Roughness-induced effects on the convection-diffusion-reaction problem in a thin domain

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Abstract

In this paper, we investigate a convection-diffusion-reaction problem in a thin domain endowed with the Robin-type boundary condition describing the reaction catalyzed by the upper wall. Motivated by the microfluidic applications, we allow the oscillating behavior of the upper boundary and analyze the resonant case where the amplitude and period of the oscillation have the same small order as the domain's thickness. Depending on the magnitude of the reaction mechanism, we rigorously derive three different asymptotic models via the unfolding operator method. In particular, we identify the critical case in which the effects of the domain's geometry and all physically relevant processes become balanced.

Key words. convection-diffusion-reaction equation; Robin boundary condition; thin domain; rough boundary; unfolding method.

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1 Introduction

The flow problems posed in thin domains (domains whose longitudinal dimension is much larger than the transverse one) are of great interest due to their practical importance. In real-life applications, the boundary of such domains are usually not perfectly smooth, i.e. they usually have some small rugosities, dents, etc. In solid mechanics, the typical examples of such structures would be thin rods, plates or shells. Lubrication devices and blood circulatory system are the obvious examples associated to fluid mechanics. No matter the context is, introducing the small parameter as the perturbation quantity in the domain boundary makes the analysis very challenging from the mathematical point of view.

Motivated by the numerous applications in which the effective flow is significantly affected by the irregular wall roughness, we suppose that the upper boundary of our thin domain has an oscillating behavior. Namely, the considered domain reads:

$$R^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}, \quad 0 < \varepsilon \ll 1. \tag{1}$$

In the sequel, we address the following elliptic boundary-value problem:

$$\begin{cases} -\kappa\Delta u^\varepsilon + Q^\varepsilon(y)\partial_x u^\varepsilon + cu^\varepsilon = f^\varepsilon \text{ in } R^\varepsilon, \\ \kappa\dfrac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \varepsilon^\alpha(g(x) - u^\varepsilon) \text{ on } \Gamma^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = \varepsilon h\left(\dfrac{x}{\varepsilon}\right)\right\}, \\ \dfrac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = 0 \text{ on } \partial R^\varepsilon \backslash \Gamma^\varepsilon. \end{cases} \tag{2}$$

Here $\kappa, c = \text{const.} > 0$, the vector $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and $\frac{\partial}{\partial \nu^\epsilon}$ is the outside normal derivative. For the function Q^ε, we assume

$$Q^\varepsilon(y) = Q\left(\frac{y}{\varepsilon}\right),$$

where $Q \in L^\infty(0, h_1)$ is a non-negative function, $h_1 = \max_{x\in\mathbb{R}} h(x)$. This assumption is reasonable from the point of view of the applications, since we are tackling the process in a thin domain and Q^ε can be interpreted as the entering (unidirectional) velocity in e.g. the solute transport problem (see [12, 13]). The boundary perturbation function h satisfies the usual assumptions listed in $(\mathbf{H_h})$, see Section 2.1. Finally, we suppose $g \in L^2(0,1)$. As you can see, the governing equation is endowed with the Robin-type boundary condition which models the reaction catalyzed by the upper wall. By taking the reaction coefficient in the form ε^α, $\alpha > 0$ (see $(2)_2$), our aim is to address different order of magnitudes of the prescribed reaction mechanism. Such type of elliptic boundary-value problems describes many processes naturally arising in chemical engineering, in particular related to microfluidic applications (see e.g. [18]). Our goal is to study the asymptotic behavior of the described problem, as $\varepsilon \to 0$.

To achieve our goal, we employ the homogenization technique based on the unfolding method proposed in [9, 10]. Due to its ability to elegantly treat the surface integrals, the unfolding method has been extensively used for derivation of lower-dimensional approximations in the last period. We refer the reader e.g. to [1, 3, 7, 15]. In this work, we adapt the variant of this method introduced by Arrieta and Villanueva-Pesqueira [5, 6] for thin domains. As a result, we obtain three different asymptotic models, depending on the value of the coefficient α. More precisely, for $0 < \alpha < 1$, the process turns out to be dominated by the function g from the Robin boundary condition, with $g \in H^1(0,1)$ (see Theorem 3.2). For $\alpha > 1$, the effective model does not depend on g (see Theorem 3.3), meaning that the reaction mechanism does not affect the process. Between those two cases, we identify the critical (and the most interesting) case $\alpha = 1$ capturing the effects of the domain's geometry and all the physical processes relevant to the problem as well (see Theorem 3.1). We firmly believe that the results presented here could prove useful in numerical simulations of the convection-diffusion-reaction problems in thin domains with irregularities.

To conclude the Introduction, let us provide more bibliographic remarks on the subject. In [8], the Neumann problem for the Laplace equation posed in a domain (of thickness $\mathcal{O}(1)$) with highly oscillating boundary has been considered via asymptotic expansion method. Using rigorous analysis in appropriate functional setting, a thin-domain situation has been addressed in [4]. It should be emphasized that, in both papers, a homogeneous Neumann boundary condition has been imposed and that the transition to a Robin-type boundary condition cannot be considered straightforward whatsoever. The present work can be viewed as the continuation of our recent work [14] in which a thin domain without boundary oscillations has been studied. Notice that introducing boundary irregularities to the problem forced us to completely change the approach.

2 Preliminary results

In this section, we state some basic results relate to our problem by introducing the functional setting as well as the unfolding method.

First we notice that the variational formulation of problem (2) reads:

$$\begin{aligned}\int_{R^\varepsilon} \kappa \nabla u^\varepsilon \nabla \varphi + Q^\varepsilon(y)\partial_x u^\varepsilon \varphi + c u^\varepsilon \varphi dxdy + \varepsilon^\alpha \int_{\Gamma^\varepsilon} u^\varepsilon \varphi dS \\ = \int_{R^\varepsilon} f^\varepsilon \varphi dxdy + \varepsilon^\alpha \int_{\Gamma^\varepsilon} g\varphi dS\,, \qquad \forall \varphi \in H^1(R^\varepsilon).\end{aligned} \tag{3}$$

The existence and uniqueness of the solution in $H^1(R^\varepsilon)$ is a direct consequence of Stampacchia Theorem and the assumption

$$c > ||Q||^2_{L^\infty(0,h_1)}/4\kappa \tag{4}$$

(see e.g. [14, Lemma 3.4]). Thus, we have a family of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ given by problem (3), and we are concerned here about the asymptotic behavior of this sequence, as ε goes to zero.

