On the interaction between partial projective representations of groups and twisted partial actions

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Let K be field, K^* group of inv. ele-ts, $\mathrm{Mat}_n K$ mult. semigrp. of $\forall$ $n \times n$-matrices over K.

Define equivalence λ on $\mathrm{Mat}_n K$: for $A, B \in \mathrm{Mat}_n K$

$$A \lambda B \iff \quad A = cB, \ \text{ some } \ c \in K^*.$$

Remark: λ is congruence of $\mathrm{Mat}_n K$.

Define

$$\mathrm{PMat}_n K = \mathrm{Mat}_n K / \lambda$$

semigroup of projective $n \times n$-matrices.

Definition 1 *<u>K-semigroup</u> is a semigrp T with* 0 *and with*

$$K \times T \to T$$

such that

$$\alpha(\beta x) = (\alpha\beta)x, \alpha(xy) = (\alpha x)y = x(\alpha y),$$

$$1x = x, 0x = 0,$$

$\forall\ \alpha, \beta \in K, x, y \in T$. *Call T <u>K-cancellative</u> if:*

$$\alpha x = \beta x \implies \alpha = \beta$$

$\forall\ \alpha, \beta \in K, 0 \neq x \in T$.

<u>Observe:</u> $\mathrm{Mat}_n K$ is K-cancellative, $\mathrm{PMat}_n K$ is not.

For a K-cancel. monoid M define congruence λ as above: for $x, y \in M$

$$x \lambda y \iff x = \alpha y \text{ some } \alpha \in K^*.$$

Set

$$\mathrm{Proj}\, M = M/\lambda.$$

Let ξ the natural $\xi : M \to \mathrm{Proj}\, M$.

Let G grp., S mond. A map $\varphi : G \to S$ is *a (unital) partial homomorphism* if $\forall x, y \in G$

$$\begin{aligned} \varphi(1) = 1, \\ \varphi(x^{-1})\varphi(x)\varphi(y) &= \varphi(x^{-1})\varphi(xy), \\ \varphi(x)\varphi(y)\varphi(y^{-1}) &= \varphi(xy)\varphi(y^{-1}). \end{aligned}$$

Definition 2 *Let M K-cancel. mond., G grp. A partial projective representation of G in M is*

$$\Gamma : G \to M$$

such that

$$\xi\Gamma : G \to \operatorname{Proj} M$$

is a partial homomorphism.

Theorem 1 *Let M a K-cancel. mond. If $\Gamma : G \to M$ is a par. proj. repr. of G then there is a (unique) partially defined map $\sigma : G \times G \to K^*$ such that*

$$\operatorname{dom} \sigma = \{(x, y) \mid \Gamma(x)\Gamma(y) \neq 0\}$$

and $\forall\, (x, y) \in \operatorname{dom} \sigma$

$$\begin{aligned} \Gamma(x^{-1})\Gamma(x)\Gamma(y) &= \Gamma(x^{-1})\Gamma(xy)\sigma(x, y), \\ \Gamma(x)\Gamma(y)\Gamma(y^{-1}) &= \Gamma(xy)\Gamma(y^{-1})\sigma(x, y). \end{aligned}$$

Definition 3 *A partial action* θ *of* G *on semigrp* S *consists of* $S_x \lhd S$ $(x \in G)$ *and iso-s* $\theta_x : S_{x^{-1}} \to S_x$ *such that* $\forall\, x, y \in G$:

(i) $S_1 = S$, $\theta_1 = \mathrm{Id}_S$;

(ii) $\theta_x(S_{x^{-1}} \cap S_y) = S_x \cap S_{xy}$;

(iii) $\theta_x \circ \theta_y(a) = \theta_{xy}(a)\ \forall a \in S_{y^{-1}} \cap S_{y^{-1}x^{-1}}$.

Definition 4 *Let* S *a* K*-mond.,* θ *a par. action of* G *on* S *such that* $\forall x \in G\ \exists\, 1_x \in S_x$ *and* $\forall \theta_x$ *is* K*-map. A* K*-valued twisting of* θ *is a function* $\sigma : G \times G \to K$:

(i) $\sigma(x, y) = 0 \Longleftrightarrow S_x \cap S_{xy} = 0\ \ (x, y \in G)$;

(ii) $\sigma(x, 1) = \sigma(1, x) = 1\ \ \forall x \in G$;

(iii) $S_x \cap S_{xy} \cap S_{xyz} \neq 0 \Longrightarrow$

$$\sigma(x, y)\sigma(xy, z) = \sigma(y, z)\sigma(x, yz)$$

$x, y, z \in G$.

