# A nonlocal Dirichlet problem with impulsive action: estimates of the growth for the solutions


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**Abstract**

Through this paper we deal with the asymptotic behaviour as $t \to +\infty$ of the solutions for the nonlocal diffusion problem with impulsive actions and Dirichlet condition. We establish a decay rate for the solutions assuming appropriate hypotheses on the impulsive functions and the nonlinear reaction.


## 1 Introduction

Our main purpose in this paper is to study the asymptotic behaviour of the solutions of the following nonlinear problem with impulsive actions

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)u(y,t)dy - u(x,t) + f(x,u(x,t)), & (x,t) \in \Gamma, \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,t_k) = g_k(u(x,t_k^-)), \quad x \in \Omega, \ \ k = 1,2,\dots \end{cases} \tag{1.1}$$

exploring the properties of the integral operator

$$K_J(u)(x,t) := \int_\Omega J(x-y)u(y,t)dy.$$

Along the whole paper we assume

**H1.** $\Omega \subset \mathbb{R}^n$ is an open bounded domain;

**H2.** $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots$ are fixed instant of times with $\lim_{k\to+\infty} t_k = +\infty$;

**H3.** $\Gamma = \bigcup_{k=0}^{\infty} \Gamma_k$ where $\Gamma_k = \{(x,t) \in \mathbb{R}^{n+1} : t \in (t_k, t_{k+1}),\ x \in \Omega\}$ for $k = 0,1,\dots$;

**H4.** The impulsive functions $g_k : \mathbb{R} \to \mathbb{R}$ are continuous satisfying $|g_k(x)| \le M_k|x|$ for positive constants $M_k$ for $k = 1,2,\dots$ . We still set $M_0 = 1$ for convenience;


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**H5.** The local reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the second variable $s \in \mathbb{R}$ uniformly in $\Omega$. Also, we assume there exist constants $C > 0$ and $D \geq 0$ such that

$$\boxed{f(\cdot, s)s \leq Cs^2 + D.}$$

**H6.** The kernel $J$ of the integral operator $K_J$ is supposed to be non-singular. It satisfies

$$\begin{gathered} J \in C(\mathbb{R}^n, \mathbb{R}) \text{ is a non-negative function with } J(0) > 0, \\ J(-x) = J(x) \text{ for every } x \in \mathbb{R}^n \text{ and } \textstyle\int_{\mathbb{R}^n} J(x)\, dx = 1. \end{gathered}$$

It is worth noticing that according to the pioneering works [2, 5, 7, 10] the nonlocal problem considered here can be seen as a nonlocal analogous to a reaction-diffusion problem given by the Laplacian with homogeneous Dirichlet boundary condition and impulsive actions

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)), \quad (x,t) \in \Gamma, \\ u(x,t) = 0, \quad \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,t_k) = g_k(u(x,t_k^-)), \quad x \in \Omega, \ k = 1, 2, \ldots \end{cases}$$

Notice that, if $u(x,t)$ sets the population density of a species at the position $x$ and time $t$, and $J(x-y)$ is treated of as the probability distribution of jumping from spot $y$ to spot $x$, then $\int_{\mathbb{R}^n} J(x-y)u(y,t)dy = (J * u)(x,t)$ is the rate at which individuals are arriving at position $x$ from all other positions $y \in \mathbb{R}^n$ and $-u(x,t) = -\int_{\mathbb{R}^n} J(x-y)u(x,t)dy$ is the rate at which they are leaving location $x$ to move to all other positions. As consequence, the density $u$ satisfies the integral equation $u_t = J * u - u$, in the absence of external or internal sources.

Additionally, one may notice situations in nature where abrupt changes such as droughts, tropical storms or many other environmental accidents may appear. Thus, the need to consider models with some kind of impulse becomes essential. Our proposal mix both situations. We combine a nonlocal reaction-diffusion equation with impulsive actions in fixed times.

The literature [4, 11, 12, 13, 16, 17, 19, 20] is recommended for more knowledge on nonlocal diffusion equations and applications. In [18], examples of integral equations whose solutions blow-up in finite time are considered. On the other side, the classical literature [14, 21] deals with the impulsive differential equations in fixed times. For recent works, we refer to [1, 6, 9] and references therein.

To the best of our knowledge, the corresponding theory for partial differential equations with impulses for the operator $K_J$ has not been investigated yet. In this paper, we intend to make some contributions in this class of problems. Our goal is to secure the existence of global solutions taking estimates for the impulsive solutions as $t \to +\infty$.

