Amenability of path groups

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The problem

open question: is $C^\infty(X, SU(n))$ amenable, where X is a smooth closed manifold, $n \geq 2$ (A. Carey, H. Grundling)?

"Yes" $\Rightarrow$ exists a "gauge-invariant vacuum state".

Abstract harmonic analysis + infinite dimensional Lie theory

Surely, not a "natural question"...

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thm. $C^1(\mathbb{I}, SU(n))$ is amenable.

(Here $\mathbb{I} = [0, 1]$.)

Why one needs amenability

thm. (André Weil) If a Polish (= separable completely metrizable) topological group admits an invariant sigma-additive measure, then it is locally compact.

Amenability = poor man's version of invariant integration with finite total volume.

Amenability

A topological group G is *amenable* if any 1 of:

– if G acts continuously on a compact space X, there is an invariant probability measure μ on X, i.e. $\mu(A) = \mu(gA)$ for all Borel $A \subseteq X$, all $g \in G$;

– $\exists$ left-invariant mean m on the space RUCB$\,(G)$ of all right uniformly continuous bounded functions on G:

$f\colon G \to \mathbb{C}$ is RUC if
$\forall \varepsilon > 0,\ \exists V \ni e,\ xy^{-1} \in V \Rightarrow |f(x) - f(y)| < \varepsilon.$

An invariant mean:
$\phi\colon$ RUCB$\,(G) \to \mathbb{C}$, linear, positive, $\phi(1) = 1$,
$\phi(f) = \phi($any left translate of $f)$.

Some observations

If G is locally compact, the list is much longer.

Abelian top. groups, compact groups are amenable ...

Amenability is closed under extensions.

F_2 with discrete topology is non-amenable.

A closed subgroup of amenable LC group is amenable.

Even for discrete countable groups, determining (non)amenability can be a hard task

(Thompson's group (F) ...)

Some infinite dimensional examples

Unions of chains of amenable groups are amenable.
If $\mathrm{cl}\, H = G$, then G is amenable $\iff$ H is amenable.

ex. $U(\ell^2)_{sot} = \mathrm{cl} \cup_{n=1}^{\infty} U(n)$, or of $SU(n)$, $\Rightarrow$ amenable.

ex. $S_\infty = \mathrm{cl} \cup_{n=1}^{\infty} S_n$, $\Rightarrow$ amenable.

ex. Aut $(X,\mu) = \cup_{n=1}^{\infty} S_n$ ("interval exchange transformations"), by Rokhlin Lemma, $\Rightarrow$ amenable.

ex. $L^0(X,\mu;SU(n))$, with the topology of convergence in measure, is the union of groups of simple functions of the form $\mathbb{T}^k$, $\Rightarrow$ amenable. Etc.

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ex. Homeo $_+[0,1]$: no compact subgroups (yet amenable...)

Difficulties in infinite dimensions

Unlike in LC case, amenability is not inherited by closed subgroups:

ex.: S_∞ contains F_2 as a closed discrete subgroup.

therefore, non-amenability is even harder to show.

- $U(\ell^2)_{uniform}$ is non-amenable (de la Harpe).
- Aut (X,μ) with topology given by the the *uniform* metric

$$d_{unif}(\sigma,\tau) = \mu\{x \in X \colon \sigma(x) \neq \tau(x)\}$$

is non-amenable (Giordano-VP).

What was known about groups of maps

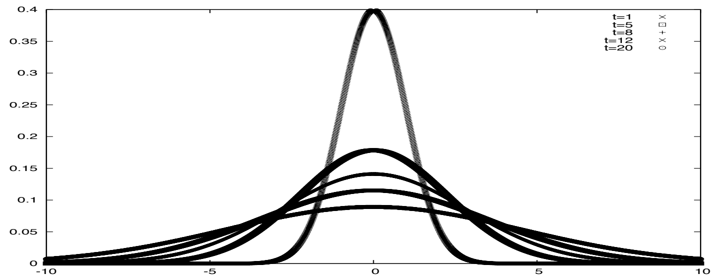
An obvious observation (J. Baez): with the *pointwise* convergence topology, $C(X, SU(n))$ is amenable: it is precompact: $\hookrightarrow SU(n)^X$.

With the *relative weak topology* induced from $C(X, \mathbb{C}^{n^2})$, the group $C(X, SU(n))$ is amenable: a consequence of deep results of A. Connes (noted by T. Giordano - VP).

