Hyperbolic Weyl Groups

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Information and Thanks To The Organizing Committee

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Definitions: Cartan Matrix, Bilinear Form, Weyl Group

Let $I = \{1, \cdots, n\}$.
Let $C = [c_{ij}]$ be an $n \times n$ symmetric hyperbolic generalized Cartan matrix, so C has signature $(n-1, 1)$.

Let $V = \mathbb{Q}^n$ with basis $\Pi = \{\alpha_1, \cdots, \alpha_n\}$, symmetric bilinear form $S(\alpha_i, \alpha_j) = c_{ij}$ and quadratic form $q(\alpha) = S(\alpha, \alpha)$.
The *simple reflections*

$$w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i, \qquad i \in I,$$

generate the *Weyl group* $\mathcal{W} = \mathcal{W}(C)$, a subgroup of the orthogonal group $O(V)$.
$\mathcal{W}$ acts on the integral lattice $\Lambda = \mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$.

Weyl Group Relations

For $1 \leq i \neq j \leq n$, define

$$m_{ij} = 2, 3, 4, 6, \infty \text{ when } c_{ij}c_{ji} = 0, 1, 2, 3, \geq 4, \text{respectively.}$$

Then the relations among the simple reflections are given by:

$$|w_i w_j| = m_{ij} \text{ for } 1 \leq i \neq j \leq n, \quad \text{and} \quad |w_i| = 2 \text{ for } 1 \leq i \leq n.$$

Our goal is to understand these hyperbolic Weyl groups in a new way as explicit matrix groups, extending the work of Feingold-Frenkel [1983], and modifying the work of Feingold-Kleinschmidt-Nicolai [2009].

They used division algebras, but we will use Clifford algebras.

Weyl Groups as Matrix Groups

The hyperbolic Weyl group studied by Feingold-Frenkel is the hyperbolic triangle group

$$\mathcal{W} = T(2,3,\infty) \cong PGL(2,\mathbb{Z})$$

coming from the Cartan matrix

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Realization of Weyl Group Action

The key point for understanding that first example was to use 2×2 real symmetric matrices,

$$V = H_2(\mathbb{R}) = \left\{ X = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

with quadratic form $q(X) = -2\det(X)$, and the lattice

$$\Lambda = H_2(\mathbb{Z}) = \left\{ X = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

with the action of

$A \in PGL(2, \mathbb{Z})$ on X given by $A \cdot X = AXA^t$. In this case,

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$w_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Division Algebra Realization of Hyperbolic Root Lattices

In [2] this was generalized to cover cases when $n = 3, 4, 6, 10$.
Let $\mathbb{K}$ be a division algebra: $\mathbb{R}$ (real), $\mathbb{C}$ (complex), $\mathbb{H}$ (quaternion) or $\mathbb{O}$ (octonian), each with a conjugation, $a \to \bar{a}$, and let

$$H_2(\mathbb{K}) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \;\middle|\; a, c \in \mathbb{R}, \quad b \in \mathbb{K} \right\}$$

be the Hermitian matrices, $X = X^\dagger = \bar{X}^t$, with Lorentzian quadratic form $q(X) = -2\det(X)$. They found a finite type root lattice Q of rank $n-2$ in $\mathbb{K}$, and obtained a hyperbolic root system $\Phi = \{X \in \Lambda(Q) \mid q(x) \leq 2\}$ in the root lattice

$$\Lambda(Q) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \;\middle|\; a, c \in \mathbb{Z}, \quad b \in Q \right\}.$$

Realization of Hyperbolic Weyl Group Action

Using the index set $I = \{-1, 0, 1, ..., n\}$, simple roots of finite type $a_1, \cdots, a_n$ in $Q \subset \mathbb{K}$, with highest root θ normalized so that $\theta\bar{\theta} = 1$, they found hyperbolic simple roots

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \alpha_0 = \begin{bmatrix} -1 & -\theta \\ -\bar{\theta} & 0 \end{bmatrix}, \ \alpha_i = \begin{bmatrix} 0 & a_i \\ \bar{a}_i & 0 \end{bmatrix}, 1 \leq i \leq n,$$

and matrices

$$M_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ M_0 = \begin{bmatrix} -\theta & 1 \\ 0 & \bar{\theta} \end{bmatrix}, \ M_i = \begin{bmatrix} \varepsilon_i & 0 \\ 0 & -\bar{\varepsilon}_i \end{bmatrix}, 1 \leq i \leq n,$$

where $\varepsilon_i = a_i / \sqrt{a_i \bar{a}_i}$, such that $w_j(X) = M_j \bar{X} M_j^\dagger$ for $-1 \leq j \leq n$.