In our analysis, the first eigenvalue $\lambda_1$ of the operator $\mathcal{D}_\Omega : L^2(\Omega) \to L^2(\Omega)$ given by

$$\mathcal{D}_\Omega(u)(x) = u(x) - K_J(u)(x), \quad x \in \Omega, \tag{1.2}$$

plays an important role. Its existence is guaranteed by [8, Theorem 2.1] and hypotheses **H1** and **H6**. It can be taken by the following expression

$$\lambda_1 = \inf\left\{ \frac{\int_\Omega \int_\Omega J(x-y)(u(y)-u(x))^2 dxdy}{\int_\Omega u^2(x)\, dx} : u \neq 0 \text{ with } u \in L^2(\Omega) \right\}. \tag{1.3}$$

We also mention [3] where recent results concerning to the eigenvalues of the operator $\mathcal{D}_\Omega$ has been considered. The main result of the paper is the following.

**Theorem 1.1.** *Let $u$ be the global solution of the impulsive nonlocal equation* (1.1) *with initial condition $u_0 \in L^2(\Omega)$. Set $\alpha = 2(\lambda_1 - C)$ and $\beta = 2D|\Omega|$ where $C$ and $D$ are the non-negative constants given by assumption* **H5**, *$\lambda_1$ is the first eigenvalue introduced by* (1.3), *and $|\Omega|$ is the Lesbegue measure of the open bounded set $\Omega$. Suppose there exists $L_k > 0$ such that*

$$\max\left\{\prod_{j=1}^{k} M_j^2, \prod_{j=2}^{k} M_j^2, \ldots, M_k^2, 1\right\} \leq L_k$$

*where $M_k$ are the constants given by condition* **H4**.

*Then, we have the following estimate*

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq e^{-\alpha t}\left[\left(\prod_{j=0}^{k} M_j^2\right)\|u_0(\cdot)\|^2_{L^2(\Omega)} + L_k\frac{\beta}{|\alpha|}\sum_{j=0}^{k} e^{\alpha t_j}\right] \quad \forall t_k \leq t < t_{k+1} \ \text{ if } \ \alpha \neq 0,$$

*and*

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq \left(\prod_{j=0}^{k} M_j^2\right)\|u_0(\cdot)\|^2_{L^2(\Omega)} + L_k\beta t \quad \forall t_k \leq t < t_{k+1} \ \text{ if } \ \alpha = 0.$$

*Furthermore, assume that there exist positive constants $L$ and $\gamma$ such that $Le^{\gamma t} \geq L_k$ for all $t \in [0, t_{k+1})$ and $k \in \mathbb{N}$, and there exists $\delta > 0$ such that $|t_i - t_j| > \delta$ for any $i, j$ where $t_i's$ are the impulsive instants. Then, if $\alpha < 0$, we have*

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq Le^{(\gamma-\alpha)t}\left[\|u_0(\cdot)\|^2_{L^2(\Omega)} + \frac{\beta}{\delta|\alpha|}\left(\frac{1}{1-e^{\alpha}}\right)\right] \quad \forall t > 0, \tag{1.4}$$

*and, when $\alpha > 0$, we get*

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq Le^{(\gamma-\alpha)t}\left[\|u_0(\cdot)\|^2_{L^2(\Omega)} + \frac{\beta}{\delta\alpha^2}\left(e^{\alpha t_{k+1}} - 1\right)\right] \quad \forall t \in [0, t_{k+1}). \tag{1.5}$$

For instance, let us assume $D = 0$ at condition **H5**. Then, $\beta = 0$ at Theorem 1.1 and $f(x, 0) = 0$ in $\Omega$ which implies that $u(x, t) \equiv 0$ is an equilibrium solution of (1.1). Now, let us suppose there exist positive constants $K$ and $\zeta$ such that $\prod_{j=0}^{k} M_j^2 \leq Ke^{\zeta t}$ for all $t \in [0, t_{k+1})$ and $k \in \mathbb{N}$. Hence, it follows from Theorem 1.1 that

$$\int_{\Omega} u^2(x,t)dx \leq Ke^{(\zeta-\alpha)t}\int_{\Omega} u_0^2(x)dx \quad \forall 0 \leq t < +\infty.$$

Thus, if $\zeta - \alpha < 0$, we can conclude that $u(x, t) \equiv 0$ is an asymptotically stable equilibrium solution for (1.1), and then, the dynamic sets by equation (1.1) is trivial. In this way, conditions on the impulsive functions can set the null equilibrium as asymptotically stable.

Next, let us mention our last result concerning to the estimates for the solutions with initial conditions in $L^\infty(\Omega)$. Here, we take $D = 0$ at assumption **H5** in order to obtain a workable estimate.

