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Nonsupercommutative Jordan superalgebras of capacity $n \geq 2$

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- An analogue of K. McCrimmon's Coordinatization Theorem
- Simplicity of the symmetrized Jordan superalgebra of capacity $n > 1$ — an analog of R. Oehmke's theorem
- The problem of classification of the simple finite dimensional nonsupercommutative Jordan superalgebras

Noncomm. J. a.: 1948: A. A. Albert;
Noncomm. J. a. $\chi = 0$: R.D.Schafer;
Flexible p.-a. a. $\chi \neq 2$: R.H.Oehmke;
Noncom. J.a. cap. > 2: K.McCrimmon;
capacity two: K.C.Smith;
positive characteristic: L.Kokoris.

The f.d. simple J. s.a. ($F = \bar{F}^0$):

V. G. Kac; I. Kantor;

J. s.a. $\chi > 0$: **I. Kaplansky;**

f.d. simple J. s.a. $\chi \neq 2$, s.s. even part:

M. Racine and E. Zelmanov;

the even part is not semisimple:

C. Martinez and E. Zelmanov.

Noncomm. J.a.

$$(x^2y)x = x^2(yx),\ (xy)x = x(yx)$$

Examples: alternative; anticommutative

$$a \cdot_\lambda b = \lambda ab + (1-\lambda)ba$$

$A(\alpha_1, \dots, \alpha_n)$ — **Cayley-Dickson**

Quadratic flexible

- **F.d. nil-s.s. possesses the unity, and it is the direct sum of simple**
- **an analog of the Wedderburn theorem on nil-radical does not hold**

1. Coordinatization Theorem

1. Defining identities. $U = U_{\bar{0}} \oplus U_{\bar{1}}$; L_x, R_x;
$(-1)^{xy} := (-1)^{p(x)p(y)}$; $(-1)^{x,y,z} := (-1)^{xy+xz+yz}$;
$[x,y] = xy - (-1)^{xy} yx$; $x \circ y = xy + (-1)^{xy} yx$.

Nonsupercomm. Jordan superalgebra:

$$
\begin{aligned}
[R_{x\circ y}, L_z] &+ (-1)^{x(y+z)} [R_{y\circ z}, L_x] \\
&+ (-1)^{z(x+y)} [R_{z\circ x}, L_y] = 0, \\
[R_x, L_y] &= [L_x, R_y] .
\end{aligned}
$$

The flexibility may be written as:

$$\begin{aligned}
&(-1)^{xy}L_{xy} - L_y L_x = R_{yx} - R_y R_x,\\
&(x,y,z) = -(-1)^{x,y,z}(z,y,x),
\end{aligned}$$

From the definition, we have in NJSA:

$$\begin{aligned}
&[E_{x\circ y}, F_z] + (-1)^{x(y+z)}[E_{y\circ z}, F_x]\\
&\qquad +(-1)^{z(x+y)}[E_{z\circ x}, F_y] = 0,\\
&[E_x, F_y] + (-1)^{xy}[E_y, F_x] = 0,
\end{aligned}$$

where $\{E,F\} = \{R,L\}$**.**

Lemma. In an arbitrary flexible s.a. A the following operator identities hold:

1) $L_{x\circ y} - L_x \circ L_y = R_{x\circ y} - R_x \circ R_y$;

2) $[R_x \circ R_y, R_z] - [R_x, R_y \circ R_z] + (-1)^{yz}[R_x \circ R_z, R_y] = 0$;

3) $[L_x \circ L_y, R_z] - [R_x, L_y \circ L_z] + (-1)^{yz}[L_x \circ L_z, R_y] = 0$;

4) $[L_x \circ L_y, L_z] - [L_x, L_y \circ L_z] + (-1)^{yz}[L_x \circ L_z, L_y] = 0$;

5) $[R_x \circ R_y, L_z] - [L_x, R_y \circ R_z] + (-1)^{yz}[R_x \circ R_z, L_y] = 0$.

