Complexity and Module Varieties for Classical Lie Superalgebras

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Background

The category of finite-dimensional modules for the classical Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ which are completely reducible over $\mathfrak{g}_{\bar{0}}$ is a highest weight category (as observed by Brundan) but, unlike the blocks of (parabolic) category $\mathcal{O}$ for a semisimple Lie algebra, there are infinitely many simple modules. Projective resolutions can have infinite length. Cohomology can be non-zero in infinitely many degrees and can grow in dimension.

This motivates studying these categories cohomologically, with ideas and tools from the modular representation theory of finite groups and restricted Lie algebras.

At the third conference in this series, at Maresias in 2007, Dan Nakano talked about our work using relative cohomology and supports to measure certain combinatorial invariants, such as the defect of a classical Lie superalgebra $\mathfrak{g}$ and the atypicality of its finite-dimensional irreducible representations.

Missing was a connection of the relative support theory to the notion of complexity of a module. Also missing was a connection to the "associated varieties" of Duflo-Serganova. Those are the topics of my talk today.

Notation

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra over $\mathbb{C}$: this means that $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector space with a bracket [,] which respects the grading and satisfies graded versions of the usual Lie algebra properties; and $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G_0)$ for a connected reductive algebraic group G_0. (We do not assume $\mathfrak{g}$ is simple!)

Let $\mathcal{F}$ be the full subcategory of $\mathfrak{g}$-modules which are finite-dimensional and completely reducible over $\mathfrak{g}_{\bar{0}}$. This is an interesting category because in general it is not semisimple, and indeed blocks often contain infinitely many simple modules; it has enough projectives; and if $\mathfrak{g}$ is Type I then $\mathcal{F}$ is a highest weight category.

Self-injectivity

A key property of $\mathcal{F}$ is that it is self-injective, in the following sense:

Proposition

A module M in $\mathcal{F}$ is projective if and only if it is injective.

Complexity

Definition

Let $\mathcal{V} = \{V_t \mid t \in \mathbb{N}\} = \{V_\bullet\}$ be a sequence of finite-dimensional $\mathbb{C}$-vector spaces. The rate of growth of $\mathcal{V}$, $r(\mathcal{V})$, is the smallest positive integer c such that there exists a constant $C > 0$ with $\dim_{\mathbb{C}} V_t \leq C \cdot t^{c-1}$ for all t. If no such integer exists then $\mathcal{V}$ is said to have infinite rate of growth.

Following Alperin (1977), we make the

Definition

Let $M \in \mathcal{F}$ and $P_\bullet \twoheadrightarrow M$ be a minimal projective resolution for M. The complexity of M, $c_{\mathcal{F}}(M) = r(\{P_n \mid n = 0, 1, 2, \ldots\})$.

Projective Resolutions

Following Kumar, we have:

Proposition

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} \Lambda^{\bullet}_{sup}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}) \twoheadrightarrow \mathbb{C}$$

is a projective resolution of the trivial module in $\mathcal{F}$.

Tensoring by any module M in $\mathcal{F}$ produces a projective resolution of M. We easily deduce:

Theorem

For any module M in $\mathcal{F}$, the complexity $c_{\mathcal{F}}(M) \leq \dim_{\mathbb{C}} \mathfrak{g}_{\bar{1}}$.

Examples

1. Consider the Lie superalgebra of type $\mathfrak{q}(1) = \left\{ \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix} \right) : a, b \in \mathbb{C} \right\}$.
The projective cover $P(\mathbb{C})$ of $\mathbb{C}$ has two composition factors, both $\cong \mathbb{C}$.
Therefore the minimal projective resolution of $\mathbb{C}$ is given by

$$\cdots \to P(\mathbb{C}) \to P(\mathbb{C}) \to P(\mathbb{C}) \to \mathbb{C} \to 0.$$

Thus $c_{\mathcal{F}}(\mathbb{C}) = 1 = \operatorname{Kr} \dim \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$.

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Thus $c_{\mathcal{F}}(\mathbb{C}) = 1 = \operatorname{Kr}\dim \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$.

2. For $\mathfrak{gl}(1|1)$ the minimal projective resolution of $\mathbb{C}$ can be written

$$\cdots \to P(2 \,|\, -2) \oplus P(0 \,|\, 0) \oplus P(-2 \,|\, 2) \to P(1 \,|\, -1) \oplus P(-1 \,|\, 1) \to P(0 \,|\, 0) \to \mathbb{C} \to 0,$$

where $\dim P(\lambda \,|\, -\lambda) = 4$. Therefore, $\dim_{\mathbb{C}} P_n = 4(n+1)$ and $c_{\mathcal{F}}(\mathbb{C}) = 2 > 1 = \operatorname{Kr}\dim \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$.

Evidently the relative cohomology ring may not be large enough to measure the complexity of modules in $\mathcal{F}$.

Projective and Periodic Modules

Proposition

Let $M \in \mathcal{F}$. Then M is projective if and only if $c_{\mathcal{F}}(M) = 0$.

Necessity is obvious. Conversely, if M has a finite projective resolution $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ then since P_n is also injective, the resolution splits. It follows that M itself is projective.

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A non-projective module is periodic if it admits a periodic projective resolution. In the context of finite group representations, M is periodic $\iff c_{kG}(M) = 1$. In our setting, necessity holds but not sufficiency. For example, if $\mathfrak{g} = \mathfrak{gl}(1|1)$, the Kac module $K(0|0)$ has minimal projective resolution

$$\cdots \to P(2\,|-2) \to P(1\,|-1) \to P(0|0) \to K(0|0) \to 0.$$

Since $\dim_{\mathbb{C}} P(\lambda\,|-\lambda) = 4$, $c_{\mathcal{F}}(K(0|0)) = 1$ even though the resolution is not periodic.