**Theorem 1.2.** *Let $u$ be the global solution of the impulsive nonlocal equation* (1.1) *with initial condition $u_0 \in L^\infty(\Omega)$. Also, let us assume $D = 0$ at assumption* **H5**.

*Then, for all $\epsilon > 0$, there exists $C_0 > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \left(\prod_{j=0}^{k} C_0\, M_j\right)\|u_0\|_{L^\infty(\Omega)}\, e^{(C_0 C + \epsilon - \lambda_1)t} \quad \text{for all } \ t_k \leq t < t_{k+1}$$

*where $C$ is the constant given by condition* **H5** *and $\lambda_1 > 0$ is the first eigenvalue of the operator $\mathcal{D}_\Omega$ which is set by* (1.3).

As we will see at the proof of Theorem 1.2, the positive constant $C_0$ comes from the estimate of the linear semigroup of the bounded linear operator $\mathcal{D}_\Omega$. It depends on the spectral set of $\mathcal{D}_\Omega$, and is needed since we are using the variation of constant formula to estimate the solutions of (1.1). Moreover, if there exist $K > 0$ and $\zeta \in \mathbb{R}$ such that $\prod_{j=0}^{k} C_0\, M_j \le K e^{\zeta t}$ for all $t \in [0, t_{k+1})$ and $k \in \mathbb{N}$, then, we can proceed as in Theorem 1.1 obtaining from Theorem 1.2 that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le K\, \|u_0(\cdot)\|_{L^\infty(\Omega)}\, e^{(\zeta + C_0 C + \epsilon - \lambda_1)t} \quad \text{for all } \ t \ge 0.$$

Hence, the null equilibrium is asymptotically stable if one can get $\zeta + C_0 C + \epsilon - \lambda_1 < 0$.

Finally, it is worth mentioning that one may combine Theorem 1.1 and Theorem 1.2 in order to obtain estimates in $L^p(\Omega)$ for any $p \in (2, +\infty)$ by interpolation. Indeed, since we are working in bounded domains, it is a direct consequence of the Riesz-Thorin Theorem, and then, it will be left to the interested reader.

The organization of this article is as follows. In Section 2, we briefly discuss the existence of global solutions for the nonlocal equation without impulses getting estimates which guarantee the existence of solutions for problem (1.1) with impulsive actions. In Section 3 we prove the main results of the paper establishing estimates for the impulsive solutions with initial conditions in $L^2(\Omega)$ and $L^\infty(\Omega)$.

## 2 Existence of solutions

In this section, we give some condition in order to guarantee the existence of solutions for the nonlocal problem with impulsive actions (1.1). First let us notice that the impulsive linear case

$$\begin{cases} u_t(x,t) = (K_J - I)u(x,t), \quad (x,t) \in \Gamma, \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,t_k) = g_k(u(x,t_k^-)), \quad x \in \Omega, \;\; k = 1, 2, \ldots, \end{cases} \tag{2.1}$$

where $u(x,t_k^-) = \lim_{t \to t_k^-} u(x,t)$ is well defined since the solutions of the non impulsive equation

$$\begin{cases} u_t(x,t) = (K_J - I)u(x,t), \quad x \in \Omega \;\text{ and }\; t > 0, \\ u(x,t) = 0, \quad x \notin \Omega, \;\; t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases}$$

are globally defined for any $u_0 \in L^p(\Omega)$ with $p \in (1, \infty]$ (see for instance [20]).

Thus, for each $k$, we have $\lim_{t \to t_k} u(x,t) = u(x,t_k)$, and then, the *solution* for the impulsive equation (2.1) is a piecewise continuous function $u$ such that in each interval $[t_k, t_{k+1})$ satisfies

$$\begin{cases} u_t(x,t) = (K_J - I)u(x,t), \quad (x,t) \in \Gamma_k, \\ u(x,t_k) = g_k(u(x,t_k^-)), \quad x \in \Omega. \end{cases}$$

Consequently, a enough condition to define the solution to the impulsive equation (1.1) is to ensure that the associated problem

$$\begin{cases} u_t(x,t) = (K_J - I)u(x,t) + f(x, u(x,t)), \quad x \in \Omega, \; t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2.2}$$

possesses solution globally defined for appropriated initial conditions.

