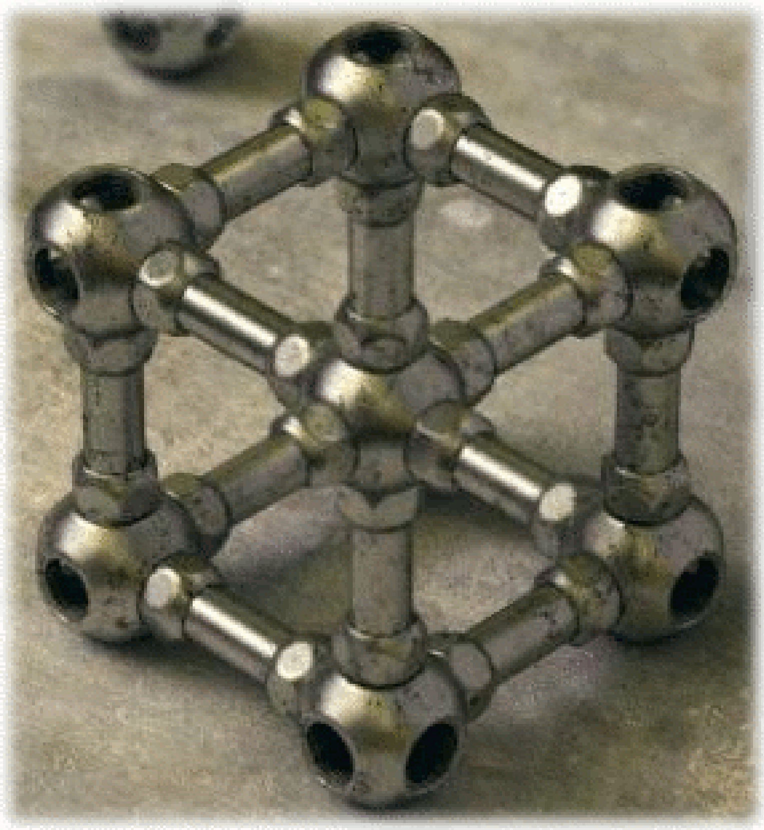

A Characterization for Quantum Logic Semantic Consequence as Algebraic Multipliers [1]

Author(s):
Mauricio Hernandes
Marcelo Finger

[1] This work was supported by Fapesp Project LogProb, grant 2008/03995-5, São Paulo, Brazil.

A Characterization for Quantum Logic Semantic Consequence as Algebraic Multipliers

Mauricio Hernandes [a*] Marcelo Finger [b †‡]

[a]*National Institute of Informatics*
School of Advanced Studies (Sokendai)
Japan
[b]*Departament of Computer Science*
Universidade de Sao Paulo
Brazil

Abstract

We prove that quantum logic (OrthoLogic) admits Algebraic Multipliers. Algebraic multipliers are an alternative form of characterizing validity in a logic system. They were shown to exist and to be computable for a class of classical, modal and multivalued logics. However, so far no such result was known for substructural logics. In this work, Orthologic is the first substructural logic in which validity in terms of algebraic multipliers has been established.

1 Introduction

We extended the results of algebraic multipliers for Substructural logics. The concept of these multipliers was introduced in [2] where only propositional classical logic and modal logics were presented.

In [3] the existence of multipliers was proposed from semantic point of view but due to inherent properties of Substructural Logics in this paper we slightly alter some key definitions to best fit the Substructural context. This modification is presented as a syntactic counterpart of Algebraic Multipliers.

To check whether a logic admits algebraic multipliers there is no other procedure than go deep in the logic's constraints (although the tools for such task were developed in [3]), hence to develop the result in the substructural level a logic was

*mauricio@nii.ac.jp

†mfinger@ime.usp.br

‡Partly supported by Fapesp Thematic Project 2008/03995-5 (LOGPROB) and 2010/51038-0, and by CNPq grant PQ 302553/2010-0.

needed and the chosen one was the Quantum Logics, more precisely, the Orthologic. The choice was quite obvious since this logic lacks distributivity, property which we don't require from our connectives.

This work contains in Section 1 the modifications that we made to be able to reach the substructural level and a fact about syntax meta-theorems for those logics. In Section 2 we present the orthologic and state the main result. In the Conclusion (Section 3) we make a comment about the answered questions and the open question here.

