Part I. Origin of the Species

Jordan algebras were conceived and grew to maturity in the landscape of physics. They were born in 1933 in a paper "Uber Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik" by the physicist Pascual Jordan; just one year later, with the help of John von Neumann and Eugene Wigner in the paper "On an algebraic generalization of the quantum mechanical formalism," they reached adulthood.

Jordan algebras arose from the search for an "exceptional" setting for quantum mechanics. In the usual interpretation of quantum mechanics (the "Copenhagen model"), the physical observables are represented by Hermitian matrices (or operators on Hilbert space), those which are self-adjoint $x^* = x$. The basic operations on matrices or operators are multiplication by a complex scalar λx, addition $x + y$, multiplication xy of matrices (composition of operators), and forming the complex conjugate transpose matrix (adjoint operator) x^*. This formalism is open to the objection that the operations are not "observable," not intrinsic to the physically meaningful part of the system: the scalar multiple λx is not again hermitian unless the scalar λ is real, the product xy is not observable unless x and y commute (or, as the physicists say, x and y are "simultaneously observable"), and the adjoint is invisible (it is the identity map on the observables, though nontrivial on matrices or operators in general). In 1932 the physicist Pascual Jordan proposed a program to *discover a new algebraic setting for quantum mechanics*, which would be freed from dependence on an invisible all-determining metaphysical matrix structure, yet would enjoy all the same algebraic benefits as the highly successful Copenhagen model.

1. The Jordan Program

• *To study the intrinsic algebraic properties of hermitian matrices, without reference to the underlying* (*unobservable*) *matrix algebra*;

• *To capture the algebraic essence of the physical situation in formal algebraic properties that seemed essential and physically significant*;

• *To consider abstract systems axiomatized by these formal properties and see what other new* (*non-matrix*) *systems satisfied the axioms.*

He wished to study the intrinsic algebraic properties of hermitian matrices, to capture these properties in formal algebraic properties, and then to see what other possible non-matrix systems satisfied these axioms. Not only was the matrix interpretation *philosophically* unsatisfactory because it derived the observable algebraic structure from an unobservable one, there were *practical* difficulties when one attempted to apply quantum mechanics to relativistic and nuclear phenomena.

The first step in analyzing the algebraic properties of hermitian matrices or operators was to decide what the basic *observable operations* were. There are many possible ways of combining hermitian matrices to get another hermitian matrix. The most natural observable operation was that of forming *polynomials*: if x was an observable, one could form an observable $p(x)$ for any real polynomial $p(t)$ with zero constant term; if one experimentally measured the value v of x in a given state, the value associated with $p(x)$ would just be $p(v)$. Breaking the operation of forming polynomials down into its basic ingredients, we have the operations of multiplication αx by a real scalar, addition $x + y$, and raising to a power x^n. By linearizing the quadratic squaring operation x^2 we obtain a symmetric bilinear operation $xy + yx$.

After some empirical experimentation Jordan decided that *they could all be expressed in terms of* **quasi-multiplication**

$$x \bullet y := \tfrac{1}{2}(xy + yx)$$

The next step in the empirical investigation of the algebraic properties enjoyed by the model was to decide what crucial formal *axioms* or *laws* the operations on hermitian matrices obey. Jordan thought the key law governing multiplication, besides its obvious commutativity $x \bullet y = y \bullet x$, was

$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$$

(we now call this equation of degree four in two variables the **Jordan identity**, in the sense of *identical relation* satisfied by all elements). For example, it was not hard to see that the powers could be defined from the Jordan product via $x^1 = x, x^{n+1} = x \bullet x^n$, so the power maps, hence all polynomials, could be derived from a single bilinear product. The product xyx (linear in y but quadratic in x, now considered the "true" Jordan product) is expressible as $2x \bullet (x \bullet y) - x^2 \bullet y$, and linearization leads to a trilinear product $xyz + zyx$ (now known as the *Jordan triple product*). Of course, like the important dog in Sherlock Holmes who did *not* bark in the night, the important product that does *not* lead back to hermitian matrices is the associative product xy.

Thus in addition to its observable linear structure as a real vector space, the model carries a basic observable product, the Jordan product, out of which more complicated observable products such as powers or Jordan triple products can be built, but it does not carry an associative product. The observables satisfied the additional "positivity" or "formal reality" condition that a sum of squares never vanishes. The outcome of all this experimentation was a distillation of the algebraic essence of quantum mechanics into an axiomatically defined algebraic system.

DEFINITION: A **Jordan algebra** consists of a real vector space equipped with a bilinear product $x \bullet y$ satisfying the commutative law and the Jordan identity:

$$\text{(JAX1)}\quad x \bullet y = y \bullet x, \qquad \text{(JAX2)}\quad (x^2 \bullet y) \bullet x = x^2 \bullet (y \bullet x).$$

A Jordan algebra is called **Euclidean** (or **formally real**) if it satisfies the formal reality axiom $x_1^2 + \cdots + x_n^2 = 0 \implies x_1 = \cdots = x_n = 0$. Jordan originally called these *r-number algebras*; the term "Jordan algebra" was first used by A.A. Albert in 1946, and caught on immediately.

Any associative algebra A over $\mathbb{R}$ gives rise to a Jordan algebra A^+ under the Jordan product: the product $x \bullet y := \frac{1}{2}(xy + yx)$ is clearly commutative, and satisfies the Jordan identity since

$$4(x^2 \bullet y) \bullet x = (x^2y + yx^2)x + x(x^2y + yx^2) = x^2yx + xyx^2 + yx^3 + x^3y = x^2(yx + xy) + (yx + xy)x^2 = 4x^2 \bullet (y \bullet x).$$

A Jordan algebra is called **special** if it can be realized as a Jordan subalgebra of some A^+. For example, if A carries an involution $*$ then the subspace $\mathcal{H}(\mathrm{A}, *)$ of hermitian elements $x^* = x$ is also closed under the Jordan product, since if $x^* = x, y^* = y$ then $(x \bullet y)^* = y^* \bullet x^* = y \bullet x = x \bullet y$, and therefore forms a special Jordan algebra. These hermitian algebras are the archetypes of all Jordan algebras. It is easy to check that the hermitian matrices over the reals, complexes, and quaternions form special Jordan algebras that are formally real.

One obtains another special formally real Jordan algebra (which we now call a **spin factor** $\mathcal{J}spin_n$) on the space $\mathbb{R}1 \oplus \mathbb{R}^n$ for $n \geq 2$, by making 1 act as unit and defining the product of vectors v, w in $\mathbb{R}^n$ to be given by the dot or inner product

$$v \bullet w := \langle v, w \rangle 1.$$

In a special Jordan algebra the algebraic structure is derived from an ambient associative structure xy via quasi-multiplication. What the physicists were looking for, of course, were Jordan algebras where there is no invisible structure xy governing the visible structure $x \bullet y$ from behind the scenes. A Jordan algebra is called **exceptional** if it is not special, i.e., does not result from quasi-multiplication.

2. The Jordan Classification

Having settled, he thought, on the basic axioms for his systems, Jordan set about trying to classify them. The algebraic setting for quantum mechanics would have to be infinite-dimensional, of course, but since even for associative algebras the study of infinite-dimensional algebras was in its infancy, there seemed no hope of obtaining a complete classification of infinite-dimensional Jordan algebras. Instead, it seemed reasonable to study first the finite-dimensional algebras, hoping to find families of simple exceptional algebras E_n parameterized by natural numbers n, so that by letting n go to infinity a suitable home could be found for quantum mechanics. The purely algebraic aspects were too much for Jordan to handle alone, so he called in the mathematical physicist Eugene Wigner and the mathematician John von Neumann.

In their fundamental 1934 paper the J–vN–W triumvirate showed that in finite-dimensions the only simple building blocks are the usual hermitian matrices and the algebra $\mathcal{JS}pin_n$, except for one small algebra of 3×3 matrices whose coordinates come from the nonassociative 8-dimensional algebra $\mathbb{K}$ of Cayley's octonions.

Jordan–von Neumann–Wigner Theorem *Every finite-dimensional formally real Jordan algebra is a direct sum of a finite number of simple ideals, and there are five basic types of simple building blocks: four types of hermitian matrix algebras $\mathcal{H}_n(\mathrm{C})$ corresponding to the four real composition division algebras* C *(the reals* $\mathbb{R}$*, the complexes* $\mathbb{C}$*, the quaternions* $\mathbb{H}$*, and (only for $n = 3$) the octonions* $\mathbb{K}$*), together with the spin factors.*

There were three surprises in this list. Two were new structures which met the Jordan axioms but weren't themselves hermitian matrices: the spin factors and $\mathcal{H}_3(\mathbb{K})$. While the spin factor was not one of the invited guests, it was related to the guest of honor: it can be realized as a certain subspace of all hermitian $2^n \times 2^n$ real matrices, so it too is special. The other uninvited guest, $\mathcal{H}_3(\mathbb{K})$, was quite a different creature. It did not *seem* to be special, since its coordinates came from the *not-associative* coordinate Cayley algebra $\mathbb{K}$, and A.A. Albert showed that it is indeed an exceptional Jordan algebra of dimension $1+1+1+8+8+8 = 27$ (each element $\begin{pmatrix} \beta_{11} & b_{12} & b_{13} \\ \overline{b_{12}} & \beta_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & \beta_{33} \end{pmatrix}$ has 3 scalar and 3 octonion parameters).We now call such 27-dimensional exceptional algebras **Albert algebras**, and denote $\mathcal{H}_3(\mathbb{K})$ by $\mathbb{A}$. The third surprise was that there were no other algebras on the list.

3. The Physical End of Exceptional Algebras

These results were deeply disappointing to physicists, since the lone exceptional algebra $\mathbb{A}$ was too tiny to provide a home for quantum mechanics, and too isolated to give a clue as to the possible existence of infinite-dimensional exceptional algebras. It was still possible that infinite-dimensional exceptional algebras existed, since there were well-known associative phenomena that appear only in infinite dimensions: in quantum mechanics, the existence of operators p, q on Hilbert space with $[p, q] = \frac{\hbar}{2\pi}\mathbf{1}$ ($\hbar$ = Planck's constant) is possible only in infinite dimensions (in finite dimensions the commutator matrix $[p, q]$ would have trace 0, hence could not be a nonzero multiple $\alpha\mathbf{1}_n$ of the identity matrix $\mathbf{1}_n$, since the trace of $\alpha\mathbf{1}_n$ is $n\alpha \neq 0$). So for a while von Neumann held out a faint hope that there might still be an exceptional home for quantum mechanics somewhere, but half a century later Efim Zel'manov quashed all remaining hopes for such an exceptional system: in 1979 he showed that even in infinite dimensions there are no simple exceptional Jordan algebras other than Albert algebras: as it is written,

. . . and there is no new thing under the sun
especially in the way of exceptional Jordan algebras;
unto mortals the Albert algebra alone is given.

In 1983 Zel'manov proved the astounding theorem that any simple Jordan algebra, of arbitrary dimension, is either (1) an algebra of Hermitian elements $\mathcal{H}(\mathrm{A}, *)$ for a $*$-simple associative algebra with involution, (2) an algebra of spin type determined by a nondegenerate quadratic form, or (3) an Albert algebra of dimension 27 over its center. Thus we have a complete classification of all simple algebras of arbitrary dimension. This brought an end to the search for an exceptional setting for quantum mechanics: it is an ineluctable fact of mathematical nature that simple algebraic systems obeying the basic laws of Jordan *must* (outside of dimension 27) have an invisible associative support behind them.

4. Special Identities

While physicists abandoned the poor orphan child of their theory, the Albert algebra, algebraists adopted it and moved to new territories. This orphan turned out to have many surprising and important connections with diverse branches of mathematics. Actually, the child should never have been conceived in the first place: it does **not** obey all the algebraic properties of the Copenhagen model, and so was in fact unsuitable as a home for quantum mechanics, not superficially due to its finite-dimensionality, but genetically because of its unsuitable algebraic structure.

We now know that Jordan was wrong in thinking that his axioms had captured the hermitian essence — he had overlooked several other natural operations and relations on hermitian matrices, so instead he had captured something slightly more general. Note first that if x, y, x_i are hermitian matrices or operators, so are the *quadratic product* xyx, the *inverse* x^{-1}, and the *n-tad products* $\{x_1, \ldots, x_n\} := x_1 \cdots x_n + x_n \cdots x_1$. The quadratic product and inverse can be defined using the Jordan product, though this wasn't noticed for another 30 years; later, each of these was used (by McCrimmon and Springer) to provide an alternate axiomatic foundation on which to base the entire Jordan theory. The n-tad for $n = 2$ is just twice the Jordan product, and the 3-tad, or Jordan triple product, can be expressed in terms of the Jordan product. On the other hand, the n-tads for $n \geq 4$ *cannot* be expressed in terms of the Jordan product. In particular, the *tetrads* $\{x_1, x_2, x_3, x_4\} := x_1x_2x_3x_4 + x_4x_3x_2x_1$ were inadvertently excluded from Jordan theory. (P.M. Cohn showed that in the presence of $\frac{1}{2}$ the n-tads are all generated by Jordan products plus tetrads, so tetrads were the only product Jordan overlooked). As we shall see, this oversight allowed two uninvited guests to join the theory, the spin factor and the Albert algebra, who were not closed under tetrads but who had influential friends and applications in many areas of mathematics.

Not only did he miss a product, he missed some laws for the bullet which cannot be derived from the Jordan identity. In 1963 Jacobson's student C.M. Glennie discovered two identities satisfied by hermitian matrices (indeed, by all special Jordan algebras) but *not* satisfied by the Albert algebra. Such identities are called **special identities** (or **s-identities**) since they are satisfied by all special algebras but not all Jordan algebras, and so serve to separate the special from the non-special. Not only did the Albert algebra not carry a tetrad operation as hermitian matrices do, but even with respect to its Jordan product it was distinguishable from hermitian matrices by its refusal to obey the s-identities. Thus it squeezed through two separate gaps in Jordan's axioms. We will see that a large part of the richness of Jordan theory is due to its exceptional algebras (with their connections to exceptional Lie algebras, and exceptional symmetric domains), and much of the power of Jordan theory is its ability to handle these exceptional objects and hermitian objects in one algebraic framework.