2.1 The unfolding operator

In order to study the convergence of the solutions u^ε, we apply the unfolding method firstly introduced in [9, 10] for oscillating coefficients and perforated domains. Here, we just give some notations and recall the main results concerning this method in the thin domain situation. The proofs and all the details can be found in [5, 6].

We consider two-dimensional thin domains defined by (1). Notice that these regions can present an oscillatory behavior at its top boundary since we are taking positive parameters ε and α, as well as the function h satisfying the following hypothesis:

$(\mathbf{H_h})$ *$h : \mathbb{R} \to \mathbb{R}$ is a strictly positive, Lipschitz continuous and L-periodic. Moreover, if we set*

$$h_0 = \min_{x \in \mathbb{R}} h(x) \quad \text{and} \quad h_1 = \max_{x \in \mathbb{R}} h(x)$$

we have $0 < h_0 \le h(x) \le h_1$ for all $x \in \mathbb{R}$.

Remark 2.1. *Here we are assuming h to be smooth due to the trace operator from $H^1(R^\varepsilon)$ into $L^2(\Gamma^\varepsilon)$. As we will see in the Section 2.2, enough smoothness is needed on the boundary in order to introduce an unfolding operator on the border.*

Throughout this paper, we use the following notations. We call Y^* the representative cell of the thin domain R^ε which is given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < g(y_1)\}. \tag{5}$$

The average of $\varphi \in L^1_{loc}(\mathbb{R}^2)$ on a measure set $\mathcal{O} \subset \mathbb{R}^2$ is denoted by

$$\langle \varphi \rangle_{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \varphi(x)\, dx\,,$$

where $|\mathcal{O}|$ sets the Lebesgue measure of any measure set $\mathcal{O}$.

We will also need to consider the following functional spaces which are defined by periodic functions in the variable $y_1 \in (0, L)$. Namely,

$$L^2_\#(Y^*) = \{\varphi \in L^2(Y^*) \,:\, \varphi(y_1, y_2) \text{ is } L\text{-periodic in } y_1 \},$$
$$L^2_\#\left((0,1) \times Y^*\right) = \{\varphi \in L^2((0,1) \times Y^*) \,:\, \varphi(x, y_1, y_2) \text{ is } L\text{-periodic in } y_1 \},$$
$$H^1_\#(Y^*) = \{\varphi \in H^1(Y^*) \,:\, \varphi|_{\partial_{left} Y^*} = \varphi|_{\partial_{right} Y^*}\}.$$

If we denote by $[a]_L$ the unique integer number such that $a = [a]_L L + \{a\}_L$ where $\{a\}_L \in [0, L)$, then for each $\varepsilon > 0$ and any $x \in \mathbb{R}$, we have

$$x = \varepsilon \left[\frac{x}{\varepsilon}\right]_L L + \varepsilon \left\{\frac{x}{\varepsilon}\right\}_L, \quad \text{where } \left\{\frac{x}{\varepsilon}\right\}_L \in [0, L).$$

Let us also denote

$$I_\varepsilon = \text{ Int } \left(\bigcup_{k=0}^{N_\varepsilon} [kL\varepsilon, (k+1)L\varepsilon] \right),$$

where N_ε is the largest integer such that $\varepsilon L(N_\varepsilon + 1) \le 1$. We also set

$$\Lambda_\varepsilon = (0,1) \backslash I_\varepsilon = [\varepsilon L(N_\varepsilon + 1), 1),$$
$$R^\varepsilon_0 = \left\{ (x, y) \in \mathbb{R}^2 : x \in I_\varepsilon, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right) \right\},$$
$$R^\varepsilon_1 = \left\{ (x, y) \in \mathbb{R}^2 : x \in \Lambda_\varepsilon, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right) \right\}.$$

Notice that we have $\Lambda_\varepsilon = \emptyset$ if $\varepsilon^\alpha L(N_\varepsilon + 1) = 1$. In this case $R^\varepsilon_0 = R^\varepsilon$ and $R^\varepsilon_1 = \emptyset$.

Definition 2.1. *Let φ be a Lebesgue-measurable function in R^ε. The unfolding operator $\mathcal{T}_\varepsilon$ acting on φ is defined as the following function in $(0,1) \times Y^*$:*

$$\mathcal{T}_\varepsilon \varphi(x, y_1, y_2) = \begin{cases} \varphi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_L L + \varepsilon y_1, \varepsilon y_2\right), \textit{ for } (x, y_1, y_2) \in I_\varepsilon \times Y^*, \\ 0, \ \textit{for } (x, y_1, y_2) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Proposition 2.2. *The unfolding operator satisfies the following properties:*

1. $\mathcal{T}_\varepsilon$ *is linear;*

2. $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)$, *for all* φ, ψ *Lebesgue mesurable in* R^ε*;*

3. *Let* φ *a Lebesgue mesurable function in* Y^* *extended periodically in the first variable. Then,* $\varphi^\varepsilon(x,y) = \varphi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ *is mesurable in* R^ε *and*
$$\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x, y_1, y_2) = \varphi(y_1, y_2), \forall (x, y_1, y_2) \in I_\varepsilon \times Y^*.$$
Moreover, if $\varphi \in L^2(Y^*)$*, then* $\varphi^\varepsilon \in L^2(R^\varepsilon)$*;*

4. *Let* $\varphi^\varepsilon \in L^1(R^\varepsilon)$*. Then,*
$$\begin{aligned}\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\varepsilon(\varphi)(x, y_1, y_2)dxdy_1dy_2 &= \frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi(x,y)dxdy \\ &= \frac{1}{\varepsilon}\int_{R^\varepsilon} \varphi(x,y)dxdy - \frac{1}{\varepsilon}\int_{R_1^\varepsilon} \varphi(x,y)dxdy;\end{aligned}$$

5. $\forall \varphi \in L^2(R^\varepsilon)$, $T_\varepsilon(\varphi) \in L^2\left((0,1)\times Y^*\right)$*. Moreover*
$$||\mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} = \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\varphi||_{L^2(R_0^\varepsilon)} \leq \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\varphi||_{L^2(R^\varepsilon)}.$$