2 Background

First we slightly alter the definitions in [3]. Then we translate the result to a syntactic context.

2.1 Definitions

Definition 2.1. *Let $\boldsymbol{M}$ be a class of matrices for many-valued logics, and let $\mathcal{A} = \langle \boldsymbol{A}, D\rangle \in \boldsymbol{M}$. If $\mathcal{A}$ satisfies the axioms (where d, d_1, d_2 and d_3 are any element of D, f, f_1, f_2 and f_3 are any elements of $\mathbf{A} \setminus D$ and $a, b, c \in \mathbf{A}$):*

(am_1) $\neg d = f$
(am_2) $\neg f = d$
(am_3) $a \cdot f_1 = f_2$
(am_4) $f_1 \cdot a = f_2$
(am_5) $d_1 \cdot d_2 = d_3$
(am_6) $a + (b + c) = d_1$ *iff* $(a + b) + c = d_2$
(am_7) $f_1 + f_2 = f_3$
(am_8) $d_1 + d_2 = f$
(am_9) $d_1 + f = d_2$
(am_{10}) $a + b = d_1$ *iff* $b + a = d_2$

then, $\mathcal{A}$ is called ***Multiplier Matrix****.*

We say that $\boldsymbol{M}$ is a ***class of multiplier matrix*** *if each $\mathcal{A} \in \boldsymbol{M}$ is a multiplier matrix.*

Definition 2.2. *(Many-valued Characteristic Polynomial). Given an entailment statement $S = a_1, ..., a_n \vDash b_1, ..., b_m$ class of multiplier matrix M satisfying ($mm_1 - mm_{10}$) above, its* ***characteristic polynomial*** *over variables $x_1, ..., x_n, y_1, ..., y_m$ is*

$$CP_S(x_1, ..., x_n, y_1, ..., y_m) = x_1 \cdot (\neg a_1) + ... + x_n \cdot (\neg a_n) + y_1 \cdot b_1 + ... + y_m \cdot b_m$$

The characteristic polynomial has D-roots if there are terms $p_1, ..., p_n, q_1, ..., q_m$ such that for all $\langle \boldsymbol{A}, D\rangle \in \boldsymbol{M}$ and for all valuation τ

$$p_1^\tau \cdot (\neg a_1^\tau) + ... + p_n^\tau \cdot (\neg a_n^\tau) + q_1^\tau \cdot b_1^\tau + ... + q_m^\tau \cdot b_m^\tau \in D$$

The terms $p_1, ..., p_n, q_1, ..., q_m$ are ***entailment multipliers****. For convenience we denote the polynomial as $CP_S(X)$.*

The following is the substructural version of the theorem which links the polynomials and the semantic consequence.

Theorem 2.3 (Algebraic Multipliers for Many-valued Logics)**.** *An entailment statement of the form $S = a_1, ..., a_n \vDash b_1, ..., b_m$ over a class of multiplier matrix M is valid iff its characteristic polynomial $CP_S(X)$ has D-roots.*

Proof. For each $\langle \mathbf{A}, D \rangle \in \mathbf{M}$ repeat the proof in [3] □

2.2 Meta-Theorems

A multiplier matrix imposes restrictions and properties in its syntactic counterpart, the following theorem express this.

Theorem 2.4. *Let L be a logic such that its semantic matrix is $\boldsymbol{M}$. $\boldsymbol{M}$ is a multiplier matrix iff the following meta-theorems hold in L:*

$$\frac{A \vdash \bot}{\vdash \neg A}\,(Sm_1) \qquad \frac{}{A + (B + C) \vdash (A + B) + C}\,(Sm_6)$$

$$\frac{A}{\neg A \vdash \bot}\,(Sm_2) \qquad \frac{}{(A + B) + C \vdash A + (B + C)}\,(Sm_6)'$$

$$\frac{A \vdash \bot}{A \cdot B \vdash \bot}\,(Sm_3) \qquad \frac{}{(A + B) \vdash (B + A)}\,(Sm_7)$$

$$\frac{A \vdash \bot}{A \cdot B \vdash \bot}\,(Sm_4) \qquad \frac{A \vdash \bot \quad B \vdash \bot}{\vdash A + B}\,(Sm_8)$$

$$\frac{\vdash A \quad \vdash B}{\vdash B \cdot A}\,(Sm_5) \qquad \frac{\vdash A \quad \vdash B}{\vdash A + B}\,(Sm_9)$$

We will call any logic L that derives those meta-theorems a ***Multiplier Logic****.*

3 Orthologic

In this section we define the orthologic (**OL**) and show that **OL**derives the meta-theorems of a Multiplier Logic in Theorem 2.4. The characterization presented here was extracted from [4].