Given (θ, σ) of G on S, define the crossed product $S *_{\theta,\sigma} G$ as follows. Let

$$L = \{a\, u_x : \ a \in S_x, x \in G\}.$$

Multiplication on L given by

$$a\, u_x \cdot b\, u_y = \theta_x({\theta_x}^{-1}(a)b)\sigma(x, y)u_{xy},$$

which is associative. Set

$$S *_{\theta,\sigma} G = L/I,$$

where

$$I = \{0\, u_x : \ x \in G\}.$$

Observe:

S is K-cancelative $\implies$ $S *_{\theta,\sigma} G$ is K -cancelative.

Let G grp., K field, M K-cancel. mond. and $\Gamma : G \to M$ a par. proj. repr. with factor set $\sigma : G \times G \to K$. Set

$$e_x = \begin{cases} \Gamma(x)\Gamma(x^{-1})\sigma(x^{-1}, x)^{-1} & \text{if } \Gamma(x) \neq 0, \\ 0 & \text{if } \Gamma(x) = 0. \end{cases}$$

Then the e_x's are pairwise commuting idempotents.

Let

$$\Gamma(G) = \langle \alpha\Gamma(x) \mid \alpha \in K, x \in G \rangle \subseteq M,$$

$$S = \langle \alpha e_x, \alpha \in K, x \in G \rangle \subseteq \Gamma(G).$$

Set $S_x = Se_x$.

Recall

Theorem 2 [2] *The maps $\theta_x : S_{x^{-1}} \to S_x$ $(x \in G)$*

$$\theta_x(a) = \begin{cases} \Gamma(x)a\Gamma(x^{-1})\sigma(x^{-1},x)^{-1} & \text{if } S_{x^{-1}} \neq 0, \\ 0 & \text{if } S_{x^{-1}} = 0, \end{cases}$$

form a par. action $\theta = \theta^{\Gamma}$ of G on S, the factor set σ is twisting for θ and

$$\psi : S *_{\theta,\sigma} G \ni au_x \mapsto a\Gamma(x) \in \Gamma(G)$$

is an epimorphism.

Let θ be a twisted par. ac. of G on a K-cancel. mond. T with twisting σ. Thus each ideal $\forall\ T_x = T1_x$. Have:

Theorem 3 [2] *The map $\Gamma_\theta : G \to T *_{\theta,\sigma} G$, defined by $\Gamma_\theta(x) = 1_x u_x$, is a proj. par. repr. whose factor set is σ.*

Theorem 2, Theorem 3 $\Longrightarrow \forall\ \Gamma : G \to M$ the following triangle is commutative:

$$\begin{array}{ccc} & G & \\ {}^{\Gamma}\swarrow & & \searrow^{\Gamma_\theta} \\ M & \xleftarrow{\ \psi\ } & S *_{\theta,\sigma} G \end{array}$$

where $\theta = \theta^\Gamma$.

Definition 5 *Let* $\Gamma : G \to M$ *and* $\Gamma' : G \to M'$ *be par. proj. repr-s. A* <u>*morphism*</u> *from* Γ *to* Γ' *is a homomorphism of* K*-monoids* $\varphi : M \to M'$ *with*

$$\begin{array}{ccc} & G & \\ {}^{\Gamma'}\swarrow & & \searrow^{\Gamma} \\ M' & \xleftarrow{\ \varphi\ } & M \end{array}$$

commutative.

<u>Observe:</u>

$$\sigma(x,y) = \sigma'(x,y) \quad \forall (x,y) \in \operatorname{dom} \sigma', \tag{1}$$

in particular $\operatorname{dom} \sigma' \subseteq \operatorname{dom} \sigma$.

Consider the category $\mathbf{Ppr}\, G$ of par. proj. repr.-s of G into K-cancel. monoids and their morphisms.

A par. proj. repr. $\Gamma : G \to M$ shall be called *adjusted* if $M = \Gamma(M)$. They (and morphisms) form a full subcategory $\mathbf{AdjPpr}\, G$.

The par. proj. repr. $\Gamma_\theta : G \to T *_{\theta,\sigma} G$ from Theorem 3 satisfies:

$$S_x\Gamma(x) \cap S_y\Gamma(y) = 0 \quad \text{for any } x, y \in G, x \neq y.$$

Such par. proj. repr-s called *strongly injective*. They (with morhisms) form a full subcategory $\mathbf{SiPpr}\, G$.