Realization of Hyperbolic Weyl Groups

This enabled them to describe the even subgroup $\mathcal{W}^+$ as a small degree extension of a matrix group $PSL(2, Q)$ where Q is a ring of integers.

In the original example, $Q = \mathbb{Z}$, but when $\mathbb{K} = \mathbb{C}$ the ring Q could be either the Eisenstein or the Gaussian integers.

When $\mathbb{K} = \mathbb{H}$ the ring Q was the Hurwitz integers, and when $\mathbb{K} = \mathbb{O}$ the ring Q was the octavians.

The most challenging case to understand was when $\mathbb{K} = \mathbb{O}$, because of the non-associativity of $\mathbb{O}$.

Further work on that case was done by Kleinschmidt-Nicolai-Palmqvist [3].

Clifford Algebra

Let $V = \mathbb{Q}^n$ with symmetric bilinear form $S(a,b)$ and quadratic form $q(a) = S(a,a)$.

Let $\mathcal{C} = \mathcal{C}(V,S) = \mathcal{C}(V,q)$ be the associated universal Clifford algebra with unit element $1_\mathcal{C}$.

Identify V as generators of $\mathcal{C}$ with the relations $v^2 = -q(v)1_\mathcal{C}$, giving $vw + wv = -2S(v,w)1_\mathcal{C}$ for all $v, w \in V$.

Identify $\mathbb{Q} = \mathbb{Q}1_\mathcal{C}$ as a subspace of $\mathcal{C}$ so $v^2 = -q(v)$.

For all $v \in V$, on $\mathcal{C}$ define:

(1) The automorphism determined by $v' = -v$,

(2) The anti-involution determined by $v^* = v$,

(3) The anti-involution determined by $\bar{v} = (v^*)' = (v')^* = -v$.

The last one is called the *Clifford conjugation*.

Clifford Group

Define the (Atiyah-Bott-Shapiro) action of $x \in \mathcal{C}^\times$ (invertibles) on $y \in \mathcal{C}$ by

$$x * y = xy(x')^{-1}$$

and define the Clifford group

$$\Gamma(V) = \{x \in \mathcal{C}^\times \mid x * V \subseteq V\}.$$

For $v \in V$, $x \in \Gamma(V)$ we have

$$x * v = -(x * v)' = -x'v'x^{-1} = x'vx^{-1}$$

which implies

$$S(x * v, x * w) = S(v, w)$$

so $v \to x * v$ is an orthogonal transformation on V.

Clifford Group

For any non-isotropic $v \in V$, $v^{-1} = -v/q(v)$ so

$$v * w = -vwv^{-1} = (wv + 2S(v, w))v^{-1}$$
$$= w - \frac{2S(v, w)}{q(v)} v = r_v(w)$$

is the reflection with respect to v.

Theorem: The map $\rho : \Gamma(V) \longrightarrow O(V)$ defined by $\rho(x)(v) = x * v$ for $x \in \Gamma(V)$, $v \in V$, has $\ker(\rho) = \mathbb{Q}^\times$ giving short exact sequence

$$1 \longrightarrow \mathbb{Q}^\times \longrightarrow \Gamma(V) \xrightarrow{\rho} O(V) \longrightarrow 1.$$

Clifford group $\Gamma(V)$ is generated by the non-isotropic vectors in V.

Abstract Pin Groups

Suppose we are given:

1. Group G and a non-singular quadratic form on V over field F.
2. A short exact sequence of groups
$$1 \longrightarrow F^{\times} \longrightarrow G \xrightarrow{\rho} O(V) \longrightarrow 1.$$
3. A group morphism $N : G \longrightarrow F^{\times}$ satisfying $N(\lambda) = \lambda^2$ whenever $\lambda \in F^{\times}$.

Then, we get a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F^{\times} & \longrightarrow & G & \xrightarrow{\rho} & O(V) & \longrightarrow & 1 \\
 & & \downarrow f & & \downarrow N & & \downarrow \vartheta & & \\
1 & \longrightarrow & F^{\times 2} & \longrightarrow & F^{\times} & \longrightarrow & F^{\times}/F^{\times 2} & \longrightarrow & 1,
\end{array}$$

where $f : F^{\times} \to F^{\times 2}$ is given by $x \mapsto f(x) = x^2$ and ϑ is induced from N so that if $\rho(g) = \sigma$, then $\vartheta(\sigma) = N(g) \cdot F^{\times 2}$.

Define the group $Pin^+(\rho, N) = \ker(N)$.