ORTHOLOGIC AXIOMS

$$\text{(OL1) } \Gamma \cup \{A\} \vdash A \text{ (identity)}$$

$$\text{(OL2) } \frac{\Gamma \vdash A \qquad \Delta \cup \{A\} \vdash B}{T \vdash A} \text{ (transitivity)}$$

$$\text{(OL3)} \Gamma \cup \{A \wedge B\} \vdash A \text{ (}\wedge\text{-elimination)}$$

$$\text{(OL4) } \Gamma \cup \{A \wedge B\} \vdash B \text{ (}\wedge\text{-elimination)}$$

$$\text{(OL5)} \frac{\Gamma \vdash A, \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{ ($\wedge$-introduction)}$$

$$\text{(OL6)} \frac{\Gamma \cup \{A, B\} \vdash C}{\Gamma \cup \{A \wedge B\} \vdash C} \text{ ($\wedge$-introduction)}$$

$$\text{(OL7)} \frac{A \vdash B \qquad A \vdash \neg B}{\vdash \neg A} \text{ (Absurdity)}$$

$$\text{(OL8) } \Gamma \cup \{A\} \vdash \neg\neg A \text{ (weak double negation)}$$

$$\text{(OL9) } \Gamma \cup \{\neg\neg A\} \vdash A \text{ (strong double negation)}$$

$$\text{(OL10) } \Gamma \cup \{A \wedge \neg A\} \vdash B \text{ (Duns Scotus)}$$

$$\text{(OL11)} \frac{A \vdash B}{\neg B \vdash \neg A} \text{ (Contraposition)}$$

Lemma 3.1 (Lemma 1). $\dfrac{\neg\neg A \vdash \neg\neg B}{A \vdash B}$ *(Lemma1)*

Proof.

$$\dfrac{\dfrac{\dfrac{}{A \vdash \neg\neg A}\text{(OL8)} \qquad \dfrac{}{\neg\neg A \vdash \neg\neg B}\text{(Hypothesis)}}{A \vdash \neg\neg B}\text{(OL2)} \qquad \dfrac{}{\neg\neg B \vdash B}\text{(OL9)}}{A \vdash B}\text{(OL2)}$$

□

Lemma 3.2 (Lemma2). $\dfrac{A \wedge B \vdash C}{\{A, B\} \vdash C}$ *(lemma2)*

Proof.

$$\dfrac{\dfrac{\{A, B\} \vdash A \qquad \{A, B\} \vdash B}{\{A, B\} \vdash A \wedge B}\text{(OL5)} \qquad \dfrac{}{A \wedge B \vdash C}\text{(Hypothesis)}}{\{A, B\} \vdash C}\text{(OL2)}$$

□

The main theorem follows

Theorem 3.3. *The following properties are hold for* ***OL****:*
(1) ***OL*** *is a multiplier matrix.*
(2) There is a fragment of ***OL*** *which is a multiplier logic.*

Proof. Let's prove (2) and (1) follows immediately. Define the following abbreviation:
(a) $A \cdot B \doteq A \wedge B$;
(b) $A + B \doteq \neg((\neg A) \cdot (\neg B))$.
And we prove that **OL** satisfy $(Sm_1) - (Sm_9)$ from theorem 2.4.

- (Sm_1) and (Sm_2) follow by $(OL11)$.
- (Sm_3)

$$\cfrac{\cfrac{B\cdot A\vdash A}{B\cdot A\ \vdash\ A}\,(OL4)\qquad \cfrac{}{A\vdash\bot}\,(Hypothesis)}{B\cdot A\ \vdash\ \bot}\,(OL2)$$

- (Sm_4)

$$\cfrac{\cfrac{A\cdot B\vdash A}{A\cdot B\ \vdash\ A}\,(OL3)\qquad \cfrac{}{A\vdash\bot}\,(Hypothesis)}{A\cdot B\ \vdash\ \bot}\,(OL2)$$