6. $\forall \varphi \in H^1(R^\varepsilon)$,
$$\partial_{y_1} T_\varepsilon(\varphi) = \varepsilon T_\varepsilon(\partial_x \varphi) \text{ and } \partial_{y_2} T_\varepsilon(\varphi) = \varepsilon T_\varepsilon(\partial_y \varphi) \text{ a.e. in } (0,1)\times Y^*;$$

7. *If* $\varphi \in H^1(R^\varepsilon)$*, then* $\mathcal{T}_\varepsilon(\varphi) \in L^2\left((0,1); H^1(Y^*)\right)$*. Besides,*
$$||\partial_{y_1}\mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} = \varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_x\varphi||_{L^2(R_0^\varepsilon)} \leq \varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_x\varphi||_{L^2(R^\varepsilon)},$$
$$||\partial_{y_2}\mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} = \varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_y\varphi||_{L^2(R_0^\varepsilon)} \leq \varepsilon\left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_y\varphi||_{L^2(R^\varepsilon)}.$$

Due to the order of the height of the thin domain, a factor $1/\varepsilon$ appears in Properties 5 and 6. Consequently, it makes sense to consider the following rescaled Lebesgue measure
$$\rho_\varepsilon(\mathcal{O}) = \varepsilon^{-1}|\mathcal{O}|$$
which is widely considered in works involving thin domains. See for instance [16, 17] and the references therein. Indeed, from now on, we will use the following rescaled norms in the thin open sets
$$\begin{aligned}|||\varphi|||_{L^2(R^\varepsilon)} &= \varepsilon^{-1/2}\,||\varphi||_{L^2(R^\varepsilon)}, \forall \varphi \in L^2(R^\varepsilon), \\ |||\varphi|||_{H^1(R^\varepsilon)} &= \varepsilon^{-1/2}\,||\varphi||_{H^1(R^\varepsilon)} \forall \varphi \in H^1(R^\varepsilon).\end{aligned}$$

From Property 6, we have
$$||\mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} \leq L^{1/2}\,|||\varphi|||_{L^2(R^\varepsilon)}.$$

Property 5 of Proposition 2.2 is essential to pass to the limit since it will allow us to transform any integral over the thin domain R^ε into an integral over the fixed set $(0,1)\times Y^*$. In this way, an important concept for the unfolding method is the following property called unfolding criterion for integrals (u.c.i.).

Definition 2.3. *A sequence (φ_ε) satisfies the unfolding criterion for integrals (u.c.i) if*

$$\frac{1}{\varepsilon}\int_{R_1^\varepsilon} |\varphi_\varepsilon| dxdy \to 0.$$

It is known that any sequence $(\varphi_\varepsilon) \subset L^2(R^\varepsilon)$ with norm $|||\cdot|||_{L^2(R^\varepsilon)}$ uniformly bounded satisfies the (u.c.i). Moreover, if we have (ψ_ε) set as

$$\psi_\varepsilon(x,y) = \psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

for any $\psi \in L^2(Y^*)$, then $(\varphi_\varepsilon \psi_\varepsilon)$ also satisfies (u.c.i).

Now, we recall some convergence properties of the unfolding operator as ε goes to zero.

Theorem 2.4. *For a measurable function f on Y^*, L-periodic in its first variable and extended by periodicity to $\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, 0 < y < g(x)\}$, define the sequence (f_ε) by*

$$f^\varepsilon(x,y) = f\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \quad a.e. \quad (x,y) \in \left\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}.$$

Then

$$\mathcal{T}_\varepsilon f^\varepsilon|_{(0,1)}(x, y_1, y_2) = \begin{cases} f(y_1, y_2), \text{ for } (x, y_1, y_2) \in I_\varepsilon \times Y^*, \\ 0, \text{ for } (x, y_1, y_2) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Moreover, if $f \in L^2_\#(Y^*)$, then $\mathcal{T}_\varepsilon f^\varepsilon \to f$ strongly in $L^2_\#\left((0,1) \times Y^*\right)$.

We also have the following convergence results:

Corollary 2.4.1. *Under previous assumptions we have:*

1. *Let $f \in L^2\left((0,1); L^p_\#(Y^*)\right)$ and extend it periodically in y_1-direction. If we set*

$$f^\varepsilon(x,y) := f\left(x, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \in L^2(R^\varepsilon), \tag{6}$$

   then $\mathcal{T}_\varepsilon f^\varepsilon \to f$ strongly in $L^2\left((0,1) \times Y^*\right)$.

2. *Let $\varphi \in L^2(0,1)$. Then, $\mathcal{T}_\varepsilon \varphi \to \varphi$ strongly in $L^2\left((0,1) \times Y^*\right)$.*

3. *Let (φ_ε) be a sequence in $L^2(0,1)$ such that $\varphi_\varepsilon \to \varphi$ strongly in $L^2(0,1)$. Then, $\mathcal{T}_\varepsilon \varphi_\varepsilon \to \varphi$ strongly in $L^2\left((0,1) \times Y^*\right)$.*

Next, we give a suitable decomposition for functions in $H^1(R^\varepsilon)$ in order to introduce other convergence results needed in the sequel. As we can see, the geometry of the thin domain plays a crucial role here. We write

$$\varphi(x,y) = V(x) + \varphi_r(x,y),$$

where V is set as

$$V(x) := \frac{1}{\varepsilon g_0}\int_0^{\varepsilon g_0} \varphi(x,s)\, ds \qquad \text{a.e. } x \in (0,1). \tag{7}$$

We define

$$\varphi_r(x,y) \equiv \varphi(x,y) - V(x).$$

Proposition 2.5. *Let $\varphi_\varepsilon \in H^1(R^\varepsilon)$ with $|||\varphi_\varepsilon|||_{H^1(R^\varepsilon)}$ uniformly bounded and $V_\varepsilon(x)$ defined as in (7). Then, there exists a function $\varphi \in H^1(0,1)$ such that, up to subsequences*

$$\begin{aligned}
&V_\varepsilon \rightharpoonup \varphi \text{ weakly in } H^1(0,1) \text{ and strongly in } L^2(0,1),\\
&\mathcal{T}_\varepsilon V_\varepsilon \to \varphi \text{ strongly in } L^2\left((0,1)\times Y^*\right),\\
&|||\varphi_\varepsilon - V_\varepsilon|||_{L^2(R^\varepsilon)} \to 0,\\
&|||\varphi_\varepsilon - \varphi|||_{L^2(R^\varepsilon)} \to 0,\\
&\mathcal{T}_\varepsilon \varphi_\varepsilon \to \varphi \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right).
\end{aligned}$$