Moreover, 2) **and** 4) **hold in every superalgebra.**

Lemma. In an arbitrary NJSA U the following hold:

1) $[R_{x\circ y}, R_z] - [R_x, R_{y\circ z}] + (-1)^{yz}[R_{x\circ z}, R_y] = 0$;

2) $[L_{x\circ y}, L_z] - [L_x, L_{y\circ z}] + (-1)^{yz}[L_{x\circ z}, L_y] = 0$.

Denote by Γ the Grassman superalgebra on generators $1, \xi_i, i \in \mathcal{I}$; we admit $\mathcal{I} = \varnothing$.

Lemma. U is a nonsupercommutative Jordan superalgebra iff U is a flexible superalgebra such that $U^{(+)}$ is a Jordan superalgebra.

(Pass to $\Gamma(U)$; use $\Gamma(U^{(+)}) = \Gamma(U)^{(+)}$)

2. Peirce decompositions. We have

$$R_{y(z\circ t)} + (-1)^{t(y+z)}(R_t + L_t)L_yL_z$$
$$+(-1)^{yz}(R_z + L_z)L_yL_t =$$
$$R_yR_{z\circ t} + (-1)^{t,y,z}(R_t + L_t)L_{zy}$$
$$+(-1)^{y(z+t)}(R_z + L_z)L_{ty}.$$

If $e \in U$**:** $e^2 = e$**,** e **is homog.** $\Rightarrow e \in U_{\bar{0}}$**, and**

$$R_e + (R_e + L_e)L_e^2 \ = \ (R_e + L_e)L_e + R_e^2.$$

Also, $L_e - L_e^2 = R_e - R_e^2$**. Therefore,**

$$(R_e + L_e)(L_e - L_e^2) \ = \ (L_e - L_e^2).$$

Put $U_i = \{x : ex + xe = ix\}$ **for** $i = 0, 1, 2$**.**

$$U = U_0 \oplus U_1 \oplus U_2$$

The spaces U_0, U_1 **and** U_2 **satisfy:**

$$U_i^2 \subseteq U_i,\ \ U_iU_1 + U_1U_i \subseteq U_1,\ \ U_0U_2 = 0,$$
$$x \in U_i \Rightarrow xe = ex = \frac{1}{2}ix,\ \ i = 0, 2,$$
$$x, y \in U_1 \Rightarrow x \circ y \in U_0 + U_2.$$

If $e = \sum_{i=1}^{n} e_i$ is the sum of orthogonal idempotents then we have the following Peirce decomposition:

$$U = \oplus_{i,j=0}^{n} U_{ij},$$

where

$$\begin{aligned}
&U_{00} = \{x \in U : e_i x = x e_i = 0 \text{ for all } i\},\\
&U_{ii} = \{x \in U : e_i x = x e_i = x, e_j x = x e_j = 0, j \neq i\},\\
&U_{i0} = \{x \in U : e_i x + x e_i = x, e_j x + x e_j = 0, j \neq i\} = U_{0i},\\
&U_{ij} = \{x \in U : e_i x + x e_i = e_j x + x e_j = x\} = U_{ji},\\
&x \in U_{ij} \Rightarrow e_i x = x e_j \text{ (even if } i = j \neq 0).
\end{aligned}$$

As before, we have the associated projections P_{ij} on U_{ij} and the inclusions:

$$U_{ii}^2 \subseteq U_{ii},\ U_{ii}U_{ij} + U_{ij}U_{ii} \subseteq U_{ij},$$
$$U_{ij}U_{jk} + U_{jk}U_{ij} \subseteq U_{ik},\ U_{ij}^2 \subseteq U_{ii} + U_{ij} + U_{jj}$$

for the pairwise different indices (non-mentioned products are zero).
Note that $U_{00} = U_{i0} = 0$ if $e = 1$.