- (Sm_5) Follows from $(OL5)$.
- (Sm_6) Note that $A+(B+C)$ is equivalent to $\neg(\neg A\cdot\neg\neg(\neg B\cdot\neg C))$, and then:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg(\neg A\cdot\neg\neg(\neg B\cdot\neg C))\vdash\neg(\neg A\cdot\neg\neg(\neg B\cdot\neg C))}{(\neg A\cdot(\neg B\cdot\neg C)\vdash\neg A\cdot(\neg B\cdot\neg C)}\,(OL11\ OL9)}{\{\neg A,(\neg B\cdot\neg C)\}\vdash\neg A\cdot(\neg B\cdot\neg C)}\,(Lemma2)}{\{\neg A,\neg B,\neg C\}\vdash\neg A\cdot(\neg B\cdot\neg C)}\,(Lemma2)}{\{(\neg A\cdot\neg B),\neg C\}\vdash\neg A\cdot(\neg B\cdot\neg C)}\,(Lemma2)}{(\neg A\cdot\neg B)\cdot\neg C\vdash\neg A\cdot(\neg B\cdot\neg C)}\,(OL6)}{\neg((\neg A\cdot(\neg B\cdot\neg C))\vdash\neg((\neg A\cdot\neg B)\cdot\neg C)}\,(OL6)}{A+(B+C)\vdash(A+B)+C}\,(OL11)$$

- $(Sm_6)'$ Is analogous to (Sm_6).
- (Sm_7)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg(\neg A\cdot\neg B)\vdash\neg(\neg A\cdot\neg B)}{\neg A\cdot\neg B\ \vdash\ \neg A\cdot\neg B}\,(OL11,\ OL9,\ Lemma1)}{\{\neg A,\neg B\}\vdash\neg A\cdot\neg B}\,(Lemma2)}{\neg B\cdot\neg A\ \vdash\ \neg A\cdot\neg B}\,(OL6)}{\neg(\neg B\cdot\neg A)\vdash\neg(\neg A\cdot\neg B)}\,(OL11)}{A+B\ \vdash\ B+A}$$

- (Sm_8)

$$\cfrac{\cfrac{\cfrac{\cfrac{A\vdash\bot}{\vdash\neg A}\,(OL11)\qquad\cfrac{B\vdash\bot}{\vdash\neg B}\,(OL11)}{\vdash\neg A\cdot\neg B}\,(OL5)}{\neg(\neg A\cdot\neg B)\vdash\bot}\,(OL11)}{A+B\ \vdash\ \bot}$$

- (Sm_9)

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg A,\neg B\vdash\neg A\qquad\neg A\vdash\bot}{\neg A,\neg B\vdash\bot}\,(OL2)}{\neg A\cdot\neg B\vdash\bot}\,(OL6)}{\vdash\ \neg(\neg A\cdot\neg B)}\,(OL11)}{\vdash A+B}$$

$\square$

4 Conclusion

With this work we obtained a positive result for the existence of multipliers in a substructural logic. In addition, condition (2) in Theorem 3.3 also says the extension is not unique. Moreover, for the first time we have a syntax counterpart in the algebraic multipliers theory, which we believe is a fundamental step in substructural context.

The question about a logic which doesn't accept multipliers remains open.

To finish we state our conjecture about Intuitionism:

Conjecture 4.1. *Intuitionistic Logic is not a multiplier Logic.*

Reasoning. Suppose that there is $\sim, +$ and $\cdot$ satisfying the axioms from (Sm_1) to (Sm_9) in Intuicionistic Calculus. Derivate a contradiction proving that the set of connectives $\{\neg, \vee, \wedge, \rightarrow\}$ is not independent. $\square$

References

[1] W.J. Blok., D. Pigozzi., *Algebrizable logics*, American mathematical society, Rhode island, Memoirs of the american mathematical society, January 1989, v. 77, Number 396(third of 4 numbers).

[2] Marcelo Finger and Mauricio S. C. Hernandes, Entailment Multipliers: An Algebraic Characterization of Validity for Classical and Modal Logics, WoLLIC, 2010, 1-18.

[3] Marcelo Finger and Mauricio S. C. Hernandes, Entailment Multipliers: an Algebraic Characterization of Validity for Classical and Many-Valued Logics (To be announced)

[4] Gabbay, D.M and Guenthner, F. Handbook of Philosophical Logic Volume 6, 2ed. Kluwer Academic. Dordrecht, Boston, London.