Furthermore, there exists $\overline{\varphi} \in L^2\left((0,1)\times Y^*\right)$ with $\partial_{y_2}\overline{\varphi} \in L^2\left((0,1)\times Y^*\right)$ such that, up to subsequences

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\varphi_r^\varepsilon) \rightharpoonup \overline{\varphi} \text{ weakly in } L^2\left((0,1)\times Y^*\right),\\
&\mathcal{T}_\varepsilon(\partial_y \varphi_\varepsilon) \rightharpoonup \partial_{y_2}\overline{\varphi} \text{ weakly in } L^2\left((0,1)\times Y^*\right),
\end{aligned}$$

where $\varphi_r^\varepsilon \equiv \varphi_\varepsilon - V_\varepsilon$.

Finally, we recall a compactness result which allows us to identify the limit of the image of the gradient of uniformly bounded sequences.

Theorem 2.6. *Let $\varphi_\varepsilon \in H^1(R^\varepsilon)$ with $|||\varphi_\varepsilon|||_{H^1(R^\varepsilon)}$ uniformly bounded. Then, there exist $\varphi \in H^1(0,1)$ and $\varphi_1 \in L^2((0,1); H^1_\#(Y^*))$ such that (up to a subsequence)*

$$\begin{aligned}
&\mathcal{T}_\varepsilon \varphi_\varepsilon \to \varphi \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right),\\
&\mathcal{T}_\varepsilon \partial_x \varphi_\varepsilon \rightharpoonup \partial_x \varphi + \partial_{y_1}\varphi_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right),\\
&\mathcal{T}_\varepsilon \partial_y \varphi_\varepsilon \rightharpoonup \partial_{y_2}\varphi_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned}$$

2.2 Boundary unfolding

In this section, we set the unfolding operator on the oscillating upper boundary of R^ε. For this sake, we adapt the one introduced in [9, 10] yielding the appropriated results to our case. Notice that under assumptions $(\mathbf{H_h})$ we have that Γ^ε is a Lipschitz border.

Definition 2.7. *Let ϕ be a measurable function on Γ_ε. The boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined by*

$$\mathcal{T}_\varepsilon^b \phi(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]L + \varepsilon y\right) & \text{a.e for } I_\varepsilon \times \partial_u Y^*,\\ 0 \quad \text{a.e for } \Lambda_\varepsilon \times \partial_u Y^* \end{cases}$$

where $\partial_u Y^*$ is the upper boundary of the representative cell Y^* given by

$$\partial_u Y^* = \{(y_1,y_2) \in \mathbb{R}^2 \ : \ y_1 \in (0,L), \text{ and } y_2 = h(y_1)\}.$$

Proposition 2.8. *The boundary unfolding satisfies the following properties:*

1. $\mathcal{T}^b_\varepsilon$ *is linear;*
2. $\mathcal{T}^b_\varepsilon(\varphi\psi) = \mathcal{T}^b_\varepsilon(\varphi)\mathcal{T}^b_\varepsilon(\psi)$, *for all* φ, ψ *Lebesgue measurable in* Γ^ε*;*
3. *For any* $\varphi \in L^1(R^\varepsilon)$,
$$\frac{1}{L}\int_{(0,1)\times\partial_u Y^*} \mathcal{T}^b_\varepsilon\varphi(x,y)dxd\sigma(y) = \int_{\Gamma^\varepsilon_0}\varphi dS = \int_{\Gamma^\varepsilon}\varphi dS - \int_{\Gamma^\varepsilon_1}\varphi dS, \tag{8}$$
where Γ^ε_i *is the upper boundary of* R^ε_i *for* $i=0,1$.
4. *Suppose that* $\varphi\in L^2(\Gamma^\varepsilon)$*. Then,*
$$||\mathcal{T}^b_\varepsilon\varphi||_{L^2((0,1)\times\partial_u Y^*)} \le \frac{1}{L}||\varphi||_{L^2(\Gamma^\varepsilon)}.$$
5. *(Unfolding criterion for integrals) Suppose that* $\varphi_\varepsilon \in L^2(\Gamma_\varepsilon)$ *is such that* $||\varphi_\varepsilon||_{L^2(\Gamma^\varepsilon)} \le c$*, with* c *independent on* ε*. Then,*
$$\int_{\Gamma^\varepsilon_1}|\varphi|dS \to 0.$$
6. *Let* $\psi_\varepsilon \in H^1(R^\varepsilon)$ *such that* $\mathcal{T}_\varepsilon\psi_\varepsilon \rightharpoonup \widehat{\psi}$ *in* $L^2((0,1);H^1(Y^*))$ *with* $\widehat{\psi}\in H^1(0,1)$*. Then,*
$$\mathcal{T}^b_\varepsilon\psi_\varepsilon \rightharpoonup \widehat{\psi} \quad in \quad L^2((0,1);H^{\frac{1}{2}}(\partial_u Y^*)).$$

Proof. It is not difficult to see that Properties 1, 2 and 4 follow immediately from the definition of the boundary unfolding operator. We discuss the remaining ones.