The local existence and uniqueness is proved using a fixed point argument with the variation of constants formula on the right side of

$$u(\cdot,t) = e^{-t}u_0 + \int_0^t e^{-(t-s)}[(K_J u)(\cdot,s) + f(\cdot,u(\cdot,s))]\,ds. \tag{2.3}$$

Considering the formula on the right side of (2.3) as an operator defined from $L^1([0,T],L^2(\Omega))$ into $\mathcal{C}([0,T],L^2(\Omega))$, one can get a $T>0$ in such way that a contraction is established since function $f$ is locally Lipschitz in the second variable. Hence, local existence and uniqueness to (2.2) is obtained applying abstract theorems from [15, p. 109]. Indeed, a strong solution is gotten in $\mathcal{C}^1([0,T],L^2(\Omega))$.

Analogously, one can show local existence and uniqueness to (2.3) in $\mathcal{C}^1([0,T],L^\infty(\Omega))$ for initial conditions in $L^\infty(\Omega)$. The next estimates ensure that the solutions are upper bounded for a function defined for all $t\geq 0$ which also guarantees that they are globally defined in $[0,+\infty)$.

**Proposition 2.1.** *Let $u:[0,T)\mapsto L^2(\Omega)$ be the solution of* (2.2) *with initial condition $u_0\in L^2(\Omega)$ and nonlinearity $f$ satisfying hypothesis **H5**. Then*

$$\|u\|^2_{L^2(\Omega)} \leq \gamma(t) \quad \forall t\in[0,T)$$

*where $\gamma(t) = (\beta/\alpha)(1-e^{-\alpha t}) + e^{-\alpha t}\|u_0\|^2_{L^2(\Omega)}$ with $\beta = 2D|\Omega|$ and $\alpha = 2(\lambda_1 - C)\neq 0$ or $\gamma(t) = \|u_0\|^2_{L^2(\Omega)} + \beta t$ if $\alpha = 0$. Consequently, the solution $u$ with initial condition $u_0$ can be extended to a global solution of* (2.2) *in* $[0,+\infty)$.

*Proof.* First, let us assume $\lambda_1\neq C$. Then,

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{1}{2}\int_\Omega u^2(x,t)dx\right] &= \frac{1}{2}\int_\Omega 2u(x,t)\frac{\partial}{\partial t}u(x,t)dx = \int_\Omega u(x,t)\left[(K_J - I)u(x,t) + f(x,u(x,t))\right]dx \\
&= -\int_\Omega u(x,t)(I-K_J)u(x,t)dx + \int_\Omega u(x,t)f(x,u(x,t))dx \\
&\leq -\lambda_1\int_\Omega u^2(x,t)dx + \int_\Omega (Cu^2(x,t)+D)dx
\end{aligned}$$

by condition **H5**. Hence,

$$\frac{\partial}{\partial t}\left[\int_\Omega u^2(x,t)dx\right] + \overbrace{(2\lambda_1 - 2C)}^{\alpha}\int_\Omega u^2(x,t)dx \leq 2\int_\Omega D dx = \overbrace{2D|\Omega|}^{\beta}.$$

Thus,

$$\begin{aligned}
&\int_0^t \frac{\partial}{\partial s}\left[e^{\alpha s}\int_\Omega u^2(x,s)dx\right]ds \leq \int_0^t \beta e^{\alpha s}ds \\
\Rightarrow\ & e^{\alpha t}\int_\Omega u^2(x,t)dx - \int_\Omega u_0^2(x)dx \leq \frac{\beta}{\alpha}e^{\alpha t} - \frac{\beta}{\alpha} \\
\Rightarrow\ & \int_\Omega u^2(x,t)dx \leq \frac{\beta}{\alpha}(1-e^{-\alpha t}) + e^{-\alpha t}\int_\Omega u_0^2(x)dx = \gamma(t).
\end{aligned}$$

In particular, we have that

$$\int_\Omega u^2(x,t)dx \leq e^{-\alpha t}\int_\Omega u_0^2(x)dx + \frac{\beta}{\alpha} \qquad \text{if } \alpha>0 \tag{2.4}$$

and

$$\int_\Omega u^2(x,t)dx \le e^{-\alpha t}\left[\int_\Omega u_0^2(x)dx - \frac{\beta}{\alpha}\right] \qquad \text{if } \alpha < 0. \tag{2.5}$$