Jordan can be excused for missing these identities, of degree 8 and 9 in 3 variables, since they cannot even be intelligibly expressed without using the (then new-fangled) **quadratic Jordan product** and **Jordan triple product**

$$U_x(y) := 2x \bullet (x \bullet y) - x^2 \bullet y, \qquad \{x, y, z\} := U_{x,z}(y) = 2\big(x \bullet (y \bullet z) + (x \bullet y) \bullet z - (x \bullet z) \bullet y\big).$$

(corresponding to xyx and $xyz + zyx$ in special Jordan algebras). In 1958 Jacobson conjectured, and I.G. Macdonald established, that this U-operator satisfied a very simple identity, the **Fundamental Formula**:

$$U_{U_x(y)} = U_x U_y U_x.$$

In terms of these products, Glennie's identities take the none-too-memorable forms

$$G_8: \ H_8(x,y,z) = H_8(y,x,z), \qquad G_9: \ H_9(x,y,z) = H_9(y,x,z),$$

i.e., the symmetry in x and y of the products

$$H_8(x,y,z) := \{U_xU_y(z), z, x\bullet y\} - U_xU_yU_z(x\bullet y), \quad H_9(x,y,z) := 2U_x(z)\bullet U_{y,x}U_z(y^2) - U_xU_zU_{x,y}U_y(z).$$

Observe that G_8, G_9 vanish in special algebras since H_8, H_9 reduce to the symmetric 8 and 9-tads $\{x,y,z,y,x,z,x,y\}, \{x,z,x,x,z,y,y,z,y\}$ respectively. In 1987 Armin Thedy discovered an operator s-identity of degree 10 in 3 variables that finally a mortal could remember:

$$T_{10}: \ U_{U_{[x,y]}(z)} = U_{[x,y]}U_zU_{[x,y]} \quad \big(U_{[x,y]} := 4U_{x\bullet y} - 2(U_xU_y + U_yU_x)\big).$$

This is just a "fundamental formula" or "structural condition" for the U-operator of the "commutator" $[x,y]$.

Notice that T_{10} vanishes on all special algebras because $U_{[x,y]}$ really *is* the map $z \mapsto (xy+yx)z(xy+yx) - 2(xyzyx + yxzxy) = [x,y]z[x,y]$. Of course, there is no such thing as a commutator in a Jordan algebra (in special Jordan algebras $\mathrm{J} \subseteq \mathrm{A}^+$ the commutators do exist in the associative envelope A), but these spiritual entities still manifest their presence by acting on the Jordan algebra. In 1999 Ivan Shestakov discovered that Glennie's identities could be rewritten in a very memorable form using commutators,

$$Sh_8(x,y,z): \ [[x,y]^3, z^2] = \{z, [[x,y]^3, z]\}, \quad Sh_9(x,y,z): \ [[x,y]^3, z^3] = \{z^2, [[x,y]^3, z]\} + U_z\big([[x,y]^3, z]\big),$$

i.e., that $Ad_{[x,y]^3}$ is a derivation of Jordan products z^2 and z^3. The quick modern proof that the Albert algebra is exceptional is to show that for *judicious* choice of x, y, z some polynomial G_8, G_9, Sh_8, Sh_9, or T_{10} does not vanish on $\mathcal{H}_3(\mathbb{K})$.

Thus it was *serendipitous* (a happy coincidence) that the Albert algebra was allowed on the mathematical stage in the first place. Many algebraic structures are famous for 15 minutes and then disappear from the action, but others go on to feature in a variety of settings and prove to be an enduring part of the mathematical landscape. So it was with the Albert algebra. In Part II we will describe some other algebraic systems that flow from the concept of Jordan algebra. In Part III we will describe some of the places where Jordan systems, especially the Albert algebra, have played a starring role. In Part IV we will describe how our views on the structure theory of Jordan algebras have evolved over time.

Part II. The Jordan River

The stream of Jordan theory originates in Jordan algebras, but soon divides into several algebraic structures (quadratic Jordan algebras, Jordan triples, Jordan pairs, and Jordan superalgebras). All these more general systems take their basic genetic structure from the parental algebras, but require their own special treatment and analysis, and result in new fields of application.

1. Quadratic Jordan Algebras

In their mature roles, Jordan algebras appear not just wearing a Jordan product, but sporting a powerful quadratic product as well. It took audiences some getting used to this new product, since (unlike quasi-multiplication, Lie bracket, dot products, or any of the familiar algebraic products) it is *not bilinear*: its polarized version is a triple product *trilinear* in 3 variables, but it is itself a binary product *quadratic* in one variable and linear in the other.

Over the years algebraists had developed a comprehensive theory of finite-dimensional Jordan algebras over arbitrary fields of characteristic different from 2. But it was clear that quasi-multiplication, with its reliance on a scalar $\frac{1}{2}$, was not sufficient for a theory of Jordan algebras in characteristic 2, or over arbitrary rings of scalars. In particular, there was no clear notion of Jordan *rings* (where the ring of scalars was the integers). For example, arithmetic investigations led naturally to hermitian matrices over the integers, and residue class fields led naturally to characteristic 2.

In the 1960s several lines of investigation revealed the crucial importance of the quadratic product $U_x(y)$ and the associated triple product $\{x, y, z\}$ in Jordan theory. Kantor, and Koecher and his students, showed that the triple product arose naturally in connections with Lie algebras and differential geometry. Nathan Jacobson and his students showed how these products facilitated many purely algebraic constructions.

The *U-operator* of "two-sided multiplication" U_x by x has the somewhat messy form $U_x = 2L_x^2 - L_{x^2}$ in terms of left multiplications $L_x(y) := x \bullet y$ in the algebra, and the important *V-operator* $V_{x,y}(z) := \{x, y, z\} := (U_{x+z} - U_x - U_z)(y)$ of "left multiplication" by x, y in the Jordan triple product has the even-messier form $V_{x,y} = 2\big(L_{x\bullet y} + [L_x, L_y]\big)$; the *brace product*, the linearization of the square, is just twice the bullet $\{x, y\} := V_x(y) := \{x, 1, y\} = 2x \bullet y$. In Jordan algebras with a unit element $1 \bullet x = x$ we have $U_1(x) = x$ and $x^2 = U_x(1)$, so we can recover linear product from the quadratic product. We will see that the operators $U_x, V_{x,y}, V_y$ are in many situations more natural than the bullet product $L_x(y) = x \bullet y$: in special Jordan algebras

$$U_x(y) = xyx, \quad V_{x,y}(z) = xyz + zyx, \quad V_x(y) = xy + yx.$$

The crucial property of the quadratic product was the *Fundamental Formula* $U_{U_x(y)} = U_x U_y U_x$; this came up in several situations, and seemed to play a role in the Jordan story like that of the associative law $L_{xy} = L_x L_y$ in the associative story. After a period of experimentation (much like Jordan's original investigation of quasi-multiplication), it was found the entire theory of unital Jordan algebras could be based on the U-operator.

A **unital quadratic Jordan algebra** is a space together with a distinguished element 1 and a product $U_x(y)$ linear in y and *quadratic* in x, which is **unital** and satisfies the **Commuting Formula** and the **Fundamental Formula**

$$U_1 = \mathbf{1}_{\mathrm{J}}, \qquad U_x V_{y,x} = V_{x,y} U_x, \qquad U_{U_x(y)} = U_x U_y U_x.$$

These are analogous to the axioms for the bilinear product $x \bullet y$ of old-fashioned unital linear Jordan algebras

$$L_1 = \mathbf{1}_{\mathrm{J}}, \qquad L_x = R_x, \qquad L_{x^2} L_x = L_x L_{x^2}$$

in terms of left and right multiplications L_x, R_x.

The quadratic approach focusing on $U_x y$ instead of $x \bullet y$ leads naturally to a generalization of automorphism. A **structural transformation** T on J satisfies $U_{T(x)} = T U_x T^*$ for some T^*; for examples, any automorphism satisfies $T^* = T^{-1}$, while any operator $T = U_u$ satisfies $T^* = T$. The **Structure group** $Strg(\mathrm{J})$ of all invertible structural transformations is a generalization of the Automorphism group.

In a commutative ring there is no distinction between left and right ideals; one of the important consequences of the quadratic approach it that the product $U_x(y) \approx xyx$ has a definite *inside* and *outside*. An **inner ideal** is a subspace $\mathrm{B} \subseteq \mathrm{J}$ closed under inner multiplication by J, $U_{B}\mathrm{J} \subseteq \mathrm{B}$; these play the role in Jordan theory that one-sided ideals play in associative theory. For well-behaved associative algebras A the inner ideals of A^+ are precisely all $\mathrm{L} \cap \mathrm{R}$ for left, right ideals L,R in A. Important examples are $\mathrm{B} = \Phi b$ for **rank-1** elements ($U_b \widehat{\mathrm{J}} = \Phi b$). The basic semisimplicity condition for Jordan algebras also depends on the U-operator: J is **nondegenerate** if it has no **trivial elements** z (with trivial U-operator $U_z \widehat{\mathrm{J}} = 0$).

Inverses and Isotopy

Another advantage of the U-operator was in providing a better understanding of inverses and isotopes in Jordan algebras, concepts cumbrous to describe using the bullet. An element x is *invertible* in J iff the operator U_x is an invertible operator on J, in which case its inverse is the operator $U_x^{-1} = U_{x^{-1}}$ for the **inverse element** $x^{-1} := U_x^{-1}(x)$. In special Jordan algebras $\mathrm{J} \subseteq \mathrm{A}^+$ an element x is invertible iff it is invertible in A and its associative inverse x^{-1} lies in J, in which case x^{-1} is also the Jordan inverse.

The fundamental tenet of **isotopy** is the belief that all invertible elements of a Jordan algebra have an equal entitlement to serve as unit element. If u is an invertible element of an associative algebra A, we can form a new associative algebra, the *associative isotope* $\mathrm{A}_{[u]}$ with new product, unit, and inverse given by

$$x_u y := xu^{-1}y, \qquad 1_u := u, \qquad x^{[-1,u]} := ux^{-1}u.$$

We can do the same thing in any Jordan algebra: the *Jordan isotope* $\mathrm{J}_{[u]}$ has new bullet, U-operator, unit, and inverse

$$x \bullet_{[u]} y := \tfrac{1}{2}\{x, u^{-1}, y\}, \qquad U_{x[u]} := U_x U_{u^{-1}}, \qquad 1_{[u]} := u, \qquad x^{[-1,u]} = U_u x^{-1}.$$

Thus each invertible u is indeed the unit in its own isotope. The structure group $Strg(\mathrm{J})$ consists precisely of all *isotopies*, isomorphisms $T : J \longrightarrow J^{[u]}$ for $u = T(1), T^* = T^{-1}U_u$.

Moreover, the inverse satisfies the **Hua Identity** used by L.K. Hua to study automorphisms of projective lines:

$$(x + U_x y^{-1})^{-1} + (x+y)^{-1} = x^{-1}.$$

One consequence of the Hua Identity is that the U-product can be built out of translations and inversions. T.A. Springer later used this to give an axiomatic description of Jordan algebras just in terms of the operation of inversion, which we may loosely describe by saying that an algebra is Jordan iff its inverse is *given by the geometric series in all isotopes,*

$$(1-x)^{[-1,u]} = \sum_{n=0}^{\infty} x^{[n,u]}.$$

This formula also suggests why the inverse might encode all the information about the Jordan algebra: it contains information about all the powers of an element, and from $x^{[2,u]} = U_x u^{-1}$ for $u = 1$ we can recover the square, hence the bullet product. Like Zeus shoving aside Kronos, the quadratic product has largely usurped governance of the Jordan domain (though to this day pockets of the theory retain the old faith in quasi-multiplication).

Moral: *The story of Jordan algebras is not the story of a nonassociative product* $x \bullet y$*, it is the story of a quadratic product* $U_x(y)$ *which is as associative as it possibly can be.*

2. Jordan Triples

The first Jordan stream to branch off from algebras was *Jordan triple systems*, whose algebraic study was initiated by Max Koecher's student Kurt Meyberg in 1969 in the process of generalizing the Tits–Kantor–Koecher construction of Lie algebras. Jordan triples are basically Jordan algebras with the unit thrown away, so there is no square or bilinear product, only a Jordan triple product $\{x, y, z\}$ which is *symmetric* and satisfies the 5-*linear Jordan identity*

$$\{x, y, z\} = \{z, y, x\}, \quad \{x, y, \{u, v, w\}\}{=}\{\{x, y, u\}, v, w\}{-}\{u, \{y, x, v\}, w\}{+}\{u, v, \{x, y, w\}\}.$$

A space with such a triple product is called a *linear Jordan triple.* This theory worked only in the presence of a scalar $\frac{1}{6}$; in 1972 Meyberg gave axioms for *quadratic Jordan triple systems* that worked smoothly for arbitrary scalars. These were based on a quadratic product $P_x(y)$ (the operators U, V are usually denoted by P, L in triples) satisfying the **Shifting Formula** , **Commuting Formula** , and the **Fundamental Formula**

$$L_{P_x(y),y} = L_{x,P_y(x)}, \quad P_xL_{y,x} = L_{x,y}P_x, \quad P_{P_x(y)} = P_xP_yP_x.$$

Any Jordan algebra J gives rise to a Jordan triple J^t by applying the forgetful functor, throwing away the unit and square and setting $P_x(y) := U_x(y),\ L_{x,y} := V_{x,y}$. In particular, any associative algebra A gives rise to a Jordan triple A^t via $P_x(y) := xyx,\ \{x, y, z\} := xyz + zyx$. A triple is *special* if it can be imbedded as a sub-triple of some A^t, otherwise it is *exceptional*.

An important example of a Jordan triple which doesn't come from a bilinear product consists of the **rectangular matrices** $\mathbb{M}_{pq}(\mathrm{A})$ under $xy^{tr}z + zy^{tr}x$; if $p \neq q$ there is no natural way to multiply two $p \times q$ matrices to get a third $p \times q$ matrix. Taking rectangular 1×2 matrices $\mathbb{M}_{12}(\mathbb{K}) = \mathbb{K}E_{11} + \mathbb{K}E_{12}$ over a Cayley algebra gives an exceptional 16-dimensional **bi-Cayley triple** (so called because it is obtained by gluing together two copies of the Cayley algebra).