Lemma. $x, y \in U_0, u \in U_2, u_i \in U_i, i = 0, 2, z, w \in U_1 \Rightarrow$

$$e(z \circ y) = ez \circ y = zy, \ \ (y \circ z)e = y \circ ze = yz;$$

$$e(u \circ z) = u \circ ez = uz, \ \ (z \circ u)e = ze \circ u = zu;$$

$$P_2(ez \circ w) = P_2(z \circ we) = P_2(zw),$$

$$P_0(w \circ ez) = P_0(we \circ z) = P_0(wz);$$

$$P_1(zw) \circ u_i = P_1(z(w \circ u_i))$$

$$= (-1)^{wu_i} P_1((z \circ u_i)w)$$

Lemma. For $x, y \in U_i,\ i = 0, 2$, the following operator equalities hold on U_1 :

1) $L_{xy} = (-1)^{xy} L_y L_x + R_x L_y$;

2) $R_{xy} = R_x R_y + (-1)^{xy} L_y R_x$;

3) $L_{xy} = (-1)^{xy} L_y L_x + L_x R_y$;

4) $R_{xy} = R_x R_y + (-1)^{xy} R_y L_x$;

5) $L_x R_y = R_x L_y$; 6) $(R_x + L_x) R_e = R_x$;

7) $(R_x + L_x) L_e = L_x$.

Lemma. **If** $N_1 \subseteq U_1$ **is such that** $U_i N_1 + N_1 U_i \subseteq N_1$ **with** $i = 0, 2$ **then**

$$N_i = P_i(U_1 N_1 + N_1 U_1) \trianglelefteq U_i.$$

Corollary. $N_1 = U_1 \Rightarrow B = N_0 + N_1 + N_2 \trianglelefteq U.$

$$\lambda \in \Phi : U_1^{[\lambda]} = \{x \in U_1 : L_e x = \lambda x\}$$

Lemma. $U_i U_1^{[\lambda]} + U_1^{[\lambda]} U_i \subseteq U_1^{[\lambda]}$ **for** $i = 0, 2$**.**

3. Algebras with connected idempotents.

$\lambda \in \Phi$, $e \in U$: $S_1^{[\phi]}(e) = U_1^{[\lambda]}(e) + U_1^{[1-\lambda]}(e)$.

$$S_{ij}^{[\phi]} = S_1^{[\phi]}(e_i) \cap S_1^{[\phi]}(e_j)$$

e_i **and** e_j: **(**$\exists \phi \in \Phi, v_{ij}, v_{ji} \in S_{ij}^{[\phi]}$**)**

even-connected: $v_{ij}v_{ji} = v_{ji}v_{ij} = e_i + e_j$;

odd-connected: $v_{ij}v_{ji} = -v_{ji}v_{ij} = e_i - e_j$;

connected; ϕ: **indicator of** U_{ij}.

Lemma. For all i, j, k such that $i \neq j$:

$$U_{jk} S_{ik}^{[\phi]} \subseteq S_{ij}^{[\phi]}$$

Lemma $\mathcal{IT}$. Assume that $1 = \sum_{i=1}^{n} e_i$ is the sum of $n \geq 3$ connected orthogonal idempotents. Then all indicators have a common value ϕ and the following are valid:

1) $U_{ij} = S_{ij}^{[\phi]}$ $(i \neq j)$;

2) $U_{ij} = U_{ik}U_{kj} + U_{kj}U_{ik}$ $(\neq i, j, k$ $)$;

3) $U_{ii} = P_{ii}(U_{ik}^2)$ $(i \neq k)$;

4) $U_{ik}^2 \subseteq U_{ii} + U_{kk}$ $(i \neq k)$.

If $e = e_k$ **then we have**

5) $U_1 = S_1^{[\phi]}$;

6) $U_i = P_i(U_1^2)(i = 0, 2)$;

7) $U_1^2 \subseteq U_0 + U_2$.

The element $\phi \in \Phi$, which is uniquely determined by U, will be called the indicator of U. We say U is of indicator type ϕ if there is a $\phi \in \Phi$ such that 1)–7) of Lemma $\mathcal{IT}$ hold for U. We say that a nonsupercomm. J. s.a. is of capacity k if it possesses k pairwise orthogonal idempotents, and U with unity of capacity k if this unity is decomposed into a sum of k primitive orthogonal idempotents.