3.: Indeed,

$$\begin{aligned}\int_{(0,1)\times\partial_u Y^*}\mathcal{T}^b_\varepsilon\varphi(x,y)dxdS &= \sum_{k=0}^{N_\varepsilon-1}\int_{kL\varepsilon}^{(k+1)L\varepsilon}\int_{\partial_u Y^*}\varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]L+\varepsilon y\right)d\sigma(y)dx\\ &= \sum_{k=0}^{N_\varepsilon-1}L\varepsilon\int_{\partial_u Y^*}\varphi\left(\varepsilon kL+\varepsilon y\right)d\sigma(y)dx. \end{aligned}\tag{9}$$

Notice that using the change of variables

$$s = \varepsilon kL + \varepsilon y,$$

we get $d\sigma(s) = \varepsilon d\sigma(y)$. Also, denoting by

$$\Gamma^\varepsilon_0 = \left\{(x,y) : 0 < x \le L\varepsilon,\, y = \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}$$

and for $k \ge 1$

$$\Gamma^\varepsilon_k = \left\{(x,y) : kL\varepsilon \le x \le (k+1)L\varepsilon,\, y = \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\},$$

it is clear that

$$\bigcup_{k=0}^{N_\varepsilon-1}\Gamma^\varepsilon_k = \Gamma^\varepsilon_0.$$

Then, by (9) we get

$$\sum_{k=0}^{N_\varepsilon-1} L\int_{\Gamma_\varepsilon^k}\varphi(s)d\sigma(s)=\int_{\Gamma_0^\varepsilon}\varphi dS. \tag{10}$$

5.: Let $\varphi_\varepsilon\in L^2(\Gamma^\varepsilon)$ such that $||\varphi_\varepsilon||_{L^2(\Gamma^\varepsilon)}\le c$, with c independent on ε. Then

$$\int_{\Gamma_1^\varepsilon}|\varphi|dS\le|\Gamma_1^\varepsilon|^{1/2}||\varphi||_{L^2(\Gamma^\varepsilon)}\le c|\Gamma_1^\varepsilon|^{1/2}\to 0.$$

6.: Straightforward from the definition of the unfolding operators and Sobolev injections.

□

3 Main results

In this Section, we prove the main results of the paper. As emphasized in the Introduction, the asymptotic behavior of the considered problem greatly depends on the value of the coefficient α appearing in the Robin boundary condition $(2)_2$. First we analyze the critical case $\alpha=1$. After that, we address two remaining characteristic cases $0<\alpha<1$ and $\alpha>1$, respectively.

3.1 Case $\alpha = 1$

Theorem 3.1. *Let u^ε be the solution of the problem (2) with $f^\varepsilon\in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Also, assume that there exists $\hat f\in L^2((0,1)\times Y^*)$ such that*

$$\mathcal{T}_\varepsilon f^\varepsilon\rightharpoonup\hat f\quad\text{weakly in } L^2((0,1)\times Y^*).$$

Then, there exist $u\in H^1(0,1)$ and $u_1\in L^2((0,1);H^1_\#(Y^*))$ such that

$$\begin{aligned}
\mathcal{T}_\varepsilon u_\varepsilon &\to u \text{ strongly in } L^2\left((0,1);H^1(Y^*)\right),\\
\mathcal{T}_\varepsilon\partial_x u_\varepsilon &\rightharpoonup \partial_x u+\partial_{y_1}u_1 \text{ weakly in } L^2((0,1)\times Y^*),\\
\mathcal{T}_\varepsilon\partial_y u_\varepsilon &\rightharpoonup \partial_{y_2}u_1 \text{ weakly in } L^2((0,1)\times Y^*).
\end{aligned}$$

Moreover, we have that u is the solution of

$$\begin{cases} -\kappa q u_{xx}+pu_x+\left(c+\dfrac{|\partial_u Y^*|}{|Y^*|}\right)u=\bar f+\dfrac{|\partial_u Y^*|}{|Y^*|}g & \text{in } (0,1),\\ u_x(0)=u_x(1)=0, \end{cases}$$

where the homogenized coefficients q and p are given by

$$q=\frac{1}{|Y^*|}\int_{Y^*}(1-\partial_{y_1}X)\,dY\qquad\text{and}\qquad p=\frac{1}{|Y^*|}\int_{Y^*}Q\,(1-\partial_{y_1}X)\,dY$$

and $X\in H^1_\#(Y^*)$ with $\int_{Y^*}XdY=0$ is the unique solution of

$$\int_{Y^*}\nabla X\nabla\varphi dY=\int_{Y^*}\partial_{y_1}\varphi dY\qquad\forall\varphi\in H^1_\#(Y^*).$$

Also, the forcing term $\bar f$ is given by

$$\bar f=\frac{1}{|Y^*|}\int_{Y^*}\hat f dY.$$

Proof. *(a)* Uniform bounds.

Take $\varphi = u^\varepsilon$ as a test function in (3). By [14, Lemma 3.4] and the assumption (4), one gets

$$|||u^\varepsilon|||^2_{H^1(R^\varepsilon)} + ||u^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \le |||f^\varepsilon|||_{L^2(R^\varepsilon)}|||u^\varepsilon|||_{L^2(R^\varepsilon)} + ||g||_{L^2(\Gamma^\varepsilon)}||u^\varepsilon||_{L^2(\Gamma^\varepsilon)}. \tag{11}$$

Thus, since we have from [11, 16] that

$$||\varphi||_{L^2(\Gamma^\varepsilon)} \le C\varepsilon^{-1/2}\|\varphi\|_{H^1(R^\varepsilon)}, \tag{12}$$

we can get that there exists $c > 0$, independent of $\varepsilon > 0$, such that

$$|||u^\varepsilon|||_{H^1(R^\varepsilon)} \le c.$$

Then, u^ε is uniformly bounded in $|||\cdot|||_{H^1}$.

(b) Limiting problem.

Let us apply Propositions 2.2 and 2.8 in (3). Then,

$$\begin{aligned}
\int_{(0,1)\times Y^*} k\mathcal{T}_\varepsilon\nabla u^\varepsilon\mathcal{T}_\varepsilon\nabla\varphi + \mathcal{T}_\varepsilon Q^\varepsilon\mathcal{T}_\varepsilon\partial_x u^\varepsilon\mathcal{T}_\varepsilon\varphi + c\mathcal{T}_\varepsilon u^\varepsilon\varphi dxdY + \int_{(0,1)\times\partial_u Y^*}\mathcal{T}^b_\varepsilon u^\varepsilon\mathcal{T}^b_\varepsilon\varphi d\sigma(y) \\
+\frac{L}{\varepsilon}\int_{R_1^\varepsilon}\kappa\nabla u^\varepsilon\nabla\varphi + Q^\varepsilon(y)\partial_x u^\varepsilon\varphi + cu^\varepsilon\varphi dxdy + L\int_{\Gamma_1^\varepsilon} u^\varepsilon\varphi dS \\
= \int_{(0,1)\times Y^*}\mathcal{T}_\varepsilon f^\varepsilon\mathcal{T}_\varepsilon\varphi dxdY + \int_{(0,1)\times\partial_u Y^*}\mathcal{T}^b_\varepsilon g\mathcal{T}^b_\varepsilon\varphi dxd\sigma(y) \\
+\frac{L}{\varepsilon}\int_{R_1^\varepsilon} f^\varepsilon\varphi dxdy + L\int_{\Gamma_1^\varepsilon} g\varphi dS,
\end{aligned} \tag{13}$$

for any $\varphi \in H^1(R^\varepsilon)$.