Experience has shown that the archetype for Jordan triple systems is A^{t*} obtained from an associative algebra A *with involution* $*$ via

$$P_x(y) := xy^*x, \qquad \{x, y, z\} := xy^*z + zy^*x.$$

The presence of the involution $*$ on the middle factor warns us to expect reversals in that position. It turns out that a triple is special iff it can be imbedded as a sub-triple of some A^{t*}, so that either model A^{t} or A^{t*} can be used to define speciality

3. Jordan Pairs

The second stream to branch off from algebras flows from the same source, the Tits–Kantor–Koecher construction of Lie algebras. Building on an offhand remark of Meyberg that the entire TKK construction would work for "verbundene Paare," two independent spaces J^+, J^- acting *on each other* like Jordan triples, a full-grown theory of Jordan pairs sprang from Ottmar Loos's mind in 1974. *Linear Jordan pairs* are pairs $\mathrm{V} = (V^+, V^-)$ of spaces with trilinear products $\{x^+, u^-, y^+\} \in V^+, \{u^-, x^+, v^-\} \in V^-$ (but incestuous triple products containing adjacent terms from the same space are strictly forbidden!) such that both products are symmetric and satisfy the 5-linear identity. *Quadratic Jordan pairs* have quadratic products $Q_{x^\varepsilon}(u^{-\varepsilon}) \in V^\varepsilon$ $(\varepsilon = \pm)$ satisfying the three quadratic Jordan triple axioms (the operators P, L are usually denoted by Q, D in Jordan pairs).

Every Jordan triple J can be *doubled* to produce a Jordan pair $\mathcal{V}(\mathrm{J}) = (J, J)$, $V^\varepsilon := J$ under $Q_{x^\varepsilon}(y^{-\varepsilon}) := P_x(y)^\varepsilon$. The double of rectangular matrices $\mathbb{M}_{pq}(F)$ could be more naturally viewed as a pair $(\mathbb{M}_{pq}(F), \mathbb{M}_{qp}(F))$. More generally, for any two vector spaces V, W over a field F we have a "rectangular" pair $(Hom_F(V, W),\ Hom_F(W, V))$ of different spaces under products xux, uxu making no reference to a transpose. This also provides an example to show that pairs are more than doubled triples. In finite dimensions all semisimple Jordan pairs have the form $\mathcal{V}(\mathrm{J})$ for a Jordan triple system J; in particular, V^+ and V^- have the same dimension, but this is quite accidental and ceases to be true in infinite dimensions. Indeed, for a vector space W of infinite dimension d the rectangular pair $\mathrm{V} := (Hom_F(W, F), Hom_F(F, W)) \cong (W^*, W)$ has $\dim(W^*) = \mid F \mid^d \geq 2^d > d = \dim(W)$.

The perspective of Jordan pairs has clarified many aspects of the theory of Jordan triples and algebras.

4. Jordan Superalgebras

The third main branching of the Jordan River leads to *Jordan superalgebras* introduced by KKK (Victor Kac, Issai Kantor, Irving Kaplansky). Once more this river springs from a physical source: Jordan superalgebras are dual to the *Lie superalgebras* invented by physicists to provide a formalism to encompass *supersymmetry,* handling bosons and fermions in one algebraic system. A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathrm{L} = L_0 \oplus L_1$ where L_0 is a Lie algebra and L_1 an L_0-module with a "Jordan-like" product into L_0. Dually, a Jordan superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathrm{J} = J_0 \oplus J_1$ where J_0 is a Jordan algebra and J_1 a J_0-bimodule with a "Lie-like" product into J_0. For example, any $\mathbb{Z}_2$-graded associative algebra $\mathrm{A} = A_0 \oplus A_1$ becomes a Lie superalgebra under the *graded Lie product*

$$[x_i, y_j] = x_i y_j - (-1)^{ij} y_j x_i$$

(reducing to the Lie bracket $xy - yx$ if at least one factor is even, but to the Jordan brace $xy + yx$ if both i, j are odd), and dually becomes a Jordan superalgebra under the *graded Jordan brace*

$$\{x_i, y_j\} = x_i y_j + (-1)^{ij} y_j x_i$$

(reducing to the Jordan brace $xy + yx$ if at least one factor is even, but to the Lie bracket $xy - yx$ if both factors are odd). Jordan superalgebras shed light on Pchelinstev Monsters, a strange class of prime Jordan algebras which seem genetically unrelated to all normal Jordan algebras.

We have indicated how these branches of the Jordan river had their origin in an outside impetus. We now want to indicate how the branches, in turn, have influenced and enriched various areas of mathematics.

Part III. Applications

1. Links with Lie Algebras and Groups

The Jordan river flows parallel to the Lie river with its extended family of Lie systems (algebras, triples, and superalgebras), and an informed view of Jordan algebras must also reflect awareness of the Lie connections. Historically, the first connection of Jordan algebras to another area of mathematics was to the theory of Lie algebras, and the remarkably fruitful interplay between Jordan and Lie theory continues to generate interest in Jordan algebras.

Recall that a *Lie algebra* is a linear algebra with product $[x, y]$ which is *anti-commutative* and satisfies the *Jacobi Identity*

$$[x, y] = -[y, x], \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Just as any associative algebra A becomes a linear Jordan algebra A^+ under the Jordan product $\{x, y\} = xy + yx$, it also becomes a Lie algebra A^- under the **Lie bracket** $[x, y] := xy - yx$. In contrast to the Jordan case, *all* Lie algebras over a field are *special* in the sense of arising as a commutator-closed subspace of some A^-.

A^- contains many subspaces which are closed under the Lie bracket but not under the ordinary product. There are three main ways of singling out such subspaces. The first is by means of the trace: if A is the algebra of $n \times n$ matrices over a field, or more abstractly all linear transformations on a finite-dimensional vector space, then the subspace of elements of **trace zero** is closed under brackets since the Lie bracket $T_1T_2 - T_2T_1$ of *any two transformations* has trace zero.

The second way is by means of an involution: in general, for any involution $*$ on an associative algebra A the subspace $\mathcal{S}kew(\mathrm{A},*)$ of skew elements $x^* = -x$ is closed under brackets since

$$[x_1, x_2]^* = [x_2^*, x_1^*] = [-x_2, -x_1] = [x_2, x_1] = -[x_1, x_2].$$

For example, a nondegenerate symmetric or skew bilinear form on a vector space V induces an *adjoint involution* $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$ and the **skew transformations** $T^* = -T$ form a Lie algebra of transformations. The third method is by means of **derivations** satisfying the *product rule* $D(xy) = D(x)y + xD(y))$ on an *arbitrary* linear algebra C. These are closed under brackets by subtracting the right from the left side of

$$(D_1D_2)(xy) - (D_1D_2)(x)y - x(D_1D_2)(y) = D_1(x)D_2(y) + D_2(x)D_1(y) = (D_2D_1)(xy) - (D_2D_1)(x)y - x(D_2D_1)(y)$$

The four great classes of simple Lie algebras (respectively groups) A_n, B_n, C_n, D_n consist of matrices of trace 0 (respectively determinant 1) or skew $T^* = -T$ (respectively isometric $T^* = T^{-1}$) with respect to a nondegenerate symmetric or skew-symmetric bilinear form. The five exceptional Lie algebras and groups G_2, F_4, E_6, E_7, E_8 appearing mysteriously in the nineteenth-century Cartan–Killing classification were originally defined in terms of a multiplication table over an algebraically closed field. The exceptional Jordan algebra (the Albert algebra), the exceptional composition algebra (the Cayley or octonion algebra), and the five exceptional Lie algebras are enduring features of the mathematical landscape, and they are all genetically related. Algebraists first became interested in the newborn Albert algebra through its unexpected connections with exceptional Lie algebras and groups. When the physicists abandoned Jordan algebras, algebraists adopted the orphans (especially the Albert algebra) when they found in the 1930s, '40s, and '50s that the exceptional Lie groups could be described in an intrinsic coordinate-free manner using the Albert algebra $\mathbb{A}$ and Cayley algebra $\mathbb{K}$. This made it possible to study these algebras over general fields.

The Lie algebra (resp. group) G_2 of dimension 14 arises as the derivation algebra (resp. automorphism group) of $\mathbb{K}$; F_4 of dimension 52 arises as the derivation algebra (resp. automorphism group) of $\mathbb{A}$; E_6 arises from the *reduced structure algebra* $\mathcal{S}trl_0(\mathbb{A}) := L(\mathbb{A}_0) + \mathcal{D}er(\mathbb{A})$ (resp. structure group $\mathcal{S}trg(\mathbb{A}) := U(\mathbb{A})\mathcal{A}ut(\mathbb{A})$) of $\mathbb{A}$ of dimension $(27-1)+52 = 78$ (the subscript 0 indicates trace zero elements); E_7 arises from the Tits–Kantor–Koecher construction $\mathcal{TKK}(\mathbb{A}) := \mathbb{A} \oplus \mathcal{S}trl(\mathbb{A}) \oplus \mathbb{A}$ (resp. $\mathcal{TKK}$ group) of $\mathbb{A}$ of dimension $27 + 79 + 27 = 133$.

E_8 of dimension 248 arises in a more complicated manner from $\mathbb{A}$ and $\mathbb{K}$. Jacques Tits discovered in 1966 a general construction of a Lie algebra $\mathcal{FT}(\mathrm{C}, \mathrm{J})$, starting from a composition algebra C and a Jordan algebra J of "degree 3," which produces E_8 when J is the Albert algebra and C the Cayley algebra. Varying the possible ingredients leads to a square arrangement that had been noticed earlier by Hans Freudenthal:

The Freudenthal–Tits Magic Square: $\mathcal{FT}(\mathrm{C}, \mathrm{J})$

$\mathrm{C} \setminus \mathrm{J}$	$\mathbb{R}$	$\mathcal{H}_3(\mathbb{R})$	$\mathcal{H}_3(\mathbb{C})$	$\mathcal{H}_3(\mathbb{H})$	$\mathcal{H}_3(\mathbb{K})$
$\mathbb{R}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	A_6	E_7
$\mathbb{K}$	G_2	F_4	E_6	E_7	E_8

Some have doubted whether this is square, but no one has ever doubted that it is magic. The ingredients for this Lie recipe are a composition algebra C ($\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$ of dimension $1, 2, 4, 8$) and a Jordan algebra J (either $\mathbb{R}$ or $\mathcal{H}_3(\mathrm{D})$ [$\mathrm{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$] of dimension 1 or $6, 9, 15, 27$). The recipe creates a space $\mathcal{FT}(\mathrm{C}, \mathrm{J}) := \mathcal{D}er(\mathrm{C}) \oplus (\mathrm{C}_0 \otimes \mathrm{J}_0) \oplus \mathcal{D}er(\mathrm{J})$ (again the subscript 0 indicates trace zero elements) with complicated Lie product requiring $\frac{1}{12}$. For example, the lower right corner of the table produces a Lie algebra of dimension $14 + (7 \times 26) + 52 = 248$, which is precisely the dimension of E_8.

The Tits–Kantor–Koecher Construction

In the late 1960's Issai Kantor and Max Koecher independently showed how to build a Lie algebra $\mathcal{TKK}(\mathrm{J}) := L_1 \oplus L_0 \oplus L_{-1}$ with "short 3-grading" $[L_i, L_j] \subseteq L_{i+j}$ by taking $L_{\pm 1}$ to be two copies of any Jordan algebra **J** glued together by the *inner structure Lie algebra* $L_0 := Instrl(\mathrm{J}) = V_{J,J}$ spanned by the V-operators: $\mathcal{TKK}(\mathrm{J}) := \mathrm{J}_1 \oplus Instrl(\mathrm{J}) \oplus \mathrm{J}_{-1}$ has an anticommutative bracket determined by

$$[T, x_1] := T(x)_1, \quad [T, y_{-1}] := -T^*(y)_{-1}, \quad [x_i, y_i] := 0 \quad (i = \pm 1), \quad [x_1, y_{-1}] := V_{x,y}, \quad [T_1, T_2] := T_1T_2 - T_2T_1.$$

for $T \in Instrl(\mathrm{J}), x, y \in J$; the natural involution $*$ on $Instrl(\mathrm{J})$ determined by $V_{x,y}^* = V_{y,x}$ extends to an involution $(x, T, y) \mapsto (y, T^*, x)$ on all of $\mathbf{L}$ (note that the 5-linear elemental identity $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} + \{u, v, \{x, y, w\}\}$ becomes the operator identity $[V_{x,y}, V_{u,v}] = V_{\{x,y,u\},v} - V_{u,\{y,x,v\}}$ acting on the element w, showing that the inner structure algebra is closed under the Lie bracket, once more indicating the close connection between the Jordan triple product and the Lie product). Later it was noticed that this was a special case of a 1953 construction by Jacques Tits of a Lie algebra built out of a Jordan algebra and a simple 3-dimensional Lie algebra of type A_1.

The Tits–Kantor–Koecher Construction is not only intrinsically important, it is historically important because it gave birth to two streams in Jordan theory. The Jacobi identity for TKK to be a Lie algebra boils down to outer-symmetry and the 5-linear identity for the Jordan triple product. This observation led Meyberg to take these two conditions as the axioms for a new algebraic system, a **Jordan triple system**, and he showed that the Tits–Kantor–Koecher construction $\mathcal{TKK}(\mathrm{J}) := \mathrm{J} \oplus Instrl(\mathrm{J}) \oplus \mathrm{J}$ produced a graded Lie algebra with reversal involution $x \oplus T \oplus y \mapsto y \oplus T^* \oplus x$ iff $\mathbf{J}$ was a linear Jordan triple system. This was the first Jordan stream to branch off the main line.

The second stream branches off from the same source, the $\mathcal{TKK}$ construction. Meyberg had mentioned that the spaces of degree L_1, L_{-1} needn't be the same spaces, they needed to be "verbundene Paare". Loos formulated the axioms for **Jordan pairs** $\mathrm{V} = (V_1, V_{-1})$ (a pair of spaces V_1, V_{-1} acting on each other like Jordan triples), and showed that they are precisely what is needed in the $\mathcal{TKK}$-Construction of Lie algebras: $\mathcal{TKK}(\mathrm{V}) := V_1 \oplus Instrl(\mathrm{V}) \oplus V_{-1}$ produces a graded Lie algebra iff $\mathrm{V} = (V_1, V_{-1})$ is a linear Jordan pair. Jordan triples arise precisely from pairs with involution, and Jordan algebras arise from pairs where the grading and involution come from a little $s\ell_2 = \{e, f, h\} = \{1_1, 1_{-1}, 2(\mathbf{1}_{\mathrm{J}})\}$.