Finally, let us suppose $\lambda_1 = C$. Arguing as before, one has

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\int_\Omega u^2(x,t)dx\right] + \overbrace{(2\lambda_1 - 2C)}^{\alpha=0}\int_\Omega u^2(x,t)dx \le 2\int_\Omega D dx = \overbrace{2D|\Omega|}^{\beta} \\ \Rightarrow \quad &\left[\frac{\partial}{\partial t}\int_\Omega u^2(x,t)dx\right] \le \beta \Rightarrow \int_\Omega u^2(x,t)dx - \int_\Omega u_0^2(x)dx \le \int_0^t \beta\, ds = \beta\, t \end{aligned}$$

which completes the proof. $\square$

It follows from hypothesis **H5** that $f(\cdot,s) \le Cs + D/s$ whenever $s > 0$ and $f(\cdot,s) \ge Cs + D/s$ if $s < 0$ for all $x \in \Omega$. Also, we have that $f$ is locally Lipschitz in the second variable uniformly in $\Omega$. Thus, there exist $D_1 > 0$ such that

$$|f(\cdot,s)| \le C|s| + D_1 \qquad \text{for all } s \in \mathbb{R}. \tag{2.6}$$

Hence, we can estimate the solutions with initial conditions in $L^\infty(\Omega)$ in the following way.

**Proposition 2.2.** *Let $u : [0,T) \mapsto L^\infty(\Omega)$ be the solution of* (2.2) *with initial condition $u_0 \in L^\infty(\Omega)$ and assume nonlinearity $f$ satisfies hypothesis **H5**.*

*Then, for all $\epsilon > 0$, there exists $C_0 > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_0\left[\|u_0(\cdot)\|_{L^\infty(\Omega)} + \frac{D_1}{(\lambda_1 - \epsilon)}\left(e^{(\lambda_1-\epsilon)t} - 1\right)\right] e^{(C_0C+\epsilon-\lambda_1)t}$$

*where $C$ and $D_1$ are the constants given by Remark* (2.6) *and $\lambda_1 > 0$ is the first eigenvalue of the linear operator $\mathcal{D}_\Omega$ given by* (1.3).

*Proof.* Let us consider the linear semigroup $e^{-\mathcal{D}_\Omega t} := \sum_{k\ge 0}\frac{t^k(-\mathcal{D}_\Omega)^k}{k!}$ of the problem (2.2). Since $D_\Omega$ set in (1.2) is a bounded linear operator in $L^\infty(\Omega)$, it follows from [15] that $e^{-\mathcal{D}_\Omega t}$ is well defined and we can rewrite $u$ as

$$u(x,t) = e^{-\mathcal{D}_\Omega t}u_0(x) + \int_0^t e^{-\mathcal{D}_\Omega(t-s)}f(x,u(x,s))\,ds$$

for all $t \in [0,T)$ and $x \in \Omega$.

Notice that $\mathrm{Re}(\sigma(-\mathcal{D}_\Omega)) \le -\lambda_1$ where $\lambda_1$ is the first eigenvalue of $\mathcal{D}_\Omega$ set in (1.3). Hence, by [20, Section 4.3] we get that for all $\epsilon > 0$ there exists $C_0 > 0$ such that

$$\|e^{-\mathcal{D}_\Omega t}u_0(\cdot)\|_{L^\infty(\Omega)} \le C_0 e^{-(\lambda_1-\epsilon)t}\|u_0(\cdot)\|_{L^\infty(\Omega)} \quad \forall t \ge 0.$$

Then, due to condition **H5** and Remark 2.6, we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le C_0 e^{-(\lambda_1-\epsilon)t}\|u_0(\cdot)\|_{L^\infty(\Omega)} + \int_0^t C_0 e^{-(\lambda_1-\epsilon)(t-s)}\|f(\cdot,u(\cdot,s))\|_{L^\infty(\Omega)}\,ds \\ &\le C_0 e^{-(\lambda_1-\epsilon)t}\|u_0(\cdot)\|_{L^\infty(\Omega)} + \int_0^t C_0 e^{-(\lambda_1-\epsilon)(t-s)}\left[C\|u(\cdot,s)\|_{L^\infty(\Omega)} + D_1\right] ds. \end{aligned}$$

Consequently,

$$\begin{aligned} e^{(\lambda_1-\epsilon)t}\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} &\le C_0\left[\left\|u_0(\cdot)\right\|_{L^\infty(\Omega)}+\frac{D_1}{(\lambda_1-\epsilon)}\left(e^{(\lambda_1-\epsilon)t}-1\right)\right] \\ &\qquad +C_0\,C\int_0^t e^{(\lambda_1-\epsilon)s}\left\|u(\cdot,s)\right\|_{L^\infty(\Omega)}\,ds, \end{aligned}$$

and then, by Grönwall's Lemma, we obtain that

$$e^{(\lambda_1-\epsilon)t}\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \le C_0\left[\left\|u_0(\cdot)\right\|_{L^\infty(\Omega)}+\frac{D_1}{(\lambda_1-\epsilon)}\left(e^{(\lambda_1-\epsilon)t}-1\right)\right]e^{C_0\,C\,t}$$