Lemma. If U is of indicator type $\phi = \frac{1}{4}$ and U possesses unity of capacity ≥ 3 then U is supercommutative.

Lemma. If U is of indicator type $\phi = 0$ and of capacity ≥ 3 then U is associative.

3. Mutations. If $U = (U, \cdot)$ is a superalgebra and $\lambda \in \Phi$ then the λ-**mutation** of U is the superalgebra $U^{(\lambda)} = (U, \cdot_\lambda)$, where

$$x \cdot_\lambda y = \lambda x \cdot y + (-1)^{xy}(1 - \lambda) y \cdot x.$$

Note that $U^{(1/2)}$ is just the symmetrized superalgebra U^{+}.

The mapping $\tau : \lambda \mapsto (\lambda + 1)/2$ is a 1-1 mapping of Φ onto itself with inverse $\tau^{-1} : \lambda \mapsto 2\lambda - 1$, so in a natural way it carries the field $\Phi = (\Phi, +, \cdot)$ isomorphically onto a field $\tilde{\Phi} = (\Phi, \oplus, \odot)$, where

$$\begin{aligned} \lambda \oplus \mu &= \lambda + \mu - \tfrac{1}{2}, \\ \lambda \odot \mu &= 2\lambda\mu - \lambda - \mu + 1. \end{aligned}$$

Consider the double mutation $(U^{(\lambda)})^{(\mu)}$;

$$(U^{(\lambda)})^{(\mu)} \;=\; U^{(\lambda\odot\mu)}.$$

If $\lambda \neq \frac{1}{2}$ then λ has an inverse μ in $\tilde{\Phi}$, so we can recover U from $U^{(\lambda)}$: $U = U^{(1)} = U^{(\lambda\odot\mu)} = (U^{(\lambda)})^{(\mu)}$. However, if $\lambda = \frac{1}{2}$ we cannot easily recover U because all mutations have the same $U^{(+)}$. Thus we cannot so easily recover an associative s.a. U from the special J. s.a. $U^{(+)}$.

Note that an ideal in U remains an ideal in $U^{(\lambda)}$, so if $\lambda \neq \frac{1}{2}$ ideals in U and $U^{(\lambda)}$ coincide. Since $L_x^{(\lambda)} = \lambda L_x + (1-\lambda)R_x, R_x^{(\lambda)} = \lambda R_x + (1-\lambda)L_x$, it is clear that a mutation of a nonsupercommutative Jordan superalgebra is again a nonsupercommutative Jordan superalgebra.

As an example, a split quasi-associative s.a. is a mutation $\mathcal{D}^{(\lambda)}$ of an associative s.a. $\mathcal{D}$. Then U is quasi-associative if there is $\Omega \supset \Phi$: U_Ω is a split quasi-assoc. s.a. over Ω: $U_\Omega = \mathcal{D}^{(\lambda)}$ for $\lambda \in \Omega$.
We say that a Jordan s.a. U is strictly nonsupercomm. if there are some homogeneous $x, y \in U$: $xy \neq (-1)^{xy} yx$.

Theorem. (*Coordinatization Theorem***) If U is a strictly nonsupercommutative J. s.a. with $n \geq 3$ connected orthogonal idempotents then $U = \mathcal{D}_n^{(\lambda)}$ is a split quasi-associative s.a. determined by the s.a. $\mathcal{D}_n$ of $n \times n$ matrices with entries in $\mathcal{D}$, where $\mathcal{D}$ is associative.**

2. Simple superalgebras

In what follows: $x \circ y = \frac{1}{2}(xy + (-1)^{xy}yx)$.

