Minimal Ideals of Jordan Systems

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For associative algebras it is well known that a minimal ideal of an associative algebra is either simple or trivial (all products zero). This was established for linear Jordan algebras (in the presence of $\frac{1}{2}$) by Medvedev in 1987 and Skosirskii in 1988. Partial results for quadratic Jordan algebras were obtained by Nam and McCrimmon in 1983. In 2007 Anquela and Cortes established that minimal ideals I of quadratic Jordan systems J (algebras, triples, or pairs) over an arbitrary ring of scalars Φ. were either simple or *trivial* in the sense that all triple products $\{I, I, I\}, U_I(I), P_I(I), Q_{I^\varepsilon}(I^{-\varepsilon})$ vanish.

Thus as systems in their own right these triples and pairs have zero products, but in the case of algebras it is not obvious that a minimal ideal I which is *cubeless* $U_I I = 0$ is also trivial as an algebra, i.e., *squareless* $I^2 = 0$ (implying $\{I, I, J\} = \{I, J, I\} = 0$ as well). We will close this gap, and prove the stronger result that trivial minimal ideals are *doubly trivial: all products of degree two* in algebras, pairs, and triples also vanish.

The Algebra Case: $U_I(\widehat{J}) = 0$.

Since J and its unital hull $\widehat{J}$ have the same ideals, may **assume that J is unital**. We will prove that $U_I J = 0$, in particular $I^2 = U_I(1) = 0$, by showing that $\boldsymbol{U_I J \neq 0}$ leads to a **contradiction.** Otherwise, since $U_I J$ is an ideal of J contained in I it equals I by minimality, hence $I = U_I(J) = U_{U_I J} J$ is spanned for $w_i \in I, a_i \in J$ by elements

$$\begin{aligned} U_{\sum_i U_{w_i} a_i} J &= \sum_i U_{U_{w_i} a_i} J + \sum_{i<j} U_{U_{w_i} a_i, U_{w_j} a_j} J \\ &\subseteq 0 + \sum_{i<j} U_{I,I} J = \{I, J, I\}, \end{aligned}$$

since all $z_i := U_{w_i} a_i$ are trivial $[U_{z_i} J = U_{w_i} U_{a_i} U_{w_i} J \subseteq U_I\big(U_J U_I J\big) \subseteq U_I I = 0$ by **cubelessness**$]$ leading to $\boldsymbol{I = \{I, J, I\} = V_{I,J}(I).}$

Choose a nonzero trivial element $z = U_w a \in U_I J$, so the ideal in J it generates is by minimality $I = \mathcal{M}(J)z$ where the multiplication algebra $\mathcal{M}(J)$ is spanned by all $U_a, a \in J$. Thus $I = V_{I,J}(I) = V_{I,J}\mathcal{M}(J)z$ and $z = T(z)$ for $T = \sum_i V_{w_i,y_i} U_{x_{i1}} \cdots U_{x_{in(i)}}$ for $w_i \in I, y_i, x_{ij} \in J$. Let X_0 be the unital subspace spanned by the finite set of all y_i, x_{ij} appearing in this sum (including $x_0 = 1$), and $\mathcal{M}_0$ the unital subalgebra generated by all $U_x, x \in X_0$. Thus $\boldsymbol{T \in V_{I,X_0}\mathcal{M}_0}$.

We have a **Migration Lemma** moving $\mathcal{M}_0$ to the left past V_{I,X_0}, $\boldsymbol{V_{I,X_0}\mathcal{M}_0 \subseteq \mathcal{M}_0 V_{I,X_0}}$ repeatedly using the fact that for $w \in I, x, y \in X_0$ we have

$$V_{w,y}U_x = -V_{w_3,x_0} + V_{w_1,x} + U_{x,x_0}V_{w_1,x_0} - U_x V_{w_2,x_0} + U_x V_{w,y}$$

for $w_1 = \{x, y, w\}, w_2 = y \circ w, w_3 = x \circ w_1 \in I$. Thus by induction on m,

$$T^m \in \overbrace{(V_{I,X_0}\mathcal{M}_0)\cdots(V_{I,X_0}\mathcal{M}_0)}^{m} \subseteq \mathcal{M}_0 \overbrace{V_{I,X_0}\cdots V_{I,X_0}}^{m}.$$

But for fixed $z_i \in I$ each $V_{z_1,y_1} \cdots V_{z_m,y_m}$ is an alternating multilinear function of $y_1, \ldots, y_m \in X_0$ modulo the ideal $\mathcal{Z}$ of multiplications which annihilate I, $Z(I) = 0$, since $V_{I,y}V_{I,y} \subseteq V_{I,U_yI} + U_{I,I}U_y \subseteq V_{I,I} + U_{I,I}U_J$ maps I into $\{I, I, I\} \subseteq U_I I = 0$ by **cubelessness**. This alternating function must vanish on the finitely-spanned subspace X_0 as soon as m exceeds the rank of the subspace, so for suitably large m we have $z = T^m(z) \subseteq Z(I) = 0$, the desired **contradiction**.