which proves the result. □

It follows from Propositions 2.1 and 2.2 that the solutions of our nonlocal problem with impulsive actions in fixed times (1.1) there exist and are unique for initial conditions in $L^2(\Omega)$ and $L^\infty(\Omega)$. Indeed, the solutions are piecewise continuous functions $u$ satisfying

$$\begin{cases} u_t(x,t)=(K_J-I)u(x,t)+f(x,u(x,t)), \quad t\in(t_k,t_{k+1}) \\ u(x,0)=u_0(x)\in L^2(\Omega)\cup L^\infty(\Omega), & x\in\Omega. \\ u(x,t_k)=g_k(u(x,t_k^-)), \quad k=1,2,\ldots \end{cases}$$

# 3 Proof of Theorems 1.1 and 1.2.

Next, we show Theorems 1.1 and 1.2.

*Proof of Theorem 1.1.* **Case** $\alpha<0$: From (2.5) in $0\le t<t_1$, we get

$$\int_\Omega u^2(x,t_1^-)dx \le e^{-\alpha t_1}\left[\int_\Omega u_0^2(x)dx-\frac{\beta}{\alpha}\right].$$

Using (2.5) again, we have by **H4** that

$$\begin{aligned} \int_\Omega u^2(x,t)dx &\le e^{-\alpha(t-t_1)}\left[\int_\Omega g_1^2(u(x,t_1^-))dx-\frac{\beta}{\alpha}\right] \quad \text{for } t_1\le t<t_2 \\ \Rightarrow \int_\Omega u^2(x,t)dx &\le e^{-\alpha(t-t_1)}\left[M_1^2\int_\Omega u^2(x,t_1^-)dx-\frac{\beta}{\alpha}\right] \\ &\le e^{-\alpha(t-t_1)}\left[M_1^2\left(e^{-\alpha t_1}\left(\int_\Omega u_0^2(x)dx-\frac{\beta}{\alpha}\right)\right)-\frac{\beta}{\alpha}\right] \\ &\le e^{-\alpha t}\left[M_1^2\left(\int_\Omega u_0^2(x)dx-\frac{\beta}{\alpha}\right)-\frac{\beta}{\alpha}e^{\alpha t_1}\right] \quad \text{for } t_1\le t<t_2. \end{aligned}$$

and then,

$$\int_\Omega u^2(x,t_2^-)dx \le e^{-\alpha t_2}\left[M_1^2\left(\int_\Omega u_0^2(x)dx-\frac{\beta}{\alpha}\right)-\frac{\beta}{\alpha}e^{\alpha t_1}\right].$$

Working as before, we obtain

$$\begin{aligned} \int_\Omega u^2(x,t)dx &\le e^{-\alpha(t-t_2)}\left[\int_\Omega g_2^2(u(x,t_2^-))dx-\frac{\beta}{\alpha}\right] \quad \text{for } t_2\le t<t_3 \\ \Rightarrow \int_\Omega u^2(x,t)dx &\le e^{-\alpha(t-t_2)}\left[M_2^2\int_\Omega u^2(x,t_2^-)dx-\frac{\beta}{\alpha}\right] \\ &\le e^{-\alpha(t-t_2)}\left[M_2^2\left(e^{-\alpha t_2}\left(M_1^2\left(\int_\Omega u_0^2(x)dx-\frac{\beta}{\alpha}\right)-\frac{\beta}{\alpha}e^{\alpha t_1}\right)\right)-\frac{\beta}{\alpha}\right] \\ &\le e^{-\alpha t}\left[M_2^2M_1^2\left(\int_\Omega u_0^2(x)dx-\frac{\beta}{\alpha}\right)-M_2^2\frac{\beta}{\alpha}e^{\alpha t_1}-\frac{\beta}{\alpha}e^{\alpha t_2}\right] \quad \text{for } t_2\le t<t_3. \end{aligned}$$

Arguing analogously, for $t_k \le t < t_{k+1}$, we conclude that

$$\int_\Omega u^2(x,t)dx \quad \le \quad e^{-\alpha t}\left[\prod_{j=1}^{k} M_j^2\left(\int_\Omega u_0^2(x)dx - \frac{\beta}{\alpha}\right) - \prod_{j=2}^{k} M_j^2\frac{\beta}{\alpha}e^{\alpha t_1} - \prod_{j=3}^{k} M_j^2\frac{\beta}{\alpha}e^{\alpha t_2} - \ldots - \frac{\beta}{\alpha}e^{\alpha t_k}\right].$$