Further diagram chasing gives the following exact sequence:

$$1 \longrightarrow \{\pm 1\} \longrightarrow Pin^+(\rho, N) \xrightarrow{\rho} O(V) \xrightarrow{\vartheta} F^\times / F^{\times 2}. \qquad (1)$$

ϑ is called the spinor norm morphism.

$Pin^+(V)$

Let $V = \mathbb{Q}^n$ have non-singular quadratic form q, let $\mathcal{C} = \mathcal{C}(V, q)$, and let $\rho : G = \Gamma(V) \to O(V)$ as before.
For $x \in \Gamma(V)$ write $x = v_1 \cdot \ldots \cdot v_m$ for $v_i \in V$ with $q(v_i) \neq 0$, so

$$\begin{aligned} x\overline{x} &= v_1 \cdot \ldots \cdot v_m \cdot \overline{v_1 \cdot \ldots \cdot v_m} \\ &= v_1 \cdot \ldots \cdot v_m \overline{v}_m \cdot \ldots \cdot \overline{v}_1 \\ &= q(v_1) \cdot \ldots \cdot q(v_m) \in \mathbb{Q}^\times. \end{aligned}$$

Define $N : \Gamma(V) \to \mathbb{Q}^\times$ by $N(x) = x\overline{x} = \overline{x}x$, so $N(\lambda) = \lambda^2$ if $\lambda \in \mathbb{Q}^\times$.
Applying the abstract construction, we let

$$Pin^+(V) = Pin^+(\rho, N)$$

and

$$O^+(V) = \rho(Pin^+(V)).$$

Vahlen Groups

Let $P = \mathbb{Q}^2$ have bilinear form S_2 and isotropic basis $\{f_1, f_2\}$ such that $S_2(f_1, f_2) = -\frac{1}{2}$. Let $V = \mathbb{Q}^n$ have positive definite bilinear form S_1 and let $W = V \perp P$ be the orthogonal direct sum with bilinear form S. For $w = v + \lambda_1 f_1 + \lambda_2 f_2 \in W$, the quadratic form $q(w) = q_1(v) - \lambda_1\lambda_2$. Define linear map $\phi : W \to M_2(\mathcal{C}(V))$ by

$$\phi(w) = \begin{pmatrix} v & \lambda_1 \\ \lambda_2 & \bar{v} \end{pmatrix}$$

so that $\phi(w)^2 = -q(w)I_2$. By the universal property of $\mathcal{C}(W)$, we get a $\mathbb{Q}$-algebra isomorphism

$$\phi : \mathcal{C}(W) \to M_2(\mathcal{C}(V))$$

so $\phi(\mathcal{C}(W)^\times) = M_2(\mathcal{C}(V))^\times$.

Define the Vahlen group of

$$\mathcal{V}(V) = \phi(\Gamma(W)) \leq M_2(\mathcal{C}(V))^{\times}.$$

The three (anti) involutions $'$,*,$\bar{\ }$ of $\mathcal{C}(W)$ correspond to (anti) involutions α, β, γ of $M_2(\mathcal{C}(V))$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{C}(V))$ we have:

$$\alpha(A) = \begin{pmatrix} a' & -b' \\ -c' & d' \end{pmatrix}, \quad \beta(A) = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}, \quad \gamma(A) = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}.$$

Then for all $x \in \mathcal{C}(W)$ we have

$$\phi(x') = \alpha(\phi(x)), \quad \phi(x^*) = \beta(\phi(x)), \quad \phi(\bar{x}) = \gamma(\phi(x)).$$

Conditions for $A \in \mathcal{V}(V)$

Theorem A: $A \in \mathcal{V}(V)$ if and only if the following are satisfied:

1. $ad^* - bc^* = d^*a - b^*c = \lambda \in \mathbb{Q}^\times$,
2. $ba^* - ab^* = cd^* - dc^* = 0$,
3. $a^*c - c^*a = d^*b - b^*d = 0$,
4. $a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbb{Q}$,
5. $b\bar{d}, a\bar{c} \in V$,
6. $av\bar{b} + b\bar{v}\bar{a}, cv\bar{d} + d\bar{v}\bar{c} \in \mathbb{Q}$ for all $v \in V$,
7. $av\bar{d} + b\bar{v}\bar{c} \in V$ for all $v \in V$.

Let $H_2(V) = \phi(W) = \left\{ X = \begin{pmatrix} v & \lambda_1 \\ \lambda_2 & \bar{v} \end{pmatrix} \middle| v \in V \text{ and } \lambda_1, \lambda_2 \in \mathbb{Q} \right\}$

with non-singular quadratic form

$$q(X) = v\bar{v} - \lambda_1 \lambda_2$$

and S the corresponding symmetric $\mathbb{Q}$-bilinear form.