Jordan algebras played a role in Zel'manov's celebrated solution of the Restricted Burnside Problem, for which he was awarded a Fields Medal in 1994. That problem, about finiteness of finitely-generated torsion groups, could be reduced to a problem in certain Lie p-rings, which in turn could be illuminated by a natural Jordan algebra structure of characteristic p. The most difficult case, where the characteristic was $p = 2$, could be most clearly settled making use of a quadratic Jordan structure. Lie algebras in characteristic 2 are weak, pitiable things; *linear* Jordan algebras aren't much better, since the Jordan product $xy + yx$ is indistinguishable from the Lie bracket $xy - yx$ in that case. The crucial extra product xyx of the *quadratic* Jordan theory was a key that helped turn the tide of battle.

Thus Jordan systems arise naturally as "coordinates" for graded Lie algebras, leading to the dictum of Kantor: ***"There are no Jordan algebras, there are only Lie algebras."*** Of course, this can be turned around: ***Nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick.***

2. Links with Projective Geometry

The second example of the serendipitous appearance of Jordan algebras is in the study of projective planes. In 1933 Ruth Moufang used an octonion division algebra to construct a projective plane which satisfied the Little Desargues's Theorem, but not Desargues's Theorem. However, this description did not allow one to describe the automorphisms of the plane, since the usual associative approach via invertible 3×3 matrices broke down for matrices with nonassociative octonion entries. In 1949 Jordan found a way to construct the real octonion plane inside the formally real Albert algebra $\mathbb{A} = \mathcal{H}_3(\mathbb{K})$ of hermitian 3×3 Cayley matrices, using the set of primitive idempotents to coordinatize both the points and the lines. This was rediscovered in 1951 by Hans Freudenthal. In 1959 this was greatly extended by T.A. Springer to reduced Albert algebras $\mathrm{J} = \mathcal{H}_3(\mathcal{O})$ for octonion division algebras over arbitrary fields of characteristic $\neq 2, 3$, obtaining a "Fundamental Theorem of Octonion Planes" describing the automorphism group of the plane in terms of the "semi-linear structure group" of the Jordan algebra. In this not-formally-real case the coordinates were the "rank 1" elements (primitive idempotents or nilpotents of index 2). Finally, in 1970 John Faulkner extended the construction to algebras over fields of any characteristic, using the newly-hatched quadratic Jordan algebras. Here the points and lines were inner ideals with inclusion as incidence; structural maps naturally preserve this relation, and so induce geometric isomorphisms. The Fundamental Theorem of Projective Geometry for octonion planes says that all isomorphisms are obtained in this way. Thus octonion planes find a natural home for their isomorphisms in Albert algebras.

Recall that an abstract *projective plane* $\Pi = (\mathcal{P}, \mathcal{L}, I)$ consists of a set of *points* $\mathcal{P}$, a set of *lines* $\mathcal{L}$, and an *incidence relation* $\mathrm{I} \subset \mathcal{P} \times \mathcal{L}$ satisfying the three axioms (I) every two distinct points P_1, P_2 are incident to a unique line (denoted by $P_1 \vee P_2$), (II) every two distinct lines L_1, L_2 are incident to a unique point (denoted by $L_1 \wedge L_2$), (III) there exists a 4-*point* (four points, no three of which are collinear).An *isomorphism* $\sigma = (\sigma_{\mathcal{P}}, \sigma_{\mathcal{L}}) : \Pi \longrightarrow \Pi'$ consisting of bijections $\sigma_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}'$ of points and $\sigma_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathcal{L}'$ of lines which preserve incidence, $P \,\mathrm{I}\, L \Leftrightarrow \sigma_{\mathcal{P}}(P) \,\mathrm{I}'\, \sigma_{\mathcal{L}}(L)$.

The most important example of a projective plane is the *vector space plane* $\mathcal{P}roj\,(V)$ determined by a 3-dimensional left vector space V over an associative division ring Δ, where the points are the 1-dimensional subspaces P, the lines are the 2-dimensional subspaces L, and incidence is inclusion $P \,\mathrm{I}\, L \Leftrightarrow P \subseteq L$. Here $P_1 \vee P_2 = P_1 + P_2,\ L_1 \wedge L_2 = L_1 \cap L_2$. These planes can also be realized by a construction directly from the underlying division ring: V is isomorphic to ${}_\Delta\Delta^3$, with dual space V^* isomorphic to the right vector space Δ^3_Δ under the nondegenerate bilinear pairing $\langle v, w \rangle = \langle (\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$. The points $P = \Delta v = [v]_*$ are coordinatized (up to a left multiple) by a nonzero vector v, the lines L are in 1-to-1 correspondence with their 1-dimensional orthogonal complements $L^\perp \leq V^*$ corresponding to a "dual point" $[w]^* = w\Delta \leq \Delta^3_\Delta$, and incidence $P \subseteq L = (L^\perp)^\perp$ reduces to *orthogonality* $P \perp L^\perp$, i.e.,

$$\mathcal{P}roj\,(\Delta): \qquad \mathcal{P} = \{\, [v]_* \,\}, \quad \mathcal{L} = \{\, [w]^* \,\}, \quad [v]_* \,\mathrm{I}\, [w]^* \Leftrightarrow \langle v, w \rangle = 0.$$

Central Automorphisms

Projective planes are often classified according to how many central automorphisms they possess. An automorphism σ is *central* with *center* C which is fixed *linewise* (the automorphism fixes all L incident to C), in which case σ automatically has an *axis* M, fixed *pointwise* (fixes all P incident to M). A plane is (C, M)-*transitive* if the subgroup of automorphisms with center C and axis M) acts transitively: any point Q off the center and axis can be moved to any other such point Q' by some (C, M)-automorphism σ as long as C, Q, Q' are collinear. A plane is (C, M)-transitive iff *Desargues's* (C, M) *Theorem* holds: whenever two triangles have their vertices in *central perspective* from C and two of their sides in *axial perspective* from M, then so is the third side. A plane is (C, M)-transitive for all centers C and *all axes* M *on* C iff the plane satisfies the *Little Desargues's Theorem*; this happens iff it is coordinatized by an *alternative division ring.* Every octonion division algebra $\mathcal{O}$ produces a *Moufang plane* $\mathcal{M}ouf(\mathcal{O})$. A plane is (C, M)-transitive for all centers C and *all axes* M *whatsoever* iff Desargues's Theorem holds and the plane is *Desarguian*; this happens precisely iff it is coordinatized by an *associative division ring* Δ, in which case the plane is isomorphic to $\mathcal{P}roj(\Delta)$. The coordinate ring is a field $\Delta = \Phi$ iff the plane satisfies Pappus's Theorem (which implies Desargues's). Every alternative division algebra is either an octonion algebra or associative, so every projective plane satisfying Little Desargues' Theorem is either a Moufang plane or is already Desarguian.

The Fundamental Theorem of Projective Geometry says that the isomorphisms of Desarguian planes $\mathcal{P}roj(\Delta^3)$ come from semilinear isomorphisms of Δ^3, represented by 3×3 matrices $A \in \mathcal{M}_3(\Delta)$ and automorphisms τ of Δ via $\sigma_A([v]_*) = [\tau(v)A]_*$. A Moufang plane $\mathcal{M}ouf(\mathcal{O})$ has a coordinatization in terms of points (x, y) and lines $[m, b], [a]$ with octonion coordinates. breaks down when the coordinates are nonassociative: the associative composition of isomorphisms cannot be faithfully captured by the nonassociative multiplication of octonion matrices in $\mathcal{M}_3(\mathcal{O})$. In order to represent the isomorphisms, we must find a more abstract representation of $\mathcal{M}ouf(\mathcal{O})$.

Let $\mathrm{J} = \mathcal{H}_3(\mathrm{O})$ be an Albert algebra for an octonion division algebra $\mathcal{O}$ over a field Φ. We construct an octonion plane $\mathcal{P}roj(\mathrm{J})$ with points $\mathcal{P}$ the 1-dimensional inner ideals B [the spaces $\mathrm{B} = \Phi b$ determined by *rank-one elements* b with $U_b\mathrm{J} = \Phi b \neq 0$] and lines $\mathcal{L}$ the 10-dimensional inner ideals C, with inclusion as incidence $B \,\mathrm{I}\, C \iff B \subseteq C$ just as in $Proj(V)$. Every Moufang plane arises (up to isomorphism) by this construction: $\mathcal{M}ouf(\mathcal{O}) \cong \mathcal{P}roj(\mathcal{H}_3(\mathcal{O}))$. If J, J' are two such Albert algebras over fields Φ, Φ', then we call a map $T : \mathrm{J} \to \mathrm{J}'$ *structural* if it is a bijective semi-linear map [$T(\alpha x) = \alpha^\tau T(x)$ for an isomorphism $\tau : \Phi \to \Phi'$ of the underlying fields] such that there exists a semi-linear bijection $T^* : \mathrm{J}' \to \mathrm{J}$ with

$$U'_{T(x)} = TU_xT^*$$

for all x. Here T is uniquely determined as $T^{-1}U'_{T(1)}$. Any such structural T induces an isomorphism $\mathcal{P}roj(T) : \mathcal{P}roj(\mathrm{J}) \to \mathcal{P}roj(\mathrm{J}')$ of projective planes, since it preserves dimension, innerness [$U'_{T(\mathrm{B})}\mathrm{J}' = TU_{\mathrm{B}}(T^*J') \subseteq T(U_{\mathrm{B}}\mathrm{J}) \subseteq T(\mathrm{B})$], and incidence.

In Faulkner's treatment, points and lines are two copies of the same set: the 10-dimensional $\{x \in \mathrm{J} \mid V_{x,c} = 0\}$ is uniquely determined by a 1-dimensional Φc, so we can take as points and lines all $[b]_*$ and $[c]^*$ for rank-one elements b, c, where incidence becomes $[b]_* \,\mathrm{I}\, [c]^* \Leftrightarrow V_{b,c} = 0$ [analogous to $\langle v, w\rangle = 0$ in $\mathcal{P}roj(\Delta)$]. A structural T induces an isomorphism of projective planes via $\sigma_{\mathcal{P}}([b]_*) := [T(b)]_*$, $\sigma_{\mathcal{L}}([b]^*) := [T^{\#}(b)]^*$ for $T^{\#} = (T^*)^{-1}$ because these preserves orthogonality: $V'_{T(b),T^{\#}(c)} = 0 \Leftrightarrow V_{b,c} = 0$ from structurality $U'_{T(x)} = TU_xT^* \Rightarrow U'_{T(x)}T^{\#}y = TU_xy \Rightarrow \{T(x), T^{\#}(y), T(z)\}' = T(\{x, y, z\}) \Rightarrow V'_{T(x),T^{\#}(y)}T = TV_{x,y} \Rightarrow V'_{T(x),T^{\#}(y)} = TV_{x,y}T^{-1}$.

Fundamental Theorem of Moufang Planes: *The isomorphisms $\mathcal{P}roj(\mathrm{J}) \to \mathcal{P}roj(\mathrm{J}')$ of Moufang planes are precisely all $\mathcal{P}roj(T)$ for structural maps $T : \mathrm{J} \to \mathrm{J}'$ of Albert algebras.*

The arguments for this make heavy use of a rich supply of structural transformations on J in the form of **Bergmann operators** $B_{x,y} = I - V_{x,y} + U_xU_y$ (known geometrically as *algebraic transvections* $T_{x,-y}$); the $B_{x,y}$ form a more useful family than the U_x to get from one rank 1 element to another. The methods are completely quadratic-Jordan rather than octonionic. Thus the octonion planes as well as their automorphisms find a natural home in the Albert algebras, which has provided shelter for so many exceptional mathematical structures.

3. Links with Differential Geometry

The last application of Jordan theory is the most recent. Though mathematical physics gave birth to Jordan algebras and superalgebras, and Lie algebras gave birth to Jordan triples and pairs, differential geometry has had a more pronounced influence on the algebraic development of Jordan theory than any other mathematical discipline. Investigations of the role played by Jordan systems in differential geometry have revealed new perspectives on purely algebraic features of the subject. We now indicate what Jordan algebras were doing in such a strange landscape.

Links with the Real World: Symmetric Spaces

A **Riemannian manifold** is a smooth connected C^∞ manifold M carrying a *Riemannian metric*, a smoothly-varying positive-definite inner product $\langle\ ,\ \rangle_p$ on the tangent space $T_p(M)$ to the manifold at each point p. An *isometry* f of a Riemannian manifold is a diffeomorphism of smooth manifolds whose differential df is *isometric* on each tangent space, i.e., preserves the inner product $\langle df_p(u), df_p(v)\rangle_{f(p)} = \langle u, v\rangle_p$. The *exponential map* $\exp_p$ maps a neighborhood of 0 in $T_p(M)$ bijectively down onto a neighborhood of p in the manifold. A **Riemannian symmetric space** is a Riemannian manifold M having at each point p a *symmetry* s_p, an involutive global isometry of the manifold having p as isolated fixed point. The existence of such symmetries is a very strong condition on the manifold. It forces the group of isometries of M to be a real Lie group G acting transitively on M, forcing the manifold to be *real analytic* instead of merely smooth.

In 1969 Loos gave an algebraic formulation of symmetric spaces which clearly revealed a Jordan connection: A symmetric space is a Hausdorff C^∞ manifold with a differentiable multiplication $x \cdot y$ whose left multiplications $s_x(y) = x \cdot y$ are involutive, satisfy the Fundamental Formula, and have isolated fixedpoint x:

$$s_x^2 = \mathbf{1}_M, \quad s_{x \cdot y} = s_x s_y s_x, \quad s_x(x) = x \text{ is isolated} .$$

Here s_x represents the symmetry at the point $x \in M$. If one fixes a basepoint $c \in M$, the maps $Q_x := s_x s_c$ satisfy the usual Fundamental Formula $Q_{Q_x y} = Q_x Q_y Q_x$. Here $Q_c = \mathbf{1}_M$, and $j = s_c$ is "inversion." For example, in any Lie group the product $x \cdot y := x y^{-1} x$ gives such a multiplication, and for $c = e$ the maps Q, j take the form $Q_x(y) = xyx, j(x) = x^{-1}$ in terms of the group operations. This already suggests a close connection with quadratic Jordan algebras.