Thus,

$$\int_\Omega u^2(x,t)dx \le e^{-\alpha t}\left[\left(\prod_{j=0}^{k} M_j^2\right)\int_\Omega u_0^2(x)dx - L_k\frac{\beta}{\alpha}\sum_{j=0}^{k} e^{\alpha t_j}\right] \quad \text{for } \; t_k \le t < t_{k+1}. \tag{3.5}$$

**Case** $\alpha > 0$: From (2.4), we get

$$\int_\Omega u^2(x,t_1^-)dx \le e^{-\alpha t_1}\int_\Omega u_0^2(x)dx + \frac{\beta}{\alpha}$$

which implies

$$\begin{aligned}\int_\Omega u^2(x,t)dx \quad &\le \quad e^{-\alpha(t-t_1)}M_1^2\left[e^{-\alpha t_1}\int_\Omega u_0^2(x)dx + \frac{\beta}{\alpha}\right] + \frac{\beta}{\alpha}\\ &\le \quad e^{-\alpha t}M_1^2\int_\Omega u_0^2(x)dx + \frac{\beta}{\alpha}\left[M_1^2 e^{-\alpha(t-t_1)} + 1\right] \quad \text{for } \; t_1 \le t < t_2.\end{aligned}$$

Analogously, one can get

$$\begin{aligned}\int_\Omega u^2(x,t)dx \quad &\le \quad e^{-\alpha(t-t_2)}M_2^2\left[e^{-\alpha t_2}M_1^2\int_\Omega u_0^2(x)dx + \frac{\beta}{\alpha}\left(M_1^2 e^{-\alpha(t-t_1)} + 1\right)\right] + \frac{\beta}{\alpha}\\ &\le \quad e^{-\alpha t}M_2^2 M_1^2\int_\Omega u_0^2(x)dx + \frac{\beta}{\alpha}\left[M_1^2 M_2^2 e^{-\alpha(t-t_1)} + M_2^2 e^{-\alpha(t-t_2)} + 1\right]\end{aligned}$$

whenever $t_2 \le t < t_3$. Hence, for any $t_k \le t < t_{k+1}$ we have

$$\begin{aligned}\int_\Omega u^2(x,t)dx \quad &\le \quad e^{-\alpha t}\prod_{j=1}^{k} M_j^2\int_\Omega u_0^2(x)dx\\ &\qquad + \frac{\beta}{\alpha}\left[\prod_{j=1}^{k} M_j^2 e^{-\alpha(t-t_1)} + \prod_{j=2}^{k} M_j^2 e^{-\alpha(t-t_2)} + \ldots + M_k^2 e^{-\alpha(t-t_k)} + 1\right]\\ &\le \quad e^{-\alpha t}\left(\prod_{j=0}^{k} M_j^2\right)\int_\Omega u_0^2(x)dx + L_k\frac{\beta}{\alpha}\sum_{j=0}^{k} e^{-\alpha(t-t_j)}.\end{aligned}$$

**Case** $\alpha = 0$: Using Proposition (2.1) again, we have that

$$\int_\Omega u^2(x,t)dx \le \int_\Omega u_0^2(x)dx + \beta\, t \;\text{ for }\; 0 \le t < t_1 \;\text{ and }\; \int_\Omega u^2(x,t_1^-)dx \le \int_\Omega u_0^2(x)dx + \beta\, t_1.$$

Hence, for $t_1 \le t < t_2$, we get

$$\begin{aligned}\int_\Omega u^2(x,t)dx \quad &\le \quad \int_\Omega g_1^2(u(x,t_1^-))dx + \beta\,(t-t_1)\\ &\le \quad M_1^2\int_\Omega u_0^2(x)dx + M_1^2\beta\, t_1 + \beta\,(t-t_1)\end{aligned}$$

which also implies for $t_2 \leq t < t_3$ that

$$\begin{aligned}
\int_\Omega u^2(x,t)dx &\leq \int_\Omega g_2^2(u(x,t_2^-))dx + \beta\,(t-t_2) \\
&\leq M_2^2\left[M_1^2\int_\Omega u_0^2(x)dx + M_1^2\beta\,t_1 + \beta\,(t_2-t_1)\right] + \beta\,(t-t_2) \\
&\leq M_2^2M_1^2\int_\Omega u_0^2(x)dx + M_2^2M_1^2\beta\,(t_1-t_0) + M_2^2\beta\,(t_2-t_1) + \beta\,(t-t_2).
\end{aligned}$$

