a uniform mesh in the x-direction and non-uniform in the y-direction, with a larger concentration of grid-points over the narrow strip (see Figure 2 for a example of the grid with $\epsilon = 0.1$, where we also present the limit solution from problem (2.5)). The discretization leads to highly anisotropic linear systems to be solved, due to the ϵ^2 term. The matrices are however diagonally dominant, ensuring the convergence of simple line relaxation schemes, which we employ here.

In the following we illustrate the convergence of the solutions when ϵ tends to 0. In Figure 3 we present the contour levels of the solution for different values of ϵ. For the larger values of ϵ we can observe the two-dimensional dependence of the solutions, which still vary significantly with y due to the forcing in the boundary strip. When ϵ is reduced the dependence of the solution on y diminishes significantly. This fact can also be observed in Figure 4, where we display cuts of the solution for $y = 0.1$, $y = 0.5$ and $y = 1$. The solutions at the different y levels are converging to the same values. However, we observe that at the boundary (and close to it) the solution presents oscillations according to the oscillatory behaviour of $H_\epsilon(x)$. This is shown clearly in Figure 5, where we present a detail of the solution for $\epsilon = 0.025$. The number of oscillations increases with $1/\epsilon$, while its amplitude is decreasing with ϵ. We can also see in Figure 5 how

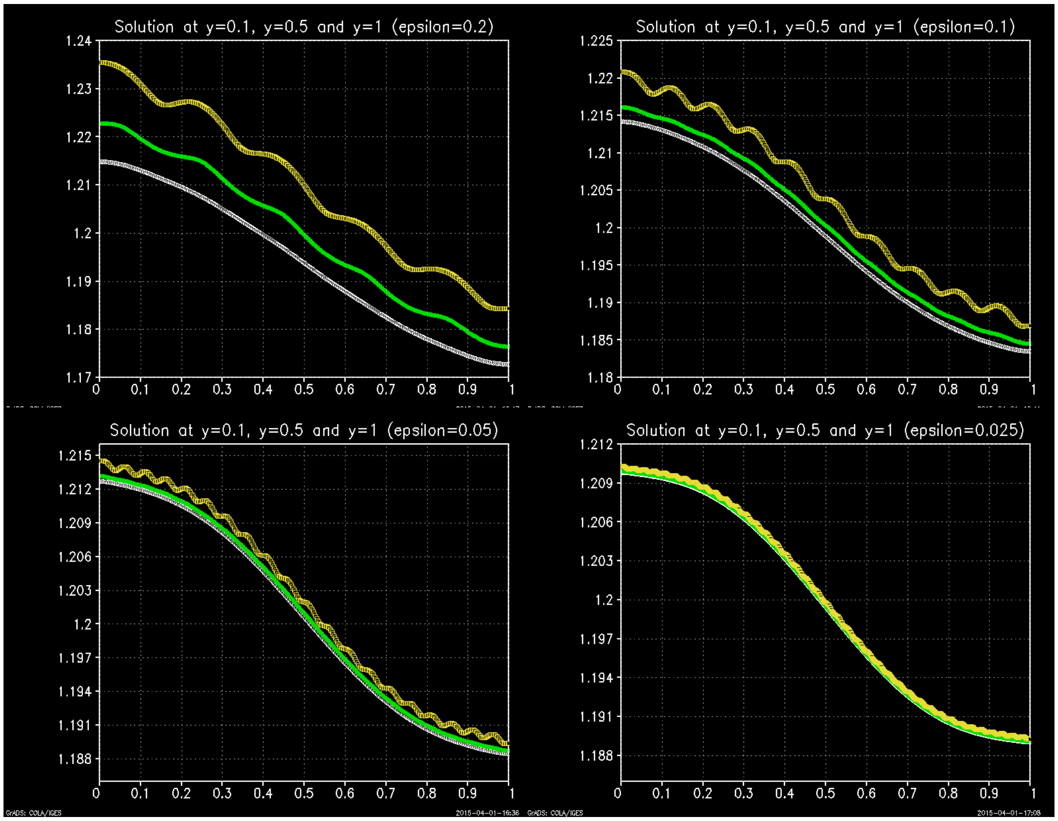

FIGURE 4. Cuts of the solution at $y = 0.1$ (in white), $y = 0.5$ (in green) and $y = 1$ (in yellow) for ϵ 0.2 (top left), 0.1 (top right), 0.05 (bottom left) and 0.025 (bottom right)