Lemma. **Let $(A, \cdot)$ be a flexible superalgebra. If $A^{(+)}$ possesses the unity $1 = \sum_{i=1}^{n} e_i$ for some orthogonal even idempotents e_i then the same holds for A.**

Lemma. **The mapping $d = [\cdot, x]$ is a superderivation in $U^{(+)}$ for every $x \in U$.**

Lemma. If U is simple then $U^{(+)}$ is differentially simple.

Theorem. A f.d. central simple non-supercomm. J. s.a. $\chi = 0$ is either
(a) of capacity 1,
(b) of capacity 2,
(c) a quasi-associative superalgebra,
(d) a Jordan superalgebra.

Proof. Since U is central $\Rightarrow U_\Omega$ is also simple for $\Omega = \bar{\Phi}$. It suffices to prove U_Ω is of type (a)–(d), so we assume $\Phi = \bar{\Phi}$. Since U^+ is a diff. simple J. s.a., by Kac-Cheng's Theorem, U^+ is the tensor product of a simple J. s.a. and a Grassman s.a. Classification of the simple f.d. J. s.a. Φ^0 implies $e \in U$.

Thus, we may assume that U is of capacity $n > 2$. Then the same argument and Lemma above say that U contains the unity, which is a sum of n orth. idemp.: $1 = \sum_{i=1}^{n} e_i$. We see $B = N_0 + U_1 + N_2$ is an ideal; by simplicity $B = U$. Thus, $P_0(U_1^2) = U_0,\ P_2(U_1^2) = U_2$, which gives 6) and 2)–3) of Lemma $\mathcal{IT}$. 4) and 7) of Lemma $\mathcal{IT}$ are proved as in Lemma $\mathcal{IT}$.

Let $\phi \in \Phi$. For $i \neq j$ set

$$B_{ij} = S_{ij}^{[\phi]},\ S_{ij} = U_{ij}B_{ij} + B_{ij}U_{ij},$$

$$B_{ii} = \sum_{i \neq j} P_{ii}(S_{ij}),\ B = \oplus B_{ij}.$$

B is an ideal. Since $\Phi = \bar{\Phi}$, L_{e_i} has a nonzero eigenvector in U_{ik}, so we may find ϕ: $S_{ik}^{[\phi]} \neq 0$, and $B \neq 0$. By simplicity, $B = U$, and $U_{ij} = B_{ij} = S_{ij}^{[\phi]}$.

Thus, U is of indicator type ϕ and with unity of capacity ≥ 3. We may assume that $\phi \neq \frac{1}{4}$ and pass to the μ-mutation of U, where μ is the inverse for ϕ in $\tilde{\Phi}$. Moreover, $U^{(\mu)}$ is a central simple non-supercomm. J. s.a. (Φ^0) with unity of capacity ≥ 3. The same argument gives: $U^{(\mu)}$ is of indicator type 0. Then $U^{(\mu)}$ is associative, and we arrive at (c).

Corollary. Let U be a f.d. simple central nonsupercomm. J. s.a. over Φ^0 and of capacity ≥ 3. Then the symmetrized Jordan superalgebra $U^{(+)}$ is simple.

Proof. We assume U quasi-associative. Then $U = A^{(\mu)}$ for an associative A. It is easy to see that $(A^{(\mu)})^{(+)} = A^{(+)}$. We may assume $\Phi = \bar{\Phi}$. Since A is simple f.d. associative s.a., $A^{(+)}$ is simple J. s.a.

3. An analog of R. Oehmke's Theorem

Let U be a nonsupercomm. J. s.a., e be an idempotent in U, and $U = \oplus U_i$ be the Peirce decomposition of U. Fix $i \in \{0, 2\}$. Define $\nu(x) = \bar{R}_x + \bar{L}_x$ for $x \in U_i$, where $\bar{R}_x, \bar{L}_x$ are the restr. of R_x, L_x on U_1. Then

$$\nu(xy) \ = \ \nu(x)\nu(y)\bar{R}_e + (-1)^{xy}\nu(y)\nu(x)\bar{L}_e.$$

Denote by Γ_1 the subalgebra in Γ generated by $\xi_i,\ i \in \mathcal{I}$. Let $(x,y,z)^+$ be the associator in $U^{(+)}$.