Let $A^\sharp = \alpha(A)^{-1} = \frac{1}{\lambda} \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \frac{1}{\lambda} \beta(A)$ for $\lambda = ad^* - bc^*$.

The Vahlen group $\mathcal{V}(V)$ acts on $H_2(V)$ by

$$A \cdot X = AXA^\sharp \quad \text{for} \quad A \in \mathcal{V}(V) \quad \text{and} \quad X \in H_2(V).$$

We then get a representation $\eta : \mathcal{V}(V) \to O(H_2(V))$. Since ϕ restricted to W gives us an isometry $W \xrightarrow{\simeq} H_2(V)$, we get an isomorphism of groups $\hat{\phi} : O(W) \longrightarrow O(H_2(V))$ given by

$$\hat{\phi}(\sigma) = \phi \circ \sigma \circ \phi^{-1} \text{ for } \sigma \in O(W).$$

In the following commutative diagram the two vertical maps are isomorphisms, and the rows are exact:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & \Gamma(W) & \xrightarrow{\rho} & O(W) & \longrightarrow & 1 \\ & & \| & & \downarrow \phi & & \downarrow \hat{\phi} & & \\ 1 & \longrightarrow & F^\times & \longrightarrow & \mathcal{V}(V) & \xrightarrow{\eta} & O(H_2(V)) & \longrightarrow & 1, \end{array}$$

If $X \in H_2(V)$ is non-isotropic then $\eta(X) = r_X$.

Define a spinor norm for Vahlen groups, $N : \mathcal{V}(V) \to \mathbb{Q}^\times$ by $N(A) = A \cdot \gamma(A)$. Applying the abstract definition of Pin^+, we get

$$\mathcal{V}^+(V) = Pin^+(\eta, N) = ker(N) = \phi(Pin^+(W)).$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{C}(V))$ we have:

$A \in \mathcal{V}^+(V)$ if and only if all conditions of Theorem A are satisfied with (1) replaced by: $ad^* - bc^* = d^*a - b^*c = 1$

Canonical Double Extensions

Let $C = [c_{ij}]$ be a finite type T_n irreducible symmetric Cartan matrix, $V = \mathbb{Q}^n$ with basis $\{\alpha_1, \cdots, \alpha_n\}$ and rescaled symmetric bilinear form $S_1(\alpha_i, \alpha_j) = c_{ij}/2$.

The root system Φ of type T_n has highest root $\theta = \sum_{i=1}^n a_i \alpha_i$ for which $S_1(\theta, \theta) = 1$.

Let P and $W = V \perp P$ be as before.

Letting

$$\alpha_{-1} = f_1 - f_2 \quad \text{and} \quad \alpha_0 = -f_1 - \theta,$$

we find that $\Pi = \{\alpha_{-1}, \alpha_0, \alpha_1, \cdots, \alpha_n\}$ determines a symmetric $(n+2) \times (n+2)$ Lorentzian Cartan matrix $C^{++} = [2S(\alpha_i, \alpha_j)]$, whose type we denote by T_n^{++}.

This class of Cartan matrix we call a “canonical double extension”.

Apply the previous Vahlen group construction to this W. Recall

$$\phi : \mathcal{C}(W) \xrightarrow{\simeq} M_2(\mathcal{C}(V))$$

induced by $\phi(v + \lambda_1 f_1 + \lambda_2 f_2) = \begin{pmatrix} v & \lambda_1 \\ \lambda_2 & \bar{v} \end{pmatrix} \in M_2(\mathcal{C}(V))$. We have $\phi(\alpha_i) = X_i$ for $\alpha_i \in \Pi$ where:

$$X_{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; X_0 = \begin{pmatrix} -\theta & -1 \\ 0 & -\bar{\theta} \end{pmatrix}, \; X_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & \bar{\alpha}_i \end{pmatrix} \text{ for } 1 \leq i \leq n.$$

We have the finite and Lorentzian lattices, respectively,

$$\Lambda = \sum_{i=1}^{n} \mathbb{Z}\alpha_i \subseteq V \quad \text{and} \quad \Lambda^{++} = \sum_{i=-1}^{n} \mathbb{Z}\alpha_i \subseteq W.$$

$$\phi(\Lambda^{++}) = H_2(\Lambda) = \left\{ \begin{pmatrix} v & n_1 \\ n_2 & \bar{v} \end{pmatrix} \middle| \, v \in \Lambda \text{ and } n_1, n_2 \in \mathbb{Z} \right\} \subseteq M_2(\mathcal{C}(V)).$$

Lattice Structures

Let

$$O(H_2(\Lambda)) = \{\sigma \in O(H_2(V)) \,|\, \sigma\,(H_2(\Lambda)) = H_2(\Lambda)\}$$

be the group of units of this lattice, and let

$$O^+(H_2(\Lambda)) = O(H_2(\Lambda)) \cap O^+(H_2(V)).$$

Let $\mathcal{O} = \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \subseteq \mathcal{C}(V)$ be the ring generated by the α_i.