the solution at the boundary is converging to the limit solution (cf. Figure 2), with diminishing values of ϵ. We have computed the global L_2-norm of the difference of the u_ϵ solutions to the limit solution u, which does not depend on y. The computed values of this error norm were 1.03×10^{-2} for $\epsilon = 0.2$, 4.59×10^{-3} for $\epsilon = 0.1$, 2.37×10^{-3} for $\epsilon = 0.05$ and 1.13×10^{-3} for $\epsilon = 0.025$, providing evidence that the error goes to zero linearly with ϵ.

6. Final conclusion

We have shown that a family of steady state solutions of a homogeneous Neumann problem for a nonlinear reaction-diffusion equation posed in a two-dimensional thin domain converge to a certain limit problem when some reaction terms are concentrated in a small neighborhood of the boundary. In our analysis we have proved that this family converges in H^1-norm to a solution u of an one-dimension equation of the same type with distinct diffusion coefficient. We see that this coefficient depends on the profile of the thin domain. Also we get in the limiting equation a nonlinear reaction term which captures both the profile and the oscillatory behavior of the oscillating strip of the border where the reaction term takes place. We present some numerical results illustrating the convergence behavior.

An important feature here is that we are dealing with the case where the small neighborhood presents a highly oscillatory behavior and is established in a thin channel. As consequence, the limit problem is not obvious from the start. We use results developed in [11, 24] combining theories to deal with thin domains and concentrated integrals in order to obtain a rigorous strong convergence result described in Theorem 2.1.

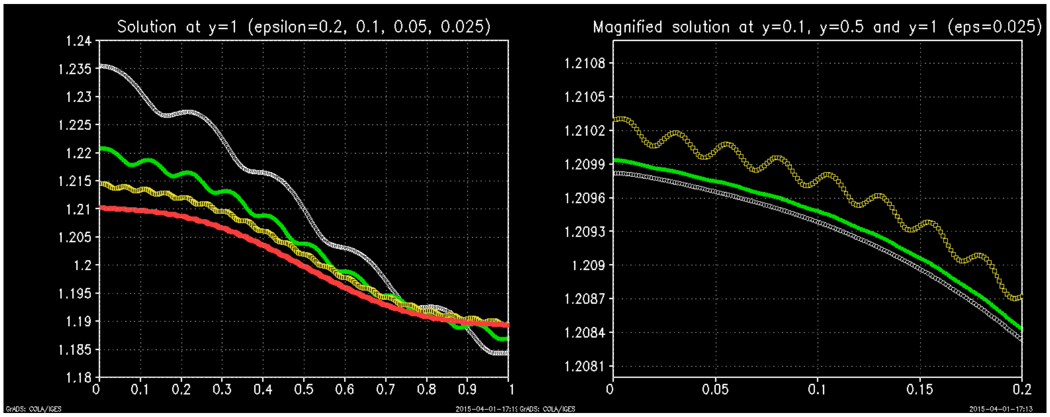

FIGURE 5. Left: Solution at the boundary (y=1) for the following values of ϵ: 0.2 (in white), 0.1 (in green), 0.05 (in yellow) and 0.025 (in red). Right: Detail of the solution for $\epsilon = 0.025$ in the x range from 0 to 0.2 (cuts at t $y = 0.1$ (in white), $y = 0.5$ (in green) and $y = 1$ (in yellow).

A natural question is whether such approximation results can be improved in order to describe the asymptotic behavior of the Dynamical System generated by the parabolic equation associated with (2.7) posed in more general thin regions of $\mathbb{R}^N$. It is our goal to investigate this question in a forthcoming paper.

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