Proposition. Let U be a nonsupercomm. J. s.a., $U \cong J \otimes \Gamma$ for some J. s.a. J, and let $U = \oplus_{i=0}^{2} U_i$ be the Peirce decomposition of U relative to an idempotent e. Assume that for every $x \in U_1$ the mapping $y \mapsto x \circ y$ is injective on U_i and $(U_i, U_1, U_i)^+ = 0$ for $i = 0, 2$. Then $I := \oplus_{i=0}^{2} (J_i \otimes \Gamma_1) \trianglelefteq U$.

$J(V, f)$: Let $V = V_0 + V_1$ be a Z_2-graded vector space with a nondeg. superform f, which is symm. on V_0 and skew-symm. on V_1, $f(V_1, V_0) = f(V_0, V_1) = 0$. Consider $J = \Phi \oplus V$. Let 1 be the unity in Φ. Define: $(\alpha + v)(\beta + w) = (\alpha\beta + f(v, w)) + (\alpha w + \beta v)$. Then $J_0 = \Phi + V_0$, $J_1 = V_1$. We may assume $J(V, f)$ has capacity two: $1 = e_1 + e_2$.

Lemma. Let U be a simple nonsuper-comm. J. s.a., J be a J. s.a.: $U^{(+)} \cong J \otimes \Gamma$. If $J \cong J(V,f), K_3, K_{10}, D_t$ then U is under condition of Proposition. In particular, $U^{(+)}$ is a simple J. s.a.

Remark. Note that the unity in K_{10} is the sum of three orthogonal idempotents. Since K_{10} is not special, Theorem says that $U \cong K_{10}$ if $U^{(+)} \cong K_{10}$.

Consider the associative superalgebra $Q(n)$, which is a subsuperalgebra in $M_{2n}(\Phi)$:

$$
\begin{aligned}
Q(n)_{\bar{0}} &= \left\langle \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A \in M_n(\Phi) \right\rangle, \\
Q(n)_{\bar{1}} &= \left\langle \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, B \in M_n(\Phi) \right\rangle.
\end{aligned}
$$

Lemma. **Let** $J = Q(2)^{(+)}$ **and** $U = J \otimes \Gamma$. **Then for all** $a \in U_1, \bar{x} = x \otimes f, \bar{y} = y \otimes g,\ x, y \in J_i,\ f, g \in \Gamma,\ i = 0, 2$, **holds**

$$(a\bar{x})\bar{y} \;=\; (-1)^{xy+\bar{x}\bar{y}}(a\bar{y})\bar{x};$$

and $\nu : U_i \mapsto End\,(U_1, U_1)$ **is injective.**

Lemma. **Let** U **be a nonsupercommutative J. s.a.:** $U^{(+)} \cong Q(2)^{(+)} \otimes \Gamma$. **Then** $U^{(+)}$ **is simple.**

Theorem. **Let U be a simple finite dimensional nonsupercommutative Jordan superalgebra of characteristic 0 and capacity > 1. Then $U^{(+)}$ is simple.**

4. Simple nonsupercommutative Jordan superalgebras of capacity 2

Lemma. For every fixed $i \in \{0, 2\}$ and for all $x, y, z \in U_1$, the following relations hold:

$$P_i(x \circ P_1(yz)) = P_i(P_1(xy) \circ z)$$
$$= (-1)^{x(y+z)} P_i(y \circ P_1(zx)).$$

The case D_t. $\alpha, t \in \Phi$. **Define** $U = D_t(\alpha)$:

$$U = U_{\bar{0}} \oplus U_{\bar{1}}, \;\; U_{\bar{0}} = \langle e_1, e_2 \rangle, \;\; U_{\bar{1}} = \langle x, y \rangle,$$

$$e_i^2 = e_i, \;\; e_1 e_2 = 0, \;\; e_1 x = \alpha x = x e_2,$$

$$x e_1 = (1-\alpha)x = e_2 x, e_1 y = (1-\alpha)y = y e_2,$$

$$e_2 y = \alpha y = y e_1, xy = 2(\alpha e_1 + (1-\alpha)t e_2),$$

$$yx = -2((1-\alpha)e_1 + \alpha t e_2), \;\; x^2 = y^2 = 0.$$

Lemma. The superalgebra $D_t(\alpha)$ is flexible; $D_t(\alpha)$ is simple $\Leftrightarrow t \neq 0$.