An important class of symmetric spaces come from Jordan algebras. A subset $\mathcal{C}$ of a real vector space V is a *cone* if it is closed under positive dilations $[t\mathcal{C} = \mathcal{C}$ for all $t > 0]$. An open convex cone is *regular* if $x \in \mathcal{C}$ implies $-x \notin \mathcal{C}$. The *dual* of a regular open convex cone is the cone of functionals in V^* which are strictly positive on the original cone. A positive-definite bilinear form σ on V allows us to identify V^* with V and the dual with the cone $\{y \in V \mid \sigma(y, \mathcal{C}) > 0\}$ in V, and we say that a cone is *self-dual* with respect to σ if under this identification it coincides with its dual.

The formally real Jordan algebras investigated by Jordan, von Neumann, and Wigner are precisely the finite-dimensional real unital Jordan algebras such that every element has a unique *spectral decomposition* $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$ for a supplementary family of orthogonal idempotents $1 = e_1 + \cdots + e_r$ and an increasing family $\lambda_1 < \cdots < \lambda_r$ of distinct *real* eigenvalues: spectral reality is equivalent to formal reality $x^2 + y^2 = 0 \Longrightarrow x = y = 0$. In analogy with Jordan canonical form for matrices, we can say that the elements all have a *diagonal* Jordan form with *real* eigenvalues.

The set of *positive elements* (those with positive spectrum, all $\lambda_i > 0$) is called the *positive cone* of the formally real Jordan algebra J. This coincides with the set of *invertible squares* $x^2 = \sum_k \lambda_k^2 e_k$ and also the *exponentials* $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_k e^{\lambda_k} e_k$, and is the connected component of the identity element in the set J^{-1} of invertible elements.

Positive Cone Theorem: *There is a* 1*-to-*1 *correspondence between the self-dual open homogeneous cones in* $\mathbb{R}^n$ *and* n*-dimensional formally real Jordan algebras, wherein the geometric structure is intimately connected with the Jordan algebraic structure living in the tangent space.*

The positive cone $\mathcal{C}$ *of an* n*-dimensional formally real Jordan algebra* J *becomes in a canonical way a Riemannian symmetric space as follows: as an open regular convex cone in* $\mathrm{J} \cong \mathbb{R}^n$*, it is naturally a smooth manifold, and it is self-dual with respect to the positive definite bilinear trace form* $\sigma(x, y) := \mathrm{tr}(V_{x,y})$*; identifying the tangent space* $T_p(\mathcal{C})$ *with* J*, taking* $\langle x, y\rangle_p := \sigma(U_p^{-1} x, y)$ *at each point gives at Riemannian metric on* $\mathcal{C}$*.*

The inversion map $j : x \mapsto x^{-1}$ *induces a diffeomorphism of* J *of period* 2 *leaving* $\mathcal{C}$ *invariant, and having there a unique fixed point* 1*, hence provides a symmetry of the Riemannian manifold* $\mathcal{C}$ *at the point* $p = 1$*; here the exponential map is the ordinary algebraic exponential* $\exp_1(x) = e^x$ *from* $T_1(\mathcal{C}) = \mathrm{J}$ *onto* $\mathcal{C} = \mathcal{C}one(\mathrm{J})$*. The Jordan product and* U*-operator arise from the inversion symmetry* $s_1 = j$ *via* $dj_x = -U_x^{-1}$ *and* $\partial_u \partial_v j|_1 = V_u(v) = 2\, u \bullet v$*.*

Any other point p *in* $\mathcal{C}one(\mathrm{J})$ *can be considered the unit element in its own algebraic system* $\mathrm{J}_{[p]}$*, and the manifold has a symmetry at the point* p *given by* $x \mapsto x^{[-1,p]}$*, so every point is an isolated fixed point of a symmetry, given by inversion in a Jordan isotope.*

For example, if $\mathrm{J} = \mathcal{H}_n(\mathbb{C})$ then the positive cone consists precisely of the positive-definite matrices (the hermitian matrices whose Jordan form has only positive real eigenvalues).

Links with the Complex World: Bounded Symmetric Domains

The complex analogue of a Riemannian manifold is a *hermitian manifold,* a connected complex analytic manifold M carrying a *hermitian metric*, a smoothly-varying positive-definite hermitian inner product $\langle\ ,\ \rangle_p$ on the tangent space $T_p(M)$ to the manifold at each point p. An *isometry* of an Hermitian manifold is a biholomorphic map of analytic manifolds whose differential is isometric on each tangent space. A *hermitian symmetric space* is a hermitian manifold having at each point p a *symmetry* s_p, an involutive global isomorphism of the manifold having p as isolated fixed point.

Every Hermitian symmetric space of "noncompact type" is isomorphic to a **bounded symmetric domain,** a down-to-earth bounded domain in $\mathbb{C}^n$ each point of which is an isolated fixed point of an involutive biholomorphic map of the domain. Initially there is no metric on such a domain, but there is a natural way to introduce one (for instance, the Bergmann metric derived from the Bergmann kernel on a corresponding Hilbert space of holomorphic functions).

In turn, every bounded symmetric domain is biholomorphically equivalent via its Harish–Chandra realization to a *bounded homogeneous circled domain,* a bounded domain containing the origin which is *circled* in the sense that the circling maps $x \mapsto e^{it}x$ are automorphisms of the domain for all real t, and *homogeneous* in the sense that the group of all biholomorphic automorphisms acts transitively on the domain. These *bounded realizations* are automatically convex, and arise as the open unit ball with respect to a certain norm on $\mathbb{C}^n$.

The classical example of an unbounded symmetric domain is the *upper half-plane* M (consisting of all $x+iy$ for real $x, y \in \mathbb{R}$ with $y > 0$). This is the home of the theory of automorphic forms and functions. The upper half-plane can be mapped by the Cayley transform $z \mapsto \frac{i-z}{i+z} = \frac{1+iz}{1-iz}$ onto its bounded realization, the *open unit disk* Δ (consisting of all $w \in \mathbb{C}$ with $|w| < 1$, i.e., $1 - w\bar{w} > 0$).

This was generalized by Carl Ludwig Siegel to *Siegel's upper half-space* (consisting of all $X+iY$ for symmetric $X,Y \in \mathcal{M}_n(\mathbb{R})$ with Y positive definite), to provide a home for the Siegel modular forms in the study of functions of several complex variables. Again, this unbounded domain is mapped by the Cayley transform $Z \mapsto (i\mathbf{1}-Z)(i\mathbf{1}+Z)^{-1} = (\mathbf{1}+iZ)(\mathbf{1}-iZ)^{-1}$ onto the *generalized unit disk* D consisting of all symmetric $W \in \mathcal{M}_n(\mathbb{C})$ with $\mathbf{1}-W\overline{W}$ positive definite.

Max Koecher began his life as an arithmetic complex analyst, but his studies of modular functions led him inexorably to Jordan algebras. He generalized Siegel's upper half-space to the case of an arbitrary formally real Jordan algebra J: *Koecher's upper half-space* $\mathcal{H}alf(\mathrm{J})$ consisting of all $Z = X+iY$ for $X,Y \in \mathrm{J}$ with Y in the positive cone $\mathcal{C}one(\mathrm{J})$. These half-spaces or *tube domains* $\mathrm{J} \oplus i\mathcal{C}$ are open and convex in the complexification $\mathrm{J}_{\mathbb{C}} := \mathrm{J} \oplus i\mathrm{J}$. The geometry of the unbounded $\mathcal{H}alf(\mathrm{J})$ is nicely described in terms of the Jordan algebra $\mathrm{J}_{\mathbb{C}}$: The biholomorphic automorphisms of $\mathcal{H}alf(\mathrm{J})$ are the linear fractional transformations generated by inversion $Z \mapsto -Z^{-1}$, translations $Z \mapsto Z+A$ $(A \in \mathrm{J})$, and $Z \mapsto \tilde{T}(Z) = T(X)+iT(Y)$ for complex-linear extensions $\tilde{T}$ of automorphisms $T \in \mathcal{A}ut(\mathrm{J})$. These half-spaces are mapped by the Cayley transform $Z \mapsto (\mathbf{1}+iZ)(\mathbf{1}-iZ)^{-1}$ onto the *generalized open unit disk* $D(\mathrm{J})$ consisting of all $Z = X+iY \in \mathrm{J}_{\mathbb{C}} = \mathrm{J} \oplus i\mathrm{J}$ with $\mathbf{1}-\frac{1}{2}V_{Z,\overline{Z}}$ positive definite with respect to the trace form $\mathrm{tr}(V_{W,\overline{Z}})$ on the Jordan algebra $\mathrm{J}_{\mathbb{C}}$.

Ottmar Loos showed that there is a natural 1-to-1 correspondence between bounded homogeneous circled domains D in $\mathbb{C}^n$ and the finite-dimensional positive hermitian Jordan triples. A *hermitian Jordan triple* is a complex Jordan triple $\mathbf{J}$ where $\{x, y, z\}$ which is symmetric and $\mathbb{C}$-linear in the outer variables x, z and conjugate-linear in the middle variable y. A finite-dimensional hermitian Jordan triple is *positive* if the trace form $\mathrm{tr}(L_{x,y})$ is a positive definite Hermitian scalar product. The *Bergmann operator* $B_{x,y} := \mathbf{1}_{\mathrm{J}} - L_{x,y} + P_x P_y$ of the triple comes up to a fixed constant κ from the *Bergmann kernel function* $K(x, y)$, the reproducing function for the Hilbert space of all holomorphic L^2-functions on the domain: $K(x, y) = \kappa/\det(B_{x,y})$. These operators, arising out of differential geometry, play a large part in the theory of Jordan triples and pairs and (as we saw) in projective geometry, where there are usually no invertible P_x or $Q'_x s$, but usually lots of invertible $B'_{x,y} s$.

We have a complete algebraic description of all these positive triples:

Hermitian Triple Classification *Every finite-dimensional positive hermitian triple system is a finite direct sum of simple triples, and there are exactly six classes of simple triples* (*together with a positive involution*)*: four great classes of special triples and two sporadic exceptional systems determined by the split octonion algebra* $\mathbb{K}_{\mathbb{C}}$ *over the complexes.*

(1) *rectangular matrices* $\mathcal{M}_{pq}(\mathbb{C})$,
(2) *skew matrices* $Skew_n(\mathbb{C})$,
(3) *symmetric matrices* $Symm_n(\mathbb{C})$,
(4) *spin factors* $\mathcal{JS}pin_n(\mathbb{C})$,
(5) *the bi-Cayley triple* $\mathbb{K}_{\mathbb{C}}^2$ *of dimension* 16,
(6) *the Albert triple* $\mathcal{H}_3(\mathbb{K}_{\mathbb{C}})$ *of dimension* 27.

The geometric properties of bounded symmetric domains are beautifully described by the algebraic properties of these triple systems.

Jordan Unit Ball Theorem: *Every positive hermitian Jordan triple system* J *gives rise to a bounded homogeneous convex circled domain* $D(\mathrm{J})$ *that is the open unit ball of* J *as a Banach space under the spectral norm, equivalently*

$$D(\mathrm{J}) := \left\{x \in \mathrm{J} \mid \mathbf{1}_{\mathrm{J}} - \tfrac{1}{2}L_{x,x} > 0\right\}.$$

Every bounded symmetric domain D *arises in this way: its bounded realization is* $D(\mathrm{J})$*, where the algebraic triple product can be recovered from the Bergmann metric* $\langle\, ,\, \rangle$ *and Bergmann kernel* K *of the domain as the logarithmic derivative of the kernel at* 0*:*

$$\langle\{u, v, w\}, z\rangle_0 := \partial_u \partial_v \partial_w \partial_z \log K(x, x)|_{x=0}.$$

The domain $D := D(\mathrm{J})$ *of the triple* J *becomes a hermitian symmetric space under the Bergmann metric* $\langle x \,|\, y\rangle_p := \mathrm{tr}(L_{B_{p,p}^{-1}x,y})$*. The automorphisms of the domain* D *fixing* 0 *are precisely the algebraic automorphisms of the triple* J*. At the origin the exponential map* $\exp_0 : T_0(D) = \mathrm{J} \to D$ *is a real analytic diffeomorphism given by the odd function* $\exp_0(v) = \tanh(v)$*.*

These examples, from the real and complex world, suggest a general Principle:

Geometric structure is often encoded in algebraic Jordan structure.

Part IV. Historical Survey of Jordan Structure Theory

1. Jordan Algebras in Physical Antiquity

The original 1933 classification of J–vN–W listed 4 matrix algebras and one spin factor.

Jordan–von Neumann–Wigner Theorem *Every finite-dimensional formally real Jordan algebra is a direct sum of a finite number of simple ideals, and there are five basic types of simple building blocks: four types of hermitian matrix algebras corresponding to the four composition division algebras: the reals* $\mathbb{R}$*, the complexes* $\mathbb{C}$*, the quaternions* $\mathbb{H}$*, and the octonions* $\mathbb{K}$ *over the reals, together with the spin factors. Every finite-dimensional simple formally real Jordan algebra is isomorphic to one of*:

$$\mathrm{H}_n^1 = \mathcal{H}_n(\mathbb{R}),\ \mathrm{H}_n^2 = \mathcal{H}_n(\mathbb{C}),\ \mathrm{H}_n^4 = \mathcal{H}_n(\mathbb{H}),\ \mathrm{H}_3^8 = \mathcal{H}_3(\mathbb{K}),\ \mathcal{JS}pin_n.$$

The notation was chosen so that H_n^k denoted the $n \times n$ hermitian matrices over the k-dimensional real composition division algebra. $\mathbb{K}$ayley's octonions $\mathbb{K} = \mathbb{H} \oplus \mathbb{H}\ell$ may be a stranger — they have basis $\{1, i, j, k\} \cup \{\ell, i\ell, j\ell, k\ell\}$ with nonassociative product $h_1(h_2\ell) = (h_2\ell)h_1^* = (h_2h_1)\ell, (h_1\ell)(h_2\ell) = -(h_2^*h_1)$, and involution $(h\ell)^* = -h\ell$ in terms of the new basic unit ℓ and old elements $h, h_1, h_2 \in \mathbb{H}$. All four of these carry a positive definite quadratic form $Q(\sum \alpha_i x_i) = \sum \alpha_i^2$ which "admits composition" in the sense that Q of the product is the product of the Q's, $Q(xy) = Q(x)Q(y)$, and for that reason are called *composition algebras*. A celebrated theorem of Hurwitz asserts that the *only* possible composition algebras over any field are the field (dimension 1), a quadratic extension (dimension 2), a quaternion algebra (dimension 4), and an octonion algebra (dimension 8).