Note that $\mathcal{O}$ is closed under the (anti) involutions $'$,*, and $^{-}$.

Define

$$\mathcal{V}(\mathcal{O}) = \mathcal{V}(V) \cap M_2(\mathcal{O})^\times \quad \text{and} \quad \mathcal{V}^+(\mathcal{O}) = \mathcal{V}^+(V) \cap M_2(\mathcal{O})^\times.$$

Conditions for $A \in \mathcal{V}(\mathcal{O})$

Theorem: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O})$. Then $A \in \mathcal{V}(\mathcal{O})$ when:

1. $ad^* - bc^* = d^*a - b^*c = \pm 1$,
2. $ba^* - ab^* = cd^* - dc^* = 0$,
3. $a^*c - c^*a = d^*b - b^*d = 0$,
4. $a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbb{Z}$,
5. $b\bar{d}, a\bar{c} \in \Lambda$,
6. $av\bar{b} + b\bar{v}\bar{a}, cv\bar{d} + d\bar{v}\bar{c} \in \mathbb{Z}$ for all $v \in \Lambda$,
7. $av\bar{d} + b\bar{v}\bar{c} \in \Lambda$ for all $v \in \Lambda$.

Proposition: With the notation as above, we have

1. $\eta(\mathcal{V}(\mathcal{O})) \subseteq O(H_2(\Lambda))$,
2. $\eta(\mathcal{V}^+(\mathcal{O})) \subseteq O^+(H_2(\Lambda))$

Since $q(X_i) = 1$ for $-1 \leq i \leq n$, the Weyl group $\mathcal{W}(C^{++})$ is a subgroup of $O^+(H_2(\Lambda))$. Letting

$$\Gamma = \langle X_i \mid -1 \leq i \leq n \rangle \leq \mathcal{V}^+(\mathcal{O})$$

we have $\eta(\Gamma) = \mathcal{W}(C^{++})$ because $\eta(X) = r_X$. Therefore, we have

$$\mathcal{W}(C^{++}) \subseteq \eta\left(\mathcal{V}^+(\mathcal{O})\right) \subseteq O^+(H_2(\Lambda)).$$

Let

$$\mathcal{PV}(\mathcal{O}) = \mathcal{V}(\mathcal{O})/\{\pm 1\} \quad \text{and} \quad \mathcal{PV}^+(\mathcal{O}) = \mathcal{V}^+(\mathcal{O})/\{\pm 1\}.$$

We finally come to our main result!
Details will appear in Feingold-Vallières [5].

Main Result

Theorem: With the notation as above, η induces an isomorphism of groups

$$\eta : \mathcal{PV}^+(\mathcal{O}) \xrightarrow{\simeq} \mathcal{W}(C^{++})$$

for the following hyperbolic canonical double extensions:

1. A_n^{++} for $n = 1, 2, 3, 4, 5, 6$,
2. D_n^{++} for $n = 5, 6, 7, 8$,
3. E_n^{++} for $n = 6, 7, 8$.

Proof of Main Result

Corollary 5.10 of Kac [4] shows that for each hyperbolic canonical double extension with symmetric Cartan matrix, one has

$$O(H_2(\Lambda)) = \pm \mathrm{Aut}(C^{++}) \ltimes \mathcal{W}(C^{++}),$$

where $\mathrm{Aut}(C^{++})$ is the group of outer automorphisms of the corresponding Dynkin diagram. It is clear that $-id \notin O^+(H_2(\Lambda))$. For each of the hyperbolic canonical double extensions with a symmetric Cartan matrix, we computed the spinor norm of $\pm a$, for each outer automorphism a. These spinor norms are non-trivial exactly in the cases listed in the theorem. Thus, in those cases, the chain of subgroups above induces the following equalities

$$\mathcal{W}(C^{++}) = \eta\left(\mathcal{V}^+(\mathcal{O})\right) = O^+(H_2(\Lambda)).$$

References

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Thank you for your kind attention!