On right alternative superalgebras

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F: field of char $\neq 2$.

A: F-algebra

A is a *superalgebra* if

$$A = A_0 \oplus A_1,$$

where A_0 and A_1 are subspaces of A such that

$$A_0^2 \subset A_0, \quad A_1^2 \subset A_0,$$

$$A_0A_1 \subset A_1, \quad A_1A_0 \subset A_1.$$

Then A_0 is an algebra over the field F and A_1 is a bimodule for A_0.

We have considered only $A_1^2 \neq 0$.

The *parity index* of a homogeneous element

$$p(a) = \begin{cases} 0, & \text{se} \quad a \in A_0, \\ 1, & \text{se} \quad a \in A_1. \end{cases}$$

A superalgebra $A = A_0 \oplus A_1$ is an *alternative superalgebra* if

$$
\begin{aligned}
(x, y, z) + (-1)^{p(y)p(z)}(x, z, y) &= 0, \\
(x, y, z) + (-1)^{p(x)p(y)}(y, x, z) &= 0,
\end{aligned}
$$

where $x, y \in A_0 \cup A_1$.

A superalgebra $A = A_0 \oplus A_1$ is a *right alternative superalgebra* if

$$
(x, y, z) + (-1)^{p(y)p(z)}(x, z, y) = 0.
$$

where $x, y \in A_0 \cup A_1$.

If A is a right alternative superalgebra, then A_0 is a right alternative algebra and A_1 is a right alternative bimodule for A_0.

Question 1. Are there infinitely many right alternative finite-dimensional superalgebras that are not alternative?

Question 2. How to classify right alternative superalgebras

$$Fa \oplus M$$

where M is an irreducible Fa-bimodule?

Question 3. How to classify right alternative superalgebras

$$M_2(F) \oplus M$$

where M is an irreducible $M_2(F)$-bimodule?

Superalgebras $Fa \oplus M$.

Fa is an one-dimensional algebra.

$a = e$, $e^2 = e$,

$a = z$, $z^2 = 0$.

M is a right alternative bimodule for Fa.

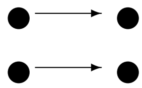
Envelope multiplicative algebra for M:

$$U(M) = \mathsf{alg}_F \langle id_M, R_a, L_a \rangle \subset \mathsf{End}(M)$$

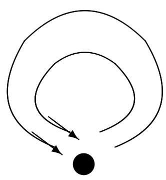
$U(M)$ is an associative algebra and $\dim U = 5$.

right alternative bimodules for Fa	$\longleftrightarrow$	associative left modules for $U(M)$

If $a = e$, then the diagram of the algebra $U(M)$ has the form



If $a = z$, then the diagram of the algebra $U(M)$ has the form



There are infinitely many right alternative bimodules for algebra Fa.

There are only five irreducible right alternative bimodule for Fa;

If $\dim M \geq 2$, then M is reducible.

If the field F is algebraically closed, then any right alternative superalgebra which is not alternative having a one-dimensional algebra as even part and a irreducible bimodule as odd part is isomorphic to the superalgebra $Fe \oplus Fm$, where m and e satisfy the relations

$$e^2 = e, \quad em = m, \quad me = 0, \quad m^2 = e.$$

Superalgebras $M_2(F) \oplus M$.

Murakami + Shestakov, 2001. The irreducible unital right alternative bimodules for $M_2(F)$ up to dimension 6 are classified.

These bimodules have even dimension and

there exists a single 2-dimensional bimodule, denoted by M_2;

there exists a single 6-dimensional bimodule, denoted by M_6;

there exists an infinite family 4-dimensional bimodule which depends on three parameters, denoted by $M(\alpha, \beta, \gamma)$.

Superalgebras $M_2(F) \oplus M_2$.

Any right alternative superalgebra $M_2(F) \oplus M_2$ is alternative.

An alternative superalgebra $M_2(F) \oplus M_2$ is not trivial if, and only if, $\text{char}(F) = 3$.

Superalgebras $M_2(F) \oplus M_6$.

The unique right alternative superalgebra $M_2(F) \oplus M_6$ is the trivial superalgebra.

Superalgebras $M_2(F) \oplus M(\alpha, \beta, \gamma)$.

There exist right alternative superalgebras which are not alternative and not trivial having the form $M_2(F) \oplus M(\alpha, \beta, \gamma)$.

If the field F is algebraically closed, then this superalgebras depends on the parameters α, β and γ. We denote

$$A(\alpha, \beta, \gamma) = M_2(F) \oplus M(\alpha, \beta, \gamma).$$

If the field F is algebraically closed, then

(1) $A\,(\alpha, \beta, \gamma) \simeq A(0, 1, \frac{1}{2})$ or
$A(\alpha, \beta, \gamma) \simeq A(0, 0, \sigma)$, for some $\sigma \in F$;

(2) $A\left(0, 1, \frac{1}{2}\right) \not\simeq A(0, 0, \sigma)$, for any $\sigma \in F$;

(3) $A(0, 0, \sigma) \simeq A(0, 0, \sigma')$ if, and only if, $\sigma = \sigma'$
or $\sigma = 1 - \sigma'$;

(4) If F is infinite, then the family
$\{A(0, 0, \sigma)\}_{\sigma \in F}$ is infinite.

There are infinitely many simple right alternative superalgebras finite-dimensional which are not alternative.