Thus, for all $t_k \leq t < t_{k+1}$ one can get

$$\begin{aligned}
\int_\Omega u^2(x,t)dx &\leq \prod_{j=1}^{k} M_j^2\int_\Omega u_0^2(x)dx \\
&\qquad + \prod_{j=1}^{k} M_j^2\beta(t_1-t_0) + \prod_{j=2}^{k} M_j^2\beta(t_2-t_1) + \ldots + M_k^2\beta(t_k-t_{k-1}) + \beta\,(t-t_k) \\
&\leq \prod_{j=0}^{k} M_j^2\int_\Omega u_0^2(x)dx + L_k\beta t.
\end{aligned}$$

Now let us see the last two estimate. Assume $\alpha < 0$ and set $\lceil t_j\rceil = \max\{n \in \mathbb{Z} : n \leq t_j\}$ the integer part of the impulsive instant $t_j$. Define $n_i$ for the cardinality of $\{t_j : i \leq t_j < i+1\}$ setting $N_k = \max_{i\in[0,\lceil t_k\rceil]} n_i$. Then,

$$\sum_{j=0}^{k} e^{\alpha t_j} \leq \sum_{j=0}^{k} e^{\alpha\lceil t_j\rceil} \leq \sum_{j=0}^{\lceil t_k\rceil} n_j e^{\alpha j} \leq N_k\sum_{j=0}^{\lceil t_k\rceil} e^{\alpha j} < \frac{N_k}{1-e^{\alpha}}.$$

Thus, it follows from (3.5) that

$$\int_\Omega u^2(x,t)dx \leq e^{-\alpha t}\left[\prod_{j=1}^{k} M_j^2\int_\Omega u_0^2(x)dx - L_k\frac{\beta}{\alpha}\left(\frac{N_k}{1-e^{\alpha}}\right)\right].$$

As $|t_i - t_j| > \delta$ for some $\delta > 0$ and all $i, j$, we ensure $N_k < \delta^{-1}$ for all $k$ . Hence

$$\int_\Omega u^2(x,t)dx \leq Le^{(\gamma-\alpha t)}\left[\int_\Omega u_0^2(x)dx - \frac{\beta}{\alpha}\delta^{-1}\left(\frac{1}{1-e^{\alpha}}\right)\right] \quad \text{for } t \geq 0 \tag{3.16}$$

since $L_k \leq Le^{\gamma t}$ for all $t \in [0, t_{k+1})$ and for all $k \in \mathbb{N}$ proving (1.4).

On the other side, if $\alpha > 0$, we have

$$\sum_{j=0}^{k} e^{\alpha t_j} \leq \delta^{-1}\int_0^{t_{k+1}} e^{\alpha s}\,ds = \frac{1}{\delta\alpha}(e^{\alpha t_{k+1}} - 1).$$

Hence, we can argue as in (3.16) to obtain (1.5) completing the proof.

□

Finally, let us show Theorem 1.2.

*Proof of Theorem 1.2.* From Proposition (2.2) taking $D_1 = 0$ we get

$$\left\|u(\cdot, t_1^-)\right\|_{L^\infty(\Omega)} \leq C_0 \left\|u_0(\cdot)\right\|_{L^\infty(\Omega)} e^{(CC_0+\epsilon-\lambda_1)t_1}.$$

Hence, from **H4** and Proposition (2.2) again, we obtain

$$\left\|u(\cdot, t)\right\|_{L^\infty(\Omega)} \leq C_0 \left\|g_1(u(\cdot, t_1^-))\right\|_{L^\infty(\Omega)} e^{(CC_0+\epsilon-\lambda_1)(t-t_1)} \text{ for } t_1 \leq t < t_2$$

which implies

$$\left\|u(\cdot, t)\right\|_{L^\infty(\Omega)} \leq C_0 M_1 \left\|u(\cdot, t_1^-)\right\|_{L^\infty(\Omega)} e^{(CC_0+\epsilon-\lambda_1)(t-t_1)} \leq C_0 M_1 \left\|u_0(\cdot)\right\|_{L^\infty(\Omega)} e^{(CC_0+\epsilon-\lambda_1)t}$$

for all $t_1 \leq t < t_2$. Consequently,

$$\left\|u(\cdot, t_2^-)\right\|_{L^\infty(\Omega)} \leq C_0 M_1 \left\|u_0(\cdot)\right\|_{L^\infty(\Omega)} e^{(CC_0+\epsilon-\lambda_1)t_2}.$$

Arguing in a similar way for $t_k \leq t < t_{k+1}$, one can have

$$\left\|u(\cdot, t)\right\|_{L^\infty(\Omega)} \leq (C_0)^k M_1 M_2 \dots M_k \left\|u_0(\cdot)\right\|_{L^\infty(\Omega)} e^{(CC_0+\epsilon-\lambda_1)t}$$

which completes the proof. □

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