Since we have uniform bounds for the solutions of (2) in the $|||.|||_{H^1(R^\varepsilon)}$ norm, we can apply Theorem 2.6. Thus, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned}
\mathcal{T}_\varepsilon u_\varepsilon \to u \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right), \\
\mathcal{T}_\varepsilon\partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1}u_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right), \\
\mathcal{T}_\varepsilon\partial_y u_\varepsilon \rightharpoonup \partial_{y_2}u_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned} \tag{14}$$

Also, by Proposition 2.8, one gets

$$\mathcal{T}^b_\varepsilon u_\varepsilon \to u \quad \text{in} \quad L^2((0,1); H^{\frac{1}{2}}(\partial_u Y^*)). \tag{15}$$

By (14) and (15), for test functions $\varphi(x,y) = \varphi(x)$, we can pass to the limit in (13) yielding

$$\begin{aligned}
\int_{(0,1)\times Y^*}\kappa\left(\partial_x u + \partial_{y_1}u_1\right)\partial_x\varphi + Q\left(\partial_x u + \partial_{y_1}u_1\right)\varphi + cu\varphi dxdY \\
+\int_{(0,1)\times\partial_u Y^*} u\varphi dxd\sigma(y) = \int_{(0,1)\times Y^*}\hat{f}\varphi dxdY + \int_{(0,1)\times\partial_u Y^*} g\varphi dxd\sigma(y).
\end{aligned} \tag{16}$$

Now we obtain the relation between u_1 and the solution of the auxiliary problem

$$\int_{Y^*}\nabla X\nabla\psi dY = \int_{Y^*}\partial_{y_1}\psi dY, \quad \int_{Y^*} XdY = 0, \quad \forall\psi \in H^1_\#(Y^*). \tag{17}$$

For this sake, consider the sequence

$$v^\varepsilon(x,y) = \varepsilon\phi(x)\psi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right), \quad (x,y)\in R^\varepsilon, \tag{18}$$

where $\phi \in C_0^\infty(0,1)$ and $\psi \in H^1_\#(Y^*)$. Then, Corollary 2.4.1 provides

$$\begin{aligned} \mathcal{T}_\varepsilon v^\varepsilon \to 0, \quad & \text{strongly in } L^2\left((0,1)\times Y^*\right), \\ \mathcal{T}_\varepsilon \partial_x v^\varepsilon \to \phi\partial_{y_1}\psi, \quad & \text{strongly in } L^2\left((0,1)\times Y^*\right), \\ \mathcal{T}_\varepsilon \partial_y v^\varepsilon \to \phi\partial_{y_2}\psi, \quad & \text{strongly in } L^2\left((0,1)\times Y^*\right). \end{aligned} \tag{19}$$

Now, we take v^ε as a test function in (13). Passing to the limit as $\varepsilon \to 0$, we get

$$\int_{(0,1)\times Y^*} \left(\partial_x u + \partial_{y_1}u_1, \partial_{y_2}u_1\right)\phi\nabla_y\psi dxdY = 0.$$

From the density of tensor product $C_0^\infty(0,1)\otimes H^1_\#(Y^*)$ in $L^2((0,1);H^1_\#(Y^*))$, we can rewrite the above equation as

$$\int_{(0,1)\times Y^*} \left(\partial_x u + \partial_{y_1}u_1, \partial_{y_2}u_1\right)\nabla_y\psi dxdY = 0, \quad \forall\psi \in L^2((0,1);H^1_\#(Y^*)). \tag{20}$$

It is not difficult to check that (20) has a unique solution in the Hilbert space $H^1(0,1)\times L^2((0,1);H^1_\#(Y^*)/\mathbb{R})$. We refer the reader to [6] for details.

Since X is the unique L-periodic solution of the problem (17) and u is independent of y_1 and y_2, we have that $-\partial_x u(x)X(y_1,y_2)$ satisfies

$$\int_{Y^*} -\partial_x u\nabla X\nabla\psi dY = \int_{Y^*} -\partial_x u\partial_{y_1}\psi dY, \quad \forall\psi\in L^2((0,1);H^1_\#(Y^*)). \tag{21}$$

Consequently, it follows from (20) that

$$u_1(x,y_1,y_2) = -\partial_x u(x)X(y_1,y_2). \tag{22}$$

Now we are in position to rewrite (16) as

$$\begin{aligned} &\int_{(0,1)\times Y^*} \kappa\left(\partial_x u - \partial_x u\partial_{y_1}X\right)\partial_x\varphi + Q\left(\partial_x u - \partial_x u\partial_{y_1}X\right)\varphi + cu\varphi dxdY \\ &+ \int_{(0,1)\times\partial_u Y^*} u\varphi dxd\sigma(y) = \int_{(0,1)\times Y^*}\hat{f}\varphi dxdY + \int_{(0,1)\times\partial_u Y^*} g\varphi dxd\sigma(y). \end{aligned}$$

Hence, since u and φ are independent on y_1 and y_2, we get

$$\begin{aligned} \int_0^1 \kappa\partial_x u\left[\int_{Y^*}\left(1-\partial_{y_1}X\right)dY\right]\partial_x\varphi + \left[\int_{Y^*} Q\left(1-\partial_{y_1}X\right)dY\right]\partial_x u\varphi dx + |Y^*|c\int_0^1 u\varphi dx \\ +|\partial_u Y^*|\int_0^1 u\varphi dx = \int_0^1\left[\int_{Y^*}\hat{f}dY\right]\varphi dxdY + |\partial_u Y^*|\int_0^1 g\varphi dx. \end{aligned}$$