Lemma. Let U be a nonsupercommutative Jordan superalgebra such that $U^{(+)} \cong D_t,\ t \neq 0$. Then $U \cong D_t(\alpha)$ for some $\alpha \in \Phi$ (after a possible quadratic ext. of Φ).

Remark. Note: $D_t(\alpha)$ is not quasi-associative, since otherwise $D_t \cong A^{(+)}$ for some associative A, which is impossible.

The case $Q(2)^{(+)}$.

Lemma. Let U be a nonsupercommutative Jordan superalgebra such that $U^{(+)} \cong Q(2)^{(+)}$. Then U is a split quasiassociative superalgebra.

The case $J(V,f)$. Here U is a nonsuper-comm J. s.a. $J := U^{(+)} \cong J(V,f)$. Then $J = \Phi + V$; fix u: $f(u,u) = \frac{1}{4}$. Then $e_1 = (\frac{1}{2}, u)$ and $e_2 = (\frac{1}{2}, -u)$ are orthog.: $1 = e_1 + e_2$. W.r.t. e_1:

$$J_0 = \langle e_2 \rangle = U_0,\ \ J_2 = \langle e_1 \rangle = U_2,$$

$$J_1 = \langle (0,w) : f(u,w) = 0 \rangle = U_1.$$

The s.a. J induces a multiplication on V:

$$x \times y = P_1((0,x)(0,y)).$$

Lemma. **If U is a nonsupercomm. J. s.a.: $J := U^{(+)} = \Phi \oplus V$ is a J. s.a. $J(V, f)$ then the multiplication in U is given by**

$$(\alpha, x)(\beta, y) = (\alpha\beta + f(x, y), \alpha y + \beta x + x \star y),$$

where $\star$ is an antisupercomm. product on V and f: $f(x \star y, z) = f(x, y \star z)$.

The case K_3. $U^{(+)} \cong K_3$ **(**K_3 **of cap. 1).**

Define s.a. $K_3(\alpha, \beta, \gamma)$**. Put** $K_3(\alpha, \beta, \gamma) :=$ $U_{\bar{0}} \oplus U_{\bar{1}}$, $U_{\bar{0}} = \langle e \rangle$, $U_{\bar{1}} = \langle z, w \rangle$. **Product:**

	e	z	w
e	e	$\alpha z + \beta w$	$\gamma z + (1-\alpha)w$
z	$(1-\alpha)z - \beta w$	$-2\beta e$	$2\alpha e$
w	$\alpha w - \gamma z$	$-2(1-\alpha)e$	$2\gamma e$

Lemma. $K_3(\alpha, \beta, \gamma)$ **is a simple flexible s.a.**

Lemma. Let U be a nonsupercomm. J. s.a.: $U^{(+)} \cong K_3$. Then $U \cong K_3(\alpha, \beta, \gamma)$.

$\Gamma(n)$ stands for the simple J. s.a. of Poisson Grassmann brackets.

Theorem. Let U be a f.d. central simple nonsupercomm. J. s.a. over Φ^0, which is neither quasi-associative nor supercomm. Then, either U is isomorphic to one of:

$i)$ $U \cong K_3(\alpha, \beta, \gamma)$;

$ii)$ $U^{(+)} \cong \Gamma(n) \otimes \Gamma$;

or $\exists P \supset \Phi$ (deg 1 or 2): $U \otimes_\Phi P \cong$ as a s.a. over P to one of:

$iii)$ $D_t(\alpha)$;

$iv)$ $U(V, f, \star)$.

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