The General Case:

All $P_I J = \{I, I, J\} = 0$ for Jordan triples and pairs

It suffices to establish the result for triples: a Jordan pair $V := (V^+, V^-)$ determines a polarized Jordan triple $T(V) = V^+ \oplus V^-$ where triple and pair products coincide via $P_{V^\varepsilon}(V^{-\varepsilon}) = Q_{V^\varepsilon}(V^{-\varepsilon})$ and $P_{V^\varepsilon}(V^\varepsilon) = \{V^\varepsilon, V^\varepsilon, V\}$ $= 0$, and with some effort one can check that $I = (I^+, I^-)$ is a minimal ideal of V iff $T(I) = I^+ \oplus I^-$ is a minimal ideal of $T(V)$. Thus it suffices to consider only trivial minimal ideals $P_I(I) = 0$ of a triple system J, and to prove *double triviality* $P_I J = \{I, I, J\} = 0$.

Using Jordan triple multiplication rules such as $L_{x,a}P_y = P_{y,\{a,x,y\}} - P_y L_{x,a}$ for $a \in I$ and $x, y \in X$, we obtain the $M_{I,X}$-**Migration Lemma** $M_{I,X}\mathcal{M}_X \subseteq \mathcal{M}_X M_{I,X}$ and the **Switching Lemma** $M_{I,x}M_{I,y} \subseteq \widehat{M_{I,y}}M_{I,x} + \mathcal{Z}$ where $M_{I,X} = L_{X,I} + L_{I,X} + P_{X,I}$ denotes all the degree-1 multiplications by I and elements of X, and $\mathcal{M}_X$ is the unital subalgebra generated by all X-multiplications $L_{X,X}, P_{X,X}, P_X$ and $\mathcal{Z}$ is the ideal of multiplications which annihilate I, $\mathcal{Z}(I) = 0$.

LEMMA. (1)$\{I, I, J\} \subseteq \{I, J, I\} \subseteq P_I(J)$, so that if $P_I J = 0$ then I is doubly trivial. (2) If $P_I J \neq 0$ then $I = P_I J = \{I, J, I\} = L_{I,J}(I)$. $\square$

THEOREM. Any trivial minimal ideal I in a Jordan triple system is doubly trivial.

PROOF: By (1) we may ASSUME $\boldsymbol{P_I J \neq 0}$**,** so some $z := P_w y \neq 0$ for $w \in I, y \in J$. Again z is trivial and $I = \mathcal{M}(J)z$. Since by (z), $z \in I = L_{I,J}(I) = L_{I,J}\mathcal{M}(J)z$, there is a finite set $X = \{x_1, \ldots, x_n\} \subseteq J$ of elements appearing in this relation and we have $z = T(z)$ for $T \in L_{I,X}\mathcal{M}_X$. Thus again

$$T^m \in \overbrace{(M_{I,X}\mathcal{M}_X)\cdots(M_{I,X}\mathcal{M}_X)}^{m} \subseteq \mathcal{M}_X \overbrace{M_{I,X}\cdots M_{I,X}}^{m}$$

by the $M_{I,X}$-Migration Lemma.

The proof for triples is more involved than that for algebras. We will prove that $T^m \in \mathcal{Z}$ for $m \geq 4n+1$, so $z = T^m(z) = 0$, contradicting our assumption. For $m \geq 4n+1$ one of the n different x_i must appear at least 5 times, and by the Switching Lemma we can move them all to the end, so it suffices to prove that $M^5_{I,x} \subseteq \mathcal{Z}$. Each string of 5 $L_{x,I}, L_{I,x}, P_{x,I}$ with the same x can be normalized modulo $\mathcal{Z}$ as follows: the Jordan triple relations and $P_{I,I}, L_{I,I} \subseteq \mathcal{Z}$ by triviality imply

(I) $L_{I,x}L_{x,I}$ can be replaced by $P_{x,I}P_{x,I}$, and vice versa

(II) $L_{I,x}P_{x,I}$ can be replaced by $P_{x,I}L_{x,I}$,

(III) $L_{I,x}L_{I,x}, L_{x,I}L_{x,I}, L_{x,I}P_{x,I}$, $P_{x,I}L_{I,x}$ can be replaced by 0.

From (I)-(III) we can assume any $L_{I,x}$ appears at the end (and by (III) there is at most one of them), so the string has an initial substring of at least 4 terms consisting only of $L_{x,I}$'s and $P_{x,I}$'s. But $L_{x,I}$ cannot be followed by $L_{x,I}$ or $P_{x,I}$ by (III), so there is at most one $L_{x,I}$ at the end. Thus there must be a string of at least 3 $P_{x,I}$, and we have $P_{x,I}P_{x,I}P_{x,I} \subseteq P_{x,I}L_{I,x}L_{x,I} + \mathcal{Z}$ [by (I)] $\subseteq \mathcal{Z}$ [by (III)]. Thus any string of 5 terms with the same x falls in $\mathcal{Z}$, leading to the **contradiction** $z = T^m(z) \in \mathcal{Z}(I) = 0$.