Dividing both sides by $|Y^*|$ gives

$$\int_0^1 \left[\kappa\, q\,\partial_x u\partial_x\varphi + p\,\partial_x u\varphi + c\,u\varphi\right]dx + \frac{|\partial_u Y^*|}{|Y^*|}\int_0^1 u\varphi dx = \int_0^1 \bar{f}\varphi dx + \frac{|\partial_u Y^*|}{|Y^*|}\int_0^1 g\varphi dx,$$

for all $\varphi\in H^1(0,1)$. It follows from [2] that the coefficients q and p are strictly positive, and then, the limit equation is a well posed problem. Thus, u^ε is a convergent sequence which leads us to the end of the proof. $\square$

Remark 3.1. *Notice the effect of the oscillating behavior on the homogenized coefficients q and p given by the auxiliary function X. Also, we emphasize the effect of the Lebesgue measure of the sets Y^* and $\partial_u Y^*$ at the limit equation, as well as, the flux condition on the border sets by g. All these ingredients can be seen at the homogenized equation.*

Remark 3.2. *See [6] for a discussing on some properties of the homogenized coefficient q. In particular, they show that $0 < q < 1$.*

3.2 Case $0 < \alpha < 1$

Theorem 3.2. *Let u^ε be the solution of the problem (2) with $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Also, assume that $g \in H^1(0,1)$. Then, as $\varepsilon \to 0$,*

$$\begin{gathered} \mathcal{T}_\varepsilon u^\varepsilon \to g \quad \textit{strongly in } L^2\left((0,1); H^1(Y^*)\right) \\ \textit{in such way that} \\ |||u^\varepsilon - g|||_{L^2(R^\varepsilon)} \to 0. \end{gathered} \tag{23}$$

Proof. Let

$$w^\varepsilon(x,y) = u^\varepsilon(x,y) - g(x) \quad \forall (x,y) \in R^\varepsilon, \tag{24}$$

where g is extended trivially to R^ε.

One can rewrite (3) as follows:

$$\begin{aligned} &\int_{R^\varepsilon} \kappa \nabla w^\varepsilon \nabla \varphi + Q^\varepsilon(y)\partial_x w^\varepsilon \varphi + c w^\varepsilon \varphi dxdy + \varepsilon^\alpha \int_{\Gamma^\varepsilon} w^\varepsilon \varphi dS \\ &+ \int_{R^\varepsilon} \kappa \nabla g \nabla \varphi + Q^\varepsilon(y)\partial_x g \varphi + c g \varphi dxdy = \int_{R^\varepsilon} f^\varepsilon \varphi dxdy, \end{aligned} \tag{25}$$

for all $\varphi \in H^1(R^\varepsilon)$.

(a) Uniform bounds.

Take $\varphi = w^\varepsilon$ as a test function in (25). Again, by [14, Lemma 3.4] and the assumption (4), one can show that there exists $\hat{c} > 0$, independent of $\varepsilon > 0$, such that

$$|||w^\varepsilon|||^2_{H^1(R^\varepsilon)} \le |||w^\varepsilon|||^2_{H^1(R^\varepsilon)} + \varepsilon^{\alpha-1}||w^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \le \hat{c}|||w^\varepsilon|||_{H^1(R^\varepsilon)}. \tag{26}$$

Thus, there exists $c > 0$ such that

$$\begin{gathered} |||w^\varepsilon|||_{H^1(R^\varepsilon)} \le \hat{c}, \\ \varepsilon^{\alpha-1}||w^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \le c. \end{gathered} \tag{27}$$

(b) Limits of w^ε.

By the uniform bound of w^ε in the $|||.|||_{H^1}$ norm, it follows from Theorem 2.6 that there exist $w \in H^1(0,1)$ and $w_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{gathered} \mathcal{T}_\varepsilon w^\varepsilon \to w \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right), \\ \mathcal{T}_\varepsilon \partial_x w^\varepsilon \rightharpoonup \partial_x w + \partial_{y_1} w_1 \text{ weakly in } L^2\left((0,1) \times Y^*\right), \\ \mathcal{T}_\varepsilon \partial_y w^\varepsilon \rightharpoonup \partial_{y_2} w_1 \text{ weakly in } L^2\left((0,1) \times Y^*\right). \end{gathered} \tag{28}$$

Also, by Proposition 2.8, one gets

$$\mathcal{T}^b_\varepsilon w^\varepsilon \to w \quad \text{in} \quad L^2((0,1); H^{\frac{1}{2}}(\partial_u Y^*)). \tag{29}$$

Now, due to Proposition 2.8 and (27), we have that

$$\begin{aligned} ||w||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} &\leq ||w - \mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} + ||\mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} \\ &\leq ||w - \mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} + c||w^\varepsilon||_{L^2(\Gamma^\varepsilon)} \\ &\leq ||w - \mathcal{T}^b_\varepsilon w^\varepsilon||_{L^2\left((0,1);H^{\frac{1}{2}}(\partial_u Y^*)\right)} + c\varepsilon^{1-\alpha}. \end{aligned} \tag{30}$$

Therefore, since $0 < \alpha < 1$ and w depends just on x-variable, we can pass to the limit in the inequality (30) as $\varepsilon \to 0$ obtaining that

$$w = 0 \quad \text{in} \quad (0,1).$$

Hence, due to (24) and (28), one can conclude that

$$\mathcal{T}_\varepsilon u^\varepsilon \to g \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right),$$

and then, it follows from Proposition 2.5 that

$$|||u^\varepsilon - g|||_{L^2(R^\varepsilon)} \to 0$$

concluding the proof of the Theorem. □

3.3 Case $\alpha > 1$

Theorem 3.3. *Let u^ε be the solution of the problem (2) with $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Assume that there exists $\hat{f} \in L^2((0,1) \times Y^*)$ such that*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \quad \text{weakly in } L^2((0,1) \times Y^*).$$

Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned} \mathcal{T}_\varepsilon u_\varepsilon &\to u \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right), \\ \mathcal{T}_\varepsilon \partial_x u_\varepsilon &\rightharpoonup \partial_x u + \partial_{y_1} u_1 \text{ weakly in } L^2((0,1) \times Y^*), \\ \mathcal{T}_\varepsilon \partial_y u_\varepsilon &\rightharpoonup \partial_{y_2} u_1 \text{ weakly in } L^2((0,1) \times Y^*). \end{aligned}$$