2. Jordan Algebras in the Algebraic Renaissance

The next stage in the history of Jordan algebras was taken over by algebraists. While the physicists lost interest in the search for an exceptional setting for quantum mechanics (the philosophical objections to the theory paling in comparison to its amazing achievements), the unexpected connections between the physicists' orphan child and other important areas of mathematics spurred algebraists to consider Jordan algebras over more general fields. By the late 1940's the J–vN–W structure theory had been extended by A.A. Albert, F. and N. Jacobson, and others to finite-dimensional Jordan algebras with essentially the same cast of characters appearing in the title roles.

The crowning achievement of the development of Jordan theory during the Algebraic Renaissance was the classification of semi-simple Jordan algebras over an arbitrary algebraically closed field Φ of characteristic not 2. The classification of simple Jordan algebras proceeds according to "degree," where the *degree* is the maximal number of supplementary orthogonal idempotents (analogous to the matrix units E_{ii}). From another point of view, the degree is the degree of the *generic minimum polynomial* of the algebra, the "generic" polynomial $m_x(\lambda) = \lambda^n - m_1(x)\lambda^{n-1} + \cdots + (-1)^n m_n(x)$ ($m_i : \mathbf{J} \to \Phi$ homogeneous polynomials of degree i) of minimal degree satisfied by all x, $m_x(x) = 0$. Degree 1 algebras are just the 1-dimensional Φ^+; the degree 2 algebras are the $\mathcal{JS}pin_n$; the degree n algebras for $n \geq 3$ are all Jordan matrix algebras $\mathcal{H}_n(\mathrm{C})$ where the coordinate $*$-algebras C are precisely the split composition algebras over Φ with their standard involutions.

Renaissance Structure Theorem: *Consider finite-dimensional Jordan algebras* J *over an algebraically closed field* Φ *of characteristic* $\neq 2$.

- *The radical of* J *is the maximal nilpotent ideal, and the quotient* $\mathrm{J}/\mathcal{R}ad(\mathrm{J})$ *is semisimple.*
- *An algebra is semisimple iff it is a finite direct sum of simple ideals. In this case, the algebra has a unit element, and its simple decomposition is unique: the simple summands are precisely the minimal ideals.*
- *Every simple algebra is automatically central-simple over* Φ.
- *An algebra is simple iff it is isomorphic to exactly one of*:

Ground Field Φ^+ *of degree* 1,

Spin Factor $\mathcal{JS}pin_n(\Phi)$ *of degree* 2, *for* $n \geq 2$,

Hermitian Matrices $\mathcal{H}_n(\mathrm{C}(\Phi))$ *of degree* $n \geq 3$

coordinatized by a split composition algebra $\mathrm{C}(\Phi)$, *yielding*

Split Unarion Matrices $\mathcal{H}_n(\Phi)$ *for* Φ *the ground field,*

Split Binarion Matrices $\mathcal{H}_n(\mathcal{B}(\Phi)) \cong \mathcal{M}_n(\Phi)^+$ *for* $\mathcal{B}(\Phi)$ *the split binarions,*

Split Quaternion Matrices $\mathcal{H}_n(\mathcal{Q}(\Phi))$ *for* $\mathcal{Q}(\Phi)$ *the split quaternions,*

Split Octonion Matrices $\mathcal{A}lb(\Phi) = \mathcal{H}_3(\mathcal{O}(\Phi))$ *for* $\mathcal{O}(\Phi)$ *the split octonions.*

Once more, the only exceptional algebra in the list is the 27-dimensional split Albert algebra. Note that the 1-dimensional algebra $\mathcal{JS}pin_0$ is the same as the ground field; the 2-dimensional $\mathcal{JS}pin_1 \cong \mathcal{B}(\Phi)$ is not simple when Φ is algebraically closed, so only $\mathcal{JS}pin_n$ for $n \geq 2$ contribute new simple algebras.

We are beginning to isolate the Albert algebras conceptually; even though the real Albert algebra discovered by Jordan, von Neumann, and Wigner appear to fit into the family of Jordan matrix algebras, we will see in the next section that their non-reduced forms really come via a completely different construction out of a cubic form.

3. Jordan Algebras in the Enlightenment

Life over an algebraically closed field Ω is split. When we move to non–algebraically–closed fields Φ we encounter modifications or "twistings" in the split algebras which produce new kinds of simple algebras. There may be several non-isomorphic algebras A over Φ which are not themselves split algebras $\mathrm{S}(\Phi)$, but "become" $\mathrm{S}(\Omega)$ over the algebraic closure when we extend the scalars: $\mathrm{A}_\Omega \cong \mathrm{S}(\Omega)$. We call such A's *forms* of the split algebra S; in some Platonic sense they are incipient S's, and are prevented from revealing their true S-ness only by deficiencies in the scalars: once they are released from the constraining field Φ they can burst forth and reveal their true split personality. If we consider $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ with the conjugate transpose involution, any diagonal matrix $u = \Gamma$ with invertible hermitian nuclear elements in D down the diagonal is invertible hermitian nuclear in A and can be used to form an isotope with twisted product; we prefer to keep within the usual Jordan matrix operations and twist the involution instead, getting an algebra isomorphic to the isotope.

TWISTED MATRIX EXAMPLE: *For an arbitrary linear $*$-algebra $(\mathrm{D}, *)$ with involution $d \mapsto d^*$ and diagonal matrix $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ whose entries γ_i are invertible hermitian elements in the nucleus of D, the twisted conjugate transpose mapping $X^{*(\Gamma)} = \Gamma^{-1}(X^*)^{tr}\Gamma$ is an involution on the algebra $\mathcal{M}_n(\mathrm{D})$ of all $n \times n$ matrices with entries from D under the usual matrix product XY. The Φ-module $\mathcal{H}_n(\mathrm{D}, *, \Gamma) := \mathcal{H}(\mathcal{M}_n(\mathrm{D}), *(\Gamma))$ of all hermitian matrices $X^{*(\Gamma)} = X$ with respect to this new involution forms a Jordan algebra under the usual matrix operation $X \bullet Y = \frac{1}{2}(XY + YX)$. This twisted matrix algebra $\mathcal{H}_n(\mathrm{D}, *, \Gamma)$ is isomorphic to the Jordan isotope $\mathcal{H}_n(\mathrm{D}, *)_{[\Gamma]} \subseteq \mathcal{M}_n(\mathrm{D})_{[\Gamma]}$.*

Generalizing the spin factors determined by the dot product, we can consider degree-2 algebras determined by an arbitrary (not-necessarily diagonalizable) quadratic form.

QUADRATIC FACTOR EXAMPLE: (1) *If Q is a quadratic norm form on a space M with basespoint c over Φ containing $\frac{1}{2}$, we obtain a degree-2 Jordan algebra* $\mathrm{J} = \mathcal{J}ord(Q, c)$ *on M with unit* $1 := c$ *and product*

$$x \bullet y := \tfrac{1}{2}\big(T(x)y + T(y)x - Q(x,y)1\big) \qquad (T(x) := Q(x,c),\ Q(c) = 1)$$

where for $y = x$ we obtain the generic minimum polynomial $x^2 - T(x)x + Q(x)1 = 0$ of degree 2. The norm determines invertibility: an element in $\mathcal{J}ord(Q,c)$ is invertible iff its norm is an invertible scalar, in which case the inverse is $x^{-1} = Q(x)^{-1}\bar{x}$ for $\bar{x} := T(x)1 - x$.

(2) *Over a field, the norm and basepoint completely determine the Jordan algebra: if we define two quadratic forms with basepoint to be* equivalent (*written* $(Q,c) \cong (Q',c')$) *if there is a linear isomorphism $\varphi : M \to M'$ which preserves norms and basepoint, $Q'(\varphi(x)) = Q(x)$ for all $x \in M$ and $\varphi(c) = c'$, then $\mathcal{J}ord(Q,c) \cong \mathcal{J}ord(Q',c') \iff (Q,c) \cong (Q',c')$.*

(3) *All isotopes remain quadratic factors, $\mathcal{J}ord(Q,c)_{[u]} = \mathcal{J}ord(Q_{[u]}, c_{[u]})$ for $Q_{[u]} = Q(u)^{-1}Q$ and $c_{[u]} = u$.*

Thus isotopy just changes basepoint and scales the norm form. The quadratic factor is simple iff Q is nondegenerate (except that a split form $Q(\alpha,\beta) = \alpha\beta$ of dimension 2 yields a semisimple $\Phi \boxplus \Phi$).

In contrast to the case of *quadratic* forms with basepoint, only certain very special *cubic* forms with basepoint can be used to construct Jordan algebras. A **cubic form** $N : M \to \Phi$ on a Φ-module M is a homogeneous polynomial of degree 3 which extends to arbitrary scalar extensions M_Ω by

$$N\Big(\sum \omega_i x_i\Big) = \sum \omega_i^3 N(x_i) + \sum \omega_i^2 \omega_j N(x_i; x_j) + \sum \omega_i \omega_j \omega_k N(x_i; x_j; x_k),$$

where the **linearization** $N(x;y)$ is quadratic in x and linear in y, and $N(x;y;z)$ is symmetric and trilinear. The test for admission to the elite circle of Jordan cubics is the existence of a unit having well-behaved trace and adjoint. A **basepoint** for N is a point $c \in M$ with $N(c) = 1$. This determines a linear **trace form** $T(x) := N(c;x)$ and a quadratic **spur form** $S(x) := N(x;c)$ with linearization $S(x,y) := S(x+y) - S(x) - S(y) = N(x;y;c)$, and a quadratic **sharp** (or **adjoint**) map $\# : M \to M$ defined uniquely by $T(x^{\#}, y) = N(x;y)$ (with linearization $x\#y := (x+y)^{\#} - x^{\#} - y^{\#}$).

JORDAN CUBIC DEFINITION: *A finite-dimensional cubic form with basepoint* (N,c) *over a field* Φ *of characteristic* $\neq 2$ *is defined to be a* **Jordan cubic** *if* (1) N *is* **nondegenerate** *at the basepoint* c*, in the sense that the* **trace bilinear form** $T(x,y) := T(x)T(y) - S(x,y)$ *is a nondegenerate bilinear form, and* (2) *the quadratic adjoint map strictly satisfies the* **Adjoint Identity**

$$(x^{\#})^{\#} = N(x)x.$$

Springer Construction: (1) *From every Jordan cubic form with basepoint we obtain a Jordan algebra* $\mathcal{J}ord(N,c)$ *with unit* $1 := c$ *and product determined from the sharp mapping by the formula*

$$x \bullet y := \tfrac{1}{2}\big(x\#y + T(x)y + T(y)x - S(x,y)1\big), \text{ equivaently } U_x y = T(x,y)x - x^{\#}\#y.$$

of generic degree 3 *with* $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$ *and sharp mapping* $x^{\#} = x^2 - T(x)x + S(x)1$.

(2) *An element is invertible iff its norm is nonzero, in which case the inverse is a multiple of the adjoint* $u^{-1} = N(u)^{-1}u^{\#}$ *(just as in matrix algebras).*

(3) *For invertible elements the isotope is obtained* (*as in quadratic factors*) *by scaling the norm and shifting the unit*: $\mathcal{J}ord(N,c)_{[u]} = \mathcal{J}ord(N_{[u]}, c_{[u]})$ *for* $c_{[u]} = u, \ \ N_{[u]}(x) = N(u)^{-1}N(x)$.

We define an **Albert algebra** over a field Φ to be an algebra $\mathcal{J}ord(N, c)$ determined by a Jordan cubic form of dimension 27. The cubic factor construction was originally introduced by H. Freudenthal for 3×3 hermitian matrices, where we have a very concrete representation of the norm.

Freudenthal Construction: *If* C *is an alternative* $*$*-algebra with involution such that* $\mathcal{H}(\mathrm{C}, *) = \Phi 1$, *then for diagonal* $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$ *with entries invertible scalars, the twisted matrix algebra* $\mathcal{H}_3(\mathrm{C}, *, \Gamma)$ *is a cubic factor* $\mathcal{J}ord(N, c)$ *with basepoint* $c := e_1 + e_2 + e_3$ *and norm given by*

$$N(x) := \alpha_1\alpha_2\alpha_3 - \sum_{cyclic} (\alpha_i\gamma_j\gamma_k n(a_i)) + \gamma_1\gamma_2\gamma_3 t(a_1a_2a_3)$$

summed over all cyclic permutations (i, j, k) *of* $(1, 2, 3)$, *where the elements* $x = \sum_{cyclic} (\alpha_i e_i + a_i[jk])$, $y = \sum_{cyclic} (\beta_i e_i + b_i[jk])$ *are given by* $\alpha_i, \beta_i \in \Phi$, $a_i, b_i \in \mathrm{C}$, $a[jk] := aE_{jk} + a^* E_{kj}$. *Then*

$$T(x) = \sum_i \alpha_i, \quad S(x) = \sum_{cyclic} (\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)), \quad T(x, y) = \sum_{cyclic} (\alpha_i\beta_i + \gamma_j\gamma_k(a_i^* b_i)).$$

Besides twisting and cubic forms, a new phenomenon that arises only over non-algebraically-closed fields is that of *division algebras*. For associative algebras, the only finite-dimensional division algebra over an algebraically closed field (like the complexes) is the field itself, but over the reals we have the central-simple Hamilton's quaternions $\mathbb{H}$ of dimension 4. In the same way, the only finite-dimensional Jordan division algebra over an algebraically closed field Ω is Ω itself, but over a general field Φ there may be others: A^+ is a Jordan division algebra iff A is an associative division algebra, the factors $\mathcal{J}ord(Q, c)$ or $\mathcal{J}ord(N, c)$ are division algebras iff the quadratic or cubic form Q or N is anisotropic, otherwise they are *reduced* [have proper idempotents]. Simple reduced $\mathcal{J}ord(N, c)$'s are always isomorphic to Jordan matrix algebras $\mathcal{H}_3(\mathrm{C}, \Gamma)$ for a composition algebra C.