Also denote

$\mathbf{AdjSiPpr}\, G = \{$adj. str. inj. par. proj. repr-s and morphisms $\}$.

Similarly define the category $\mathbf{Pa}_G$ of the partial actions of G on K-cancel. monoids and their morphisms:

Definition 6 *Let* $\theta = \{\theta_x : T_{x^{-1}} \to T_x \ (x \in G)\}$ *and* $\theta' = \{\theta'_x : T'_{x^{-1}} \to T'_x \ (x \in G)\}$ *be par. ac.-s of* G *on* T *and* T' *respectively. A morphism from* $\theta \to \theta'$ *is a homomorphism of* K*-monoids*

$$\varphi : T \to T'$$

such that

$$\varphi(T_x) \subseteq T'_x$$

$\forall\, x \in G$ *and*

$$\begin{array}{ccc} T_{x^{-1}} & \xrightarrow{\theta_x} & T_x \\ \downarrow \varphi & & \downarrow \varphi \\ T'_{x^{-1}} & \xrightarrow{\theta'_x} & T'_x \end{array}$$

is commutative.

Also consider the category $\mathbf{TwPa}_G$ of twisted partial actions of grps on K-cancel. monoids in which the morphisms are defined as follows:

Definition 7 *A morphism of tw. par. actions*

$$\varphi : (\theta, \sigma) \to (\theta', \sigma'),$$

with $\theta = \{\theta_x : T_{x^{-1}} \to T_x \ (x \in G)\}$ *and* $\theta' = \{\theta'_x : T'_{x^{-1}} \to T'_x \ (x \in G)\}$, *is a morphism of par. ac.-s*

$$\varphi : \theta \to \theta'$$

such that $\forall$ *restriction*

$$\varphi : T_x \to T'_x \quad (x \in G)$$

is homom. of monds and

$$\varphi(T_x \cap T_{xy}) \neq 0 \Longrightarrow \sigma(x, y) = \sigma'(x, y) \quad (\forall x, y \in G).$$

A tw. par. action of G on T is called *adjusted* if T is generated by $\alpha 1_x$ ($\alpha \in K, x \in G$). They (and morphisms) form a subcategory $\mathbf{AdjTwPa}_G$.

Describe the interaction:

Theorem 4 *(i)* $\exists$ *a functor* $\mathbf{Ppr}\, G \to \mathbf{AdjTwPa}_G$ *which takes* $\Gamma \mapsto \theta^\Gamma$.

(ii) $\exists$ *a functor* $\mathbf{TwPa}_G \to \mathbf{SiPpr}\, G$ *which takes* $\theta \mapsto \Gamma_\theta$. *Moreover,* Γ_θ *is adjusted if* θ *is adjusted.*

(iii) $\forall\ \Gamma \in \mathrm{Ob}\,\mathbf{Ppr}\, G\ \exists$ *morphism*

$$\Gamma_{\theta^\Gamma} \to \Gamma$$

which is $\cong$ *if* $\Gamma \in \mathrm{Ob}\,\mathbf{AdjSiPpr}\, G$.

(iv) $\forall\ \theta \in \mathrm{Ob}\,\mathbf{TwPa}_G\ \exists$ *mono.*

$$\theta^{\Gamma_\theta} \to \theta,$$

which is $\cong$ *if* θ *is adjusted.*

(v) Functors from (i) and (ii) give

$$\mathbf{AdjSiPpr}\, G \sim \mathbf{AdjTwPa}_G.$$

(vi) The restriction

$$\mathbf{AdjPpr}\, G \to \mathbf{AdjTwPa}_G$$

is right adjoint to restriction

$$\mathbf{AdjTwPa}_G \to \mathbf{AdjPpr}\, G.$$

References

[1] M. Dokuchaev, R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, *Trans. Amer. Math. Soc.*, **357**, (5), (2005), 1931-1952.

[2] M. Dokuchaev, B. Novikov, Partial projective representations and partial actions, *J. Pure Appl. Algebra* (to appear).

[3] R. Exel, Partial actions of groups and actions of semigroups, *Proc. Amer. Math. Soc.*, **126**, (12), (1998), 3481-3494.

[4] B. V. Novikov, On projective representations of semigroups, Dopovidi AN USSR, 1979, N6, 472-475 (in Russian). [MR 80i:20041]