Moreover, u is the solution of

$$\begin{cases} -\kappa q u_{xx} + p u_x + c u = \bar{f} \text{ in } (0,1) \\ u_x(0) = u_x(1) = 0, \end{cases}$$

where the homogenized coefficients q and p are given by

$$q = \frac{1}{|Y^*|}\int_{Y^*} (1 - \partial_{y_1} X)\, dY \quad \text{and} \quad p = \frac{1}{|Y^*|}\int_{Y^*} Q\,(1 - \partial_{y_1} X)\, dY$$

and $X \in H^1_\#(Y^*)$ with $\int_{Y^*} X dY = 0$ is the unique solution of

$$\int_{Y^*} \nabla X \nabla \varphi dY = \int_{Y^*} \partial_{y_1} \varphi dY \qquad \forall \varphi \in H^1_\#(Y^*).$$

Also, the forcing term $\bar{f}$ is given by

$$\bar{f} = \frac{1}{|Y^*|} \int_{Y^*} \hat{f} dY.$$

Proof. The proof follows the same arguments as the proof of Theorem 3.1.

(a) Uniform bounds.

Take $\varphi = u^\varepsilon$ as a test function in (3). By [14, Lemma 3.4] and the assumption (4), one gets

$$|||u^\varepsilon|||^2_{H^1(R^\varepsilon)} + \varepsilon^{\alpha-1} ||u^\varepsilon||^2_{L^2(\Gamma^\varepsilon)} \leq |||f^\varepsilon|||_{L^2(R^\varepsilon)} |||u^\varepsilon|||_{L^2(R^\varepsilon)} + \varepsilon^{\alpha-1} ||g||_{L^2(\Gamma^\varepsilon)} ||u^\varepsilon||_{L^2(\Gamma^\varepsilon)}. \tag{31}$$

Thus, since we have from [11, 16] that

$$||\varphi||_{L^2(\Gamma^\varepsilon)} \leq C \varepsilon^{-1/2} \|\varphi\|_{H^1(R^\varepsilon)}, \tag{32}$$

we can get that there exists $c > 0$, independent of $\varepsilon > 0$, such that

$$|||u^\varepsilon|||_{H^1(R^\varepsilon)} \leq |||f^\varepsilon|||_{L^2(R^\varepsilon)} |||u^\varepsilon|||_{L^2(R^\varepsilon)} + c\, \varepsilon^{\alpha-1} ||g||_{L^2(0,1)} |||u^\varepsilon|||_{H^1(R^\varepsilon)}.$$

Since $\alpha > 1$, for $\varepsilon < 1$, we get u^ε is uniformly bounded in $||| \cdot |||_{H^1}$.

(b) Limiting problem

Rewrite (3) using Proposition 2.2:

$$\begin{aligned}
\int_{(0,1)\times Y^*} k \mathcal{T}_\varepsilon \nabla u^\varepsilon \mathcal{T}_\varepsilon \nabla \varphi + \mathcal{T}_\varepsilon Q^\varepsilon \mathcal{T}_\varepsilon \partial_x u^\varepsilon \mathcal{T}_\varepsilon \varphi + c \mathcal{T}_\varepsilon u^\varepsilon \varphi dx dY + \varepsilon^{\alpha-1} \int_{(0,1)\times \partial_u Y^*} \mathcal{T}^b_\varepsilon u^\varepsilon \mathcal{T}^b_\varepsilon \varphi d\sigma(y) \\
+ \frac{L}{\varepsilon} \int_{R^\varepsilon_1} \kappa \nabla u^\varepsilon \nabla \varphi + Q^\varepsilon(y) \partial_x u^\varepsilon \varphi + c u^\varepsilon \varphi dx dy + L \varepsilon^{\alpha-1} \int_{\Gamma^\varepsilon_1} u^\varepsilon \varphi dS \\
= \int_{(0,1)\times Y^*} \mathcal{T}_\varepsilon f^\varepsilon \mathcal{T}_\varepsilon \varphi dx dY + \varepsilon^{\alpha-1} \int_{(0,1)\times \partial_u Y^*} \mathcal{T}^b_\varepsilon g \mathcal{T}^b_\varepsilon \varphi dx d\sigma(y) \\
+ \frac{L}{\varepsilon} \int_{R^\varepsilon_1} f^\varepsilon \varphi dx dy + L \varepsilon^{\alpha-1} \int_{\Gamma^\varepsilon_1} g \varphi dS,
\end{aligned} \tag{33}$$

Since we have uniform bounds for the solutions of (2) in the $|||.|||_{H^1(R^\varepsilon)}$ norm, we can apply Theorem 2.6. Thus, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned}
\mathcal{T}_\varepsilon u_\varepsilon \to u \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right), \\
\mathcal{T}_\varepsilon \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1 \text{ weakly in } L^2\left((0,1) \times Y^*\right), \\
\mathcal{T}_\varepsilon \partial_y u_\varepsilon \rightharpoonup \partial_{y_2} u_1 \text{ weakly in } L^2\left((0,1) \times Y^*\right).
\end{aligned} \tag{34}$$

Also, by Proposition 2.8, one gets

$$\mathcal{T}^b_\varepsilon u_\varepsilon \to u \quad \text{in} \quad L^2((0,1); H^{\frac{1}{2}}(\partial_u Y^*)). \tag{35}$$

By (34), (15) and $\alpha > 1$, for test functions $\varphi(x, y) = \varphi(x)$, we can pass to the limit (33) getting

$$\int_{(0,1)\times Y^*} \kappa \left(\partial_x u + \partial_{y_1} u_1\right) \partial_x \varphi + Q \left(\partial_x u + \partial_{y_1} u_1\right) \varphi + cu\varphi dx dY = \int_{(0,1)\times Y^*} \hat{f} \varphi dx dY. \quad (36)$$

If one proceedes as in Theorem 2.6, it is not difficult to obtain

$$\int_0^1 \left[\kappa\, q\, \partial_x u \partial_x \varphi + p\, \partial_x u \varphi + c\, u\varphi\right] dx = \int_0^1 \bar{f} \varphi dx, \quad \forall \varphi \in H^1(0,1)$$

which completes the proof. □

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