If J is finite-dimensional simple over Φ, its center Ω is a finite extension field of Φ, and J is finite-dimensional central-simple over Ω, so it suffices to classify all central-simple algebras. This leads to the final classification of finite-dimensional algebras over a general field of characteristic not 2.

Enlightenment Structure Theorem *Consider finite-dimensional Jordan algebras* J *over a field* Φ *of characteristic* $\neq 2$.

- *The radical of* J *is the maximal nilpotent ideal, and the quotient* $\mathrm{J}/\mathcal{R}ad(\mathrm{J})$ *is semisimple.*
- *An algebra is semisimple iff it is a finite direct sum of simple ideals. In this case, it has a unit element, and the simple decomposition is unique: the simple summands are the minimal ideals.*
- *Every simple algebra is central-simple over its center, which is a field.*
- *An algebra is central-simple over* Φ *iff it is isomorphic to exactly one of:*

DIVISION TYPE: *a finite-dimensional central Jordan division algebra over* Φ;

QUADRATIC TYPE: $\mathcal{J}ord(Q, c)$ *for an isotropic nondegenerate quadratic form with basepoint of finite dimension* ≥ 3 *over* Φ;

HERMITIAN TYPE: $\mathcal{H}_n(\mathrm{D}, *, \Gamma)$ *for* $n \geq 3$ *and* $(\mathrm{D}, *)$, *which is isomorphic to one of:*

ORTHOGONAL TYPE: $\mathcal{H}_n(\Delta, *, \Gamma)$ *for a finite-dimensional central associative division algebra* Δ *with involution over* Φ;

EXCHANGE TYPE: $\mathcal{H}_n(\mathcal{E}x(\Delta)) \cong \mathcal{M}_n(\Delta)^+$ *for a finite-dimensional central associative division algebra* Δ *over* Φ *with exchange involution* $\mathcal{E}(x, y) = (y, x)$ *on* $\mathrm{D} = \Delta \boxplus \Delta$;

SYMPLECTIC TYPE: $\mathcal{H}_n(\mathcal{Q}(\Phi))$ *for* $\mathcal{Q}(\Phi)$ *the split quaternion algebra over* Φ *with the standard involution*;

ALBERT TYPE: $\mathcal{J}ord(N, c) = \mathcal{H}_3(\mathcal{O}, \Gamma)$ *of dimension* 27 *for* $\mathcal{O}$ *an octonion algebra over* Φ *with standard involution* (*only for* $n = 3$).

Once more, the only exceptional algebras in the list are the 27-dimensional Albert algebras and, possibly, some exceptional division algebras, and the listed types of simple algebras are complete and non-overlapping.

At this stage of development there was no way to classify the division algebras, especially to decide whether there were any exceptional ones which were not Albert algebras determined by anisotropic cubic forms. In fact, to this very day *there is no general classification of all finite-dimensional associative division algebras*: there is a general construction (crossed products) which yields all the division algebras over many important fields (including all algebraic number fields), but in 1972 Amitsur gave the first construction of a non–crossed–product division algebra, and there is as yet no general characterization of these.

4. Jordan Algebras in the Classical Age

Artin–Wedderburn–Jacobson Structure Theorem

In associative theory there are many different radicals designed to remove different sorts of pathology and create different sorts of "niceness." The more difficult a "pathology" is to remove, the larger the corresponding radical will have to be. Nondegeneracy, the absence of *trivial elements,* is the useful Jordan version of the associative concept of semiprimeness, the absence of *trivial ideals* $\mathrm{BB} = \mathbf{0}$. For associative algebras, trivial elements z are the same as trivial ideals $\mathrm{B} = \widehat{\mathrm{A}}z\widehat{\mathrm{A}}$, since $z\widehat{\mathrm{A}}z = \mathbf{0} \iff \mathrm{BB} = (\widehat{\mathrm{A}}z\widehat{\mathrm{A}})(\widehat{\mathrm{A}}z\widehat{\mathrm{A}}) = \mathbf{0}$. A major difficulty in Jordan theory is that there is no convenient characterization of the Jordan ideal generated by a single element z. Because element-conditions are much easier to work with than ideal-conditions, the element-condition of nondegeneracy has proven much more useful than semiprimeness.

Inner ideals were first introduced by David M. Topping in his 1965 A.M.S. Memoir on Jordan algebras of self-adjoint operators; he called them *quadratic ideals*, and explicitly motivated them as analogues of one-sided ideals in associative operator algebras. Jacobson was quick to realize the significance of this concept, and in 1966 used it to define artinian Jordan algebras in analogy with artinian associative algebras, and to obtain for them a beautiful Artin–Wedderburn Structure Theorem. Later on Jacobson proposed the terms "inner" and "outer ideal," which won immediate acceptance. A Jordan algebra J is **artinian** if it has minimum condition on inner ideals, equivalently if it has the descending chain condition (d.c.c.) on inner ideals.

Artin–Wedderburn–Jacobson Structure Theorem: *Consider Jordan algebras* J *with minimum condition on inner ideals over a ring of scalars* Φ *containing* $\frac{1}{2}$.

• J *is nondegenerate with minimum condition iff it is a finite direct sum of ideals which are simple nondegenerate with minimum condition*; *in this case* J *has a unit, and the decomposition into simple summands is unique.*

• J *is simple nondegenerate with minimum condition iff it is isomorphic to one of the following*:

DIVISION TYPE: *a Jordan division algebra;*

QUADRATIC TYPE: *a quadratic factor* $\mathcal{J}ord(Q, c)$ *determined by a nondegenerate quadratic form* Q *with basepoint* c *over a field* Ω, *such that* Q *is not split of dimension* 2 *and has no infinite-dimensional totally isotropic subspaces;*

HERMITIAN TYPE: *an algebra* $\mathcal{H}(\mathrm{A}, *)$ *for a* $*$*-simple artinian associative algebra* A;

ALBERT TYPE: *an exceptional Albert algebra* $\mathcal{J}ord(N, c)$ *of dimension* 27 *over a field* Ω, *given by a Jordan cubic form* N.

In more detail, there are three algebras of Hermitian Type, the standard Jordan matrix algebras coming from the three standard types of involutions on simple artinian algebras:

EXCHANGE TYPE: $\mathcal{M}_n(\Delta)^+$ *for an associative division ring* Δ(*when* A *is* $*$*-simple but not simple,* $\mathrm{A} = \mathcal{E}x(\mathrm{B}) = \mathrm{B} \boxplus \mathrm{B}^{op}$ *under the exchange involution for a simple artinian algebra* $\mathrm{B} = \mathcal{M}_n(\Delta)$);

ORTHOGONAL TYPE: $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division ring* Δ *with involution*
(*when* $\mathrm{A} = \mathcal{M}_n(\Delta)$ *simple artinian with involution of orthogonal type*);

SYMPLECTIC TYPE: $\mathcal{H}_n(\mathrm{Q}, \Gamma)$ *for a quaternion algebra* Q *over a field* Ω *with standard involution*
(*when* ($\mathrm{A} = \mathcal{M}_n(\mathrm{Q})$ *simple artinian with involution of symplectic type*).

Past dimension 2 the nondegenerate Quadratic Types are always central-simple and have at most two orthogonal idempotents, but any infinite-dimensional *totally-isotropic* subspace B (where every vector is isotropic, $Q(B) = 0$) is an inner ideal as are all its subspaces, so there is no d.c.c on inner ideals. Loos showed that such a J always has d.c.c. on *principal* inner ideals, so it just barely misses being artinian. These poor $\mathcal{J}ord(Q, c)$'s are left outside while their siblings party inside with the Artinian simple algebras. The final classical formulation in the next section will revise the entrance requirements, allowing these to join the party too.

The question of the structure of Jordan division algebras remained open: since they had no proper idempotents $e \neq 0, 1$ and no proper inner ideals, the classical techniques were powerless to make a dent in their structure. The nature of the radical also remained open. From the associative theory one expected the nondegenerate radical to be nilpotent in algebras with minimum condition, but a proof seemed exasperatingly elusive.

The Final Classical Formulation: Capacity

The pinnacle of the classical structure theory in Jacobson's 1983 Arkansas Lecture Notes, where he reformulated his structure theory in terms of algebras with capacity. This proved serendipitous, for it was precisely the algebras with capacity, not the (slightly more restrictive) algebras with minimum condition, that arose naturally in Zel'manov's study of arbitrary infinite-dimensional algebras. The key to this approach lies in *division idempotents*. An element e of a Jordan algebra J is called an **idempotent** if $e^2 = e$ (then all its powers are equal to itself, hence the name meaning"same-powered"). In this case the principal inner ideal $U_e\mathrm{J}$ forms a unital subalgebra. A **division idempotent** is one such that this subalgebra is a Jordan division algebra. Two idempotents e, f in J are **orthogonal**, written $e \perp f$, if $e \bullet f = 0$, in which case their sum $e+f$ is again idempotent; an **orthogonal family** $\{e_\alpha\}$ is a family of mutually orthogonal idempotents ($e_\alpha \perp e_\beta$ for all $\alpha \neq \beta$). A finite orthogonal family is **supplementary** if the idempotents sum to the unit, $\sum_{i=1}^n e_i = 1$. Two orthogonal idempotents e_i, e_j in a Jordan algebra are **connected** if there is a *connecting element* $u_{ij} \in U_{e_i,e_j}\mathrm{J}$ which is invertible in the subalgebra $U_{e_i+e_j}\mathrm{J}$. If the element u_{ij} can be chosen such that $u_{ij}^2 = e_i + e_j$, then we say that u_{ij} is a *strongly connecting element* and e_i, e_j are **strongly connected**. A Jordan algebra has **capacity** n if it has a unit 1 which can be written as a finite sum of n orthogonal division idempotents: $1 = e_1 + \cdots + e_n$; it has **connected capacity** if each pair e_i, e_j for $i \neq j$ is connected. It has **finite capacity** (or simply **capacity**) if it has capacity n for some n.

Classical Structure Theorem *Consider Jordan algebras over a ring of scalars containing $\frac{1}{2}$.*

• *A nondegenerate Jordan algebra with minimum condition on inner ideals has finite capacity.*

• *A Jordan algebra is nondegenerate with finite capacity iff it is a finite direct sum of algebras with finite connected capacity.*

• *A Jordan algebra is nondegenerate with finite connected capacity iff it is simple, which happens if it is either of*

DIVISION TYPE: *a Jordan division algebra;*

QUADRATIC TYPE: *a quadratic factor $\mathcal{J}ord(Q, c)$ determined by a nondegenerate quadratic form Q with basepoint c over a field Ω (not split of dimension 2);*

ALBERT TYPE: *an exceptional Albert algebra $\mathcal{J}ord(N, c)$ of dimension 27 over a field Ω, determined by a Jordan cubic form N.*

HERMITIAN TYPE: *an algebra $\mathcal{H}(\mathrm{A}, *)$ for a $*$-simple artinian associative algebra A;*

In more detail, the algebras of Hermitian Type are twisted Jordan matrix algebras:

EXCHANGE TYPE: $\mathcal{M}_n(\Delta)^+$ *for an associative division ring Δ (when A is $*$-simple but not simple, $\mathrm{A} = \mathcal{E}x(\mathrm{B})$ with exchange involution for a simple artinian algebra $\mathrm{B} = \mathcal{M}_n(\Delta)$);*

ORTHOGONAL TYPE: $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division ring Δ with involution (when $\mathrm{A} = \mathcal{M}_n(\Delta)$ simple artinian with involution of orthogonal type);*

SYMPLECTIC TYPE: $\mathcal{H}_n(\mathrm{Q}, \Gamma)$ *for a quaternion algebra Q over a field Ω with standard involution (when $\mathrm{A} = \mathcal{M}_n(\mathrm{Q})$ simple artinian with involution of symplectic type).*

In Jordan theory the d.c.c. artificially excludes the quadratic factors with infinite-dimensional totally isotropic subspaces. Zel'manov's structure theory shows that, in contrast, imposing *finite capacity* is a natural act: finite capacity grows *automatically* out of the finite degree of any non-vanishing s-identity in a prime Jordan algebra.

The classical structure theory of nondegenerate algebras with capacity proceeds by finding as many orthogonal division idempotents as possible, using their Peirce decompositions to refine the structure. Peirce decompositions are also the key tool in the classical structure theory of associative rings: the *Wedderburn Coordinatization Theorem* says that if an associative algebra has a supplementary family of $n \times n$ *associative matrix units*, then it is itself a matrix algebra, $\mathrm{A} \cong \mathcal{M}_n(\mathrm{D})$ coordinatized by $\mathrm{D} = \mathrm{A}_{11}$. Jacobson found an important analogue for Jordan algebras: a Jordan algebra with enough Jordan matrix units is a Jordan matrix algebra $\mathcal{H}_n(\mathrm{D}, *)$ coordinatized by $\mathrm{D} = \mathrm{J}_{12}$ (one needs $n \geq 3$ in order to recover the product in D). Often in mathematics, in situations with low degrees of freedom we may have many "sporadic" objects, but once we get enough degrees of freedom to maneuver we reach a very stable situation. A good example is projective geometry. Projective n-spaces for $n = 1, 2$ (lines and planes) are bewilderingly varied; the *Moufang plane* with octonion coordinates satisfies the Little but not the Big Desargues' Axiom. But as soon as you get past 2, projective n-spaces for $n \geq 3$ automatically satisfy Desargues's Axiom. The situation is the same in Jordan algebras: degrees 1 and 2 are complicated, but once we reach degree $n = 3$ the algebras are Jordan matrix algebras coordinatized by alternative algebras, and once $n \geq 4$ by associative algebras. Note that no "niceness" conditions such as nondegeneracy are needed here: the result is a completely general structural result.