Gaussian measure and amenability of $\mathbb{R}$

Gaussian measure with mean zero and variance $t > 0$:

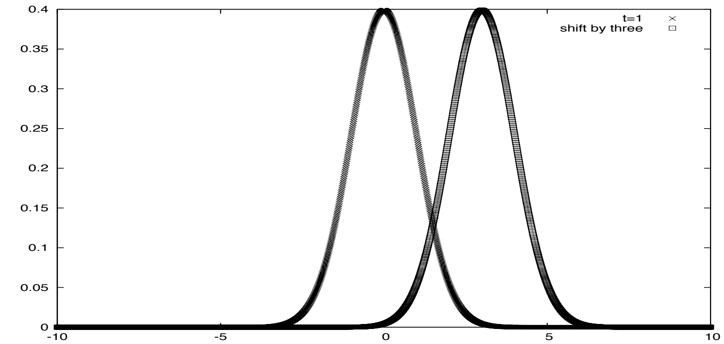
$$\gamma_t(a,b) = \frac{1}{\sqrt{2\pi t}} \int_a^b \exp(-x^2/2t)dx.$$



What happens to $\int_{-\infty}^{\infty} f(x)\, d\gamma_t(x)$ if we translate γ_t, say by $x = 3$?

Gaussian measure and amenability of $\mathbb{R}$

Turns out: $\int_{-\infty}^{\infty} f(x)\, d\gamma_t(x) - \int_{-\infty}^{\infty} f(x+3)\, d\gamma_t(x) \to 0$ as $t \to \infty$



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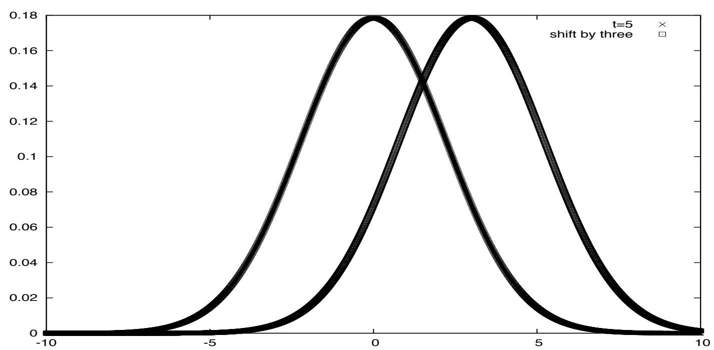

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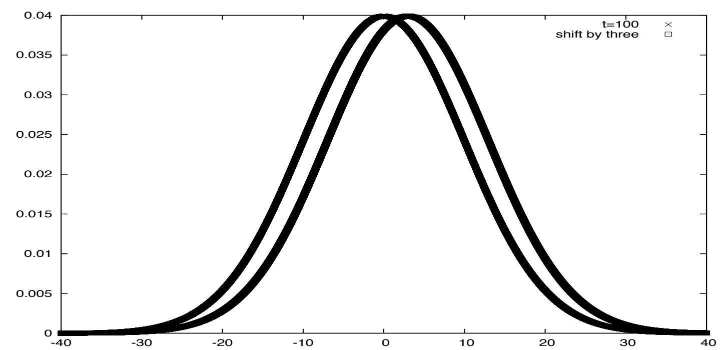

A weak* cluster point of integrals $\int_{-\infty}^{\infty} d\gamma_t$, $t \to \infty$, gives an invariant mean for functions on $\mathbb{R}$.

Main result

thm. $C^1(\mathbb{I}, SU(n))$ is amenable.

Enough to prove for group $G = C^1_e(\mathbb{I}, SU(n))$ of paths $p(0) = e$.

$C^1_e(\mathbb{I}, SU(n))$ is a Banach-Lie group, contractible.

Isomorphic to $C(\mathbb{I}, su(n))$ with the group operation

$u * v(t) = u(t) + Ad\left(\prod_0^t \exp u(\tau)\, d\tau\right) v(t)$,

where $\prod$ is the product integral inside $SU(n)$.

$\prod$ is the inverse to the left logarithmic derivative:

$u(t) \mapsto u'(t)u(t)^{-1}$, curve in $SU(n) \mapsto$ curve in $su(n)$.

Classical Wiener measure

The Wiener measure w_t with mean zero and variance $t > 0$ on the space $C_0(\mathbb{I}, \ell^2(N))$.

Here $\ell^2(N) = su(n)$ with an Ad-invariant inner product.

Was used to construct representations of path groups (Albeverio—Høegh-Krohn).

It turns out that for every $f \in$ RUCB (G) and each $k \in G$, $\int f(u)\, dw_t(u) - \int f(ku)\, dw_t(u) \to 0$ as $t \to \infty$.

A weak* cluster point of the family (w_t), $t > 0$ gives an inariant mean on the path group.