Functional identities and their applications to graded algebras

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Example 1

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$f = 0$ or R "very special" (its left annihilator is nonzero: $aR = 0$ with $a \neq 0$)

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$f, g : R \rightarrow R$

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Proof: Algebraic manipulations + structure theory of PI-rings

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In the context of prime rings,the **maximal** (left or right) **ring of quotients** is suitable.

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Applications! Hint: interprete $f(x,x)x = xf(x,x)$ as

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where $*$ is another (nonassociative) product on R.

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R does not satisfy a PI of degree $2d-2$ (i.e., R cannot be embedded in $M_{d-1}(F)$).

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Other important examples of d-free sets:

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M. Brešar, M. A. Chebotar, W. S. Martindale, *Functional Identities*, Birkhäuser Verlag, 2007.

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Jordan case: similar results, but less restrictions

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Bahturin-Brešar: extending a Lie superhomomorphism to the Grassman envelope makes it possible to use FI's.