Jacobson Coordinatization Theorem: (1) *If a Jordan algebra* J *has a supplementary family* $1 = e_1 + \cdots + e_n$ *of* $n \geq 3$ *for orthogonal idempotents* e_i strongly connected *by* u_{ij}, *then* J *is isomorphic to a Jordan matrix algebra* $\mathcal{H}_n(\mathrm{D}, *)$ *under an isomorphism*

$$\mathrm{J} \to \mathcal{H}_n(\mathrm{D}, *) \quad \textit{via} \quad e_i \mapsto E_{ii}, \quad u_{ij} \mapsto E_{ij} + E_{ji}.$$

The coordinate $*$*-algebra* D *with unit and involution is given by* $\mathrm{D} := \mathrm{J}_{12} = U_{e_1,e_2}\mathrm{J},\ 1 := u_{12},\ x^* :=$

(2) *A Jordan algebra whose unit is a sum of* $n \geq 3$ *orthogonal* connected *idempotents is isomorphic to a twisted Jordan matrix algebra: there is an isotope* $\mathrm{J}_{[u]}$ *which is strongly connected,* $\mathrm{J}_{[u]} \cong \mathcal{H}_n(\mathrm{D}, *)$, *and* $\mathrm{J} = (\mathrm{J}_{[u]})_{[u^{-2}]} \cong \mathcal{H}_n(\mathrm{D}, *)_{[\Gamma]} \cong \mathcal{H}_n(\mathrm{D}, *, \Gamma)$.

The Coordinates

Now we must find what the possible coordinate algebras are for an algebra with capacity. Here $U_{e_1}\mathrm{J} \cong \mathcal{H}(\mathrm{D}, *)E_{11}$ must be a division algebra, and D must be nondegenerate if J is. These algebras are completely described by a theorem due to I.N. Herstein, Erwin Kleinfeld, and J. Marshall Osborn. Note that again there are no explicit simplicity or finiteness conditions imposed, only nondegeneracy; nevertheless, the only possibilities are associative division rings and *finite-dimensional* composition algebras.

Herstein–Kleinfeld–Osborn Theorem: *A nondegenerate alternative* $*$*-algebra has all its nonzero hermitian elements invertible and nuclear iff it is isomorphic to one of*:

Exchange Type: *the direct sum* $\Delta \boxplus \Delta^{op}$ *of an associative division algebra* Δ *and its opposite, under the exchange involution*;

Division Type: *an associative division algebra* Δ *with involution*:

Split Quaternion Type: *a split quaternion algebra of dimension* 4 *over its center* Ω *with standard involution; equivalently,* 2×2 *matrices* $\mathcal{M}_2(\Omega)$ *under the symplectic involution* $x^{sp} := sx^{tr}s^{-1}$ *for symplectic* $s := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$;

Octonion Type: *an octonion algebra* $\mathcal{O}$ *of dimension* 8 *over its center* Ω *with standard involution.*

Once we have division idempotents, we can assemble them carefully into a capacity.

Capacity Theorem: (1) *A nondegenerate Jordan algebra with minimum condition on inner ideals has a finite capacity.*
(2) *A nondegenerate algebra with capacity splits into the direct sum of a finite number of nondegenerate ideals having connected capacity.*
(3) *Any nondegenerate algebra with connected capacity is simple.*
(4) Capacity 1 Theorem: *A Jordan algebra has capacity* 1 *iff it is a division algebra.*
(5) Osborn's Capacity Two Theorem: *A Jordan algebra is nondegenerate with connected capacity* 2 *iff it is isomorphic to one of*:

FULL 2×2 $\mathcal{M}_2(\Delta)^+ \cong \mathcal{H}_2(\mathcal{E}x(\Delta))$ *for a noncommutative associative division algebra* Δ;
HERMITIAN 2×2 $\mathcal{H}_2(\Delta, \Gamma)$ *for an associative division algebra* Δ *with non-central involution*;
QUADRATIC FACTOR $\mathcal{J}ord(Q, c)$ *for a nondegenerate isotropic quadratic form* Q *with basepoint over a field* Ω.

(6) Capacity ≥ 3 Theorem: *A Jordan algebra is nondegenerate with connected capacity* $n \geq 3$ *iff it is isomorphic to one of the following algebras of* $n \times n$ *hermitian matrices*:

$\mathcal{H}_n(\mathcal{E}x(\Delta)) \cong \mathcal{M}_n(\Delta)^+$ *for an associative division algebra* Δ;
$\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division* $*$*-algebra* Δ;
$\mathcal{H}_n(\mathcal{Q}, \Gamma)$ *for a quaternion algebra* $\mathcal{Q}$ *over a field*;
$\mathcal{H}_3(\mathcal{O}, \Gamma)$ *for an octonion algebra* $\mathcal{O}$ *over a field.*

Crudely put, the reason that the nice Jordan algebras of degree $n \geq 3$ are what they are is because Jordan algebras are naturally coordinatized by alternative algebras, and the only nice alternative algebras are associative or octonion algebras.

5. The Russian Revolution: 1977–1983

The Lull Before the Storm

At the end of the year 1977 the classical theory was essentially complete, but held little promise for a general structure theory. One indication that infinite-dimensional simple exceptional algebras are inherently of "degree 3" was the result that a simple algebra with more than three orthogonal idempotents is necessarily special. The classical methods relied so unavoidably on finiteness or idempotents that there seemed no point of attack on the structure of infinite-dimensional algebras: they offered no hope for a Life After Idempotents. In particular, the following Frequently Asked Questions on the structure of general Jordan algebras seemed completely intractable:

(FAQ1) *Is the nondegenerate radical nilpotent in the presence of the minimum condition on inner ideals? Does every finite set of trivial elements in a Jordan algebra generate a nilpotent subalgebra?*

(FAQ2) *Do there exist simple exceptional algebras which are not Albert algebras of disappointing dimension* 27 *over their center?*

(FAQ3) *Do there exist special algebras which are simple or division algebras but are not of the classical types* $\mathcal{J}ord(Q, c)$ *or* $\mathcal{H}(\mathrm{A}, *)$?

(FAQ4) *Can one develop a theory of Jordan* PI-algebras (*those strictly satisfying a* polynomial identity)?

(FAQ5) *Does the free Jordan algebra on three or more generators have trivial elements, or is it a domain imbeddable in a division algebra* (*which would be exceptional yet infinite-dimensional*)?

Yet within the next six years all of these FAQs became settled FAQTs.

The First Tremors

The first warnings of the imminent eruption reached the West in 1978. Rumors from visitors to Novosibirsk and a brief mention at an Oberwolfach conference attended only by associative ring theorists, claimed that Arkady M. Slin'ko and Efim I. Zel'manov had in 1977 settled the first part of FAQ1: the radical is nilpotent when $\mathbf{J}$ has maximum or minimum condition. A crucial role was played by the concept of the **Zel'manov annihilator** of a set X in a linear Jordan algebra $\mathbf{J}$, $\mathcal{Z}ann_{\mathbf{J}}(X) = \{z \in \mathbf{J} \mid \{z, X, \widehat{\mathbf{J}}\} = \mathbf{0}\}$. This annihilator is always *inner* for any set X, and is an *ideal* if X is an ideal. Annihilation is an order-reversing operation (the bigger a set the smaller its annihilator), so a decreasing chain of inner ideals has an increasing chain of annihilator inner ideals, and vice versa. He showed that in any Jordan algebra, a finite set of trivial elements generates a nilpotent subalgebra. Here the methods used were extremely involved and "Lie-theoretic," based on the notion of "thin sandwiches" (the Lie analogue of trivial elements), developed originally by A.I. Kostrikin to attack the Burnside Problem and extended by Zel'manov in his Fields–medal–winning conquest of that problem.

The Main Quake

In 1979 Zel'manov flabbergasted the Jordan community by proving that there were no new exceptional algebras in infinite dimensions, settling once and for all FAQ2 which had motivated the original investigation of Jordan algebras by physicists in the 1930s, and showing that there is no way to avoid an invisible associative structure behind quantum mechanics.

Zelmanov's Exceptional Theorem: *There are no simple exceptional Jordan algebras but the Albert algebras: any simple exceptional Jordan algebra is an Albert algebra of dimension* 27 *over its center. Indeed, any prime exceptional Jordan algebra is a form of an Albert algebra: its central closure is a simple Albert algebra.*

Equally flabbergasting was his complete classification of Jordan division algebras: there was nothing new under the sun in this region either, answering the division algebra part of (FAQ3).

Zel'manov's Division Theorem: *The Jordan division algebras are just those of classical type*:

QUADRATIC TYPE: $\mathcal{J}ord(Q,c)$ *for an anisotropic quadratic form* Q *over a field*;
FULL ASSOCIATIVE TYPE: Δ^+ *for an associative division algebra* Δ;
HERMITIAN TYPE: $\mathcal{H}(\Delta,*)$ *for an associative division algebra* Δ *with involution* $*$;
ALBERT TYPE: $\mathcal{J}ord(N,c)$ *for an anisotropic Jordan cubic form* N *in* 27 *dimensions.*

As a coup de grâce, administered in 1983, he classified all possible simple Jordan algebras in arbitrary dimensions, settling (FAQ2) and (FAQ3).

Zel'manov's Simple Theorem: *The simple Jordan algebras are just those of classical type*:

QUADRATIC FACTOR TYPE: $\mathcal{J}ord(Q,c)$ *for a nondegenerate quadratic form* Q *over a field*;
HERMITIAN TYPE: $\mathcal{H}(\mathrm{B},*)$ *for a* $*$*-simple associative algebra* B *with involution* $*$,
(*hence either* A^+ *for a simple* A, *or* $\mathcal{H}(\mathrm{A},*)$ *for a simple* A *with involution*);
ALBERT TYPE: $\mathcal{J}ord(N,c)$ *for a Jordan cubic form* N *in* 27 *dimensions.*

As if this weren't enough, he actually classified the prime algebras (those with no orthogonal ideals).

Zel'manov's Prime Theorem: *The prime nondegenerate Jordan algebras are precisely*:

QUADRATIC FACTOR FORMS: *special algebras with central closure a simple quadratic factor* $\mathcal{J}ord(Q, c)$ (Q *a nondegenerate quadratic form with basepoint over a field*);

HERMITIAN FORMS*: special algebras* J *of hermitian elements squeezed between two full hermitian algebras,* $\mathcal{H}(\mathrm{A}, *) \triangleleft \mathrm{J} \subseteq \mathcal{H}(\mathcal{Q}(\mathrm{A}), *)$ *for a* $*$*-prime associative algebra* A *with involution* $*$ *and its Martindale ring of symmetric quotients* $\mathcal{Q}(\mathrm{A})$;

ALBERT FORMS: *exceptional algebras with central closure a simple Albert algebra* $\mathcal{J}ord(N, c)$ (N *a Jordan cubic form in* 27 *dimensions*).

This can be considered the Mother of All Classification Theorems. Notice the final tripartite division of the Jordan landscape into Quadratic, Hermitian, and Albert Types. The original Jordan–von Neumann–Wigner classification considered the Hermitian and Albert types together because they were represented by hermitian matrices, but we now know that this is a misleading feature (due to the "reduced" nature of the Euclidean algebras). The final three types represent genetically different strains of Jordan algebras. They are distinguished among themselves by the sorts of identities they do or do not satisfy: Albert fails to satisfy the s-identities (Glennie's or Thedy's), Hermitian satisfies the s-identities but not Zel'manov's Eater Identity, and Quadratic satisfies s-identities as well as the Eater Identity.

The restriction to *nondegenerate* algebras is important; while all simple algebras are automatically nondegenerate, the same is not true of prime algebras. Sergei Pchelintsev was the first to construct prime special Jordan algebras which have trivial elements (and therefore *cannot* be quadratic or Hermitian forms); in his honor, such algebras are now called **Pchelintsev monsters**.

Once one had such an immensely powerful tool as this theorem, it could be used to bludgeon most FAQs into submission. We first start with FAQ4.

Zel'manov's PI Theorem: *Each nonzero ideal of a nondegenerate Jordan PI-algebra has nonzero intersection with the center* (*so if the center is a field, the algebra is simple*). *The central closure of a prime nondegenerate PI-algebra is central-simple. Any primitive PI-algebra is simple. Each simple PI-algebra is either finite-dimensional, or a quadratic factor* $\mathcal{J}ord(Q, c)$ *over its center.*

Next, Ivan Shestakov settled part of (FAQ4), a question raised by Jacobson in analogy with the associative PI theory.

Shestakov's PI Theorem: *If* J *is a special Jordan PI-algebra, then its universal special envelope is locally finite* (*so if* J *is finitely generated, its envelope is an associative PI-algebra*).

The example of an infinite-dimensional simple $\mathcal{J}ord(Q, c)$, which as a degree 2 algebra satisfies lots of polynomial identities yet has the infinite-dimensional simple Clifford algebra $\mathcal{C}liff(Q, c)$ as its universal special envelope, shows that we cannot expect the envelope to be globally finite.

The structure theory has surprising consequences for the free algebra, settling FAQ5.

Free Consequences Theorem: *The free Jordan algebra on three or more generators has trivial elements.*

In fact, Yuri Medvedev explicitly exhibited trivial elements.

These results also essentially answer FAQ6: just as Alfsen–Shultz–Størmer indicated, the Glennie Identity G_8 is the "only" s-identity, or put another way, all s-identities which survive the Albert algebra are "equivalent."

i-Speciality Theorem: *All s-identities do equally well at separating special algebras from exceptional algebras: a semiprimitive algebra will be special as soon as it satisfies any particular identity* ($such\ as\ G_8, G_9, Sh_8, Sh_9,\ or\ T_{11}$), *which does not vanish on the Albert algebra.*

Zel'manov has gone on to classify all simple and prime Jordan triple systems and Jordan pairs in a similar manner; once more all the resulting objects are of classical type. With co-workers he has also classified all finite-dimensional Jordan superalgebras in characteristic not 2. The study of infinite-dimensional Jordan superalgebras, however, is not yet finished, and already provides many new simple objects which do not exist in finite dimensions. We can expect in the future many more exciting new results about strange Jordan creatures, who seem so widely dispersed across the mathematical landscape.