Kac Modules

From now on, assume that our classical superalgebra $\mathfrak{g}$ is Type I, meaning it has a consistent $\mathbb{Z}$-grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

This includes $\mathfrak{gl}(m|n)$ and the simple Lie superalgebras of types $A(m,n),\ C(n)$, and $P(n)$.

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Put $\mathfrak{p}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{p}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. If $L_0(\lambda)$ is a simple finite-dimensional $\mathfrak{g}_0$-module, let

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} L_0(\lambda) \quad \text{and} \quad K^-(\lambda) = \mathrm{Hom}_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), L_0(\lambda))$$

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A module in $\mathcal{F}$ has a (dual) Kac filtration if it has a filtration with all nonzero subquotients isomorphic to (dual) Kac modules. A tilting module is one which admits both a Kac and a dual Kac filtration.

Support Varieties

Given $M \in \mathcal{F}$, define the $\mathfrak{g}_{\pm 1}$-support variety of M as

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \{x \in \mathfrak{g}_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

These can also be defined cohomologically, as the maximal ideal spectrum of the quotient of the cohomology ring of $\mathfrak{g}_{\pm 1}$ modulo the annihilator of $\operatorname{Ext}^{\bullet}(M, M)$. The cohomological and rank variety definitions agree because $\mathfrak{g}_{\pm 1}$ are abelian Lie superalgebras.

The operations $\mathcal{V}_{\mathfrak{g}_{\pm 1}}$ satisfy the usual properties of a support variety theory, such as the tensor product rule and detecting $\mathfrak{g}_{\pm 1}$-projectivity.

Theorem

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra, and $M \in \mathcal{F}$.

- *M has a Kac filtration* $\iff \mathcal{V}_{\mathfrak{g}_{-1}}(M) = 0$
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Corollary

M is a tilting module if and only if $\mathcal{V}_{\mathfrak{g}_1}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}(M) = 0$.

Projectivity Test

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Let $\mathfrak{g}$ be a Type I classical Lie superalgebra, and $M \in \mathcal{F}$. Then M is projective if and only if $\mathcal{V}_{\mathfrak{g}_1}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}(M) = 0$.

The proof uses a tensor-product "trick" due to Cline-Parshall-Scott.

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Corollary

M is a tilting module if and only if M is projective.

This result is implicit in the work of Jon Brundan (2004).

Connection with Duflo-Serganova Associated Varieties

Consider the following subvariety of $\mathfrak{g}_{\bar{1}}$:

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} \mid [x, x] = 0\}.$$

For $M \in \mathcal{F}$, Duflo and Serganova (2005) define the associated variety

$$\mathcal{X}_M = \{x \in \mathcal{X} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

In Type I, $\mathfrak{g}_{\pm 1} \subset \mathcal{X}$ since $\mathfrak{g}_{\pm 1}$ is abelian. It follows that $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \mathcal{X}_M \cap \mathfrak{g}_{\pm 1}$.

Using our previous theorem, we can recover the following theorem of Duflo-Serganova (in Type I):

Theorem (Duflo-Serganova)

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra, and $M \in \mathcal{F}$. Then M is projective if and only if $\mathcal{X}_M = 0$.

Our proof uses the fact that a projective module M is $\mathbb{Z}$-graded. If there were a nonzero element in $\mathcal{X}_M$, a limiting process produces a nonzero element of either $\mathcal{X}_M \cap \mathfrak{g}_1 = \mathcal{V}_{\mathfrak{g}_1}(M)$ or $\mathcal{X}_M \cap \mathfrak{g}_{-1} = \mathcal{V}_{\mathfrak{g}_{-1}}(M)$, contradicting the projectivity of M.

Refinement of the Projectivity Test

Recall the connected reductive algebraic group G_0 with $\mathrm{Lie}(G_0) = \mathfrak{g}_{\bar{0}}$. Let $\{x_i \mid i \in I\}$ (resp. $\{y_j \mid j \in J\}$) be a set of orbit representatives for the minimal orbits[1] of the action of G_0 on $\mathfrak{g}_1$ (resp. on $\mathfrak{g}_{-1}$).

Theorem

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra and $M \in \mathcal{F}$. Then M is projective if and only if M is projective on restriction to $U(\langle x_i \rangle)$ for all $i \in I$ and to $U(\langle y_j \rangle)$ for all $j \in J$.

A similar test detects the existence of a (dual) Kac filtration on M.

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The point is that, since M is a $\mathfrak{g}_{\bar{0}}$-module, $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ are closed G_0-stable varieties.

[1] By minimal orbit, we mean minimal non-zero orbit with respect to the partial order on orbits given by containment in closures.

Example: $\mathfrak{gl}(m|n)$

Consider $\mathfrak{g} = \mathfrak{gl}(m|n)$. The action of $G_0 \cong GL(m) \times GL(n)$ on $\mathfrak{g}_1$ is given by $(A, B) \cdot x = AxB^{-1}$ for $A \in GL(m),\ B \in GL(n),\ x \in \mathfrak{g}_1$. The orbits are

$$(\mathfrak{g}_1)_r = \{\, x \in \mathfrak{g}_1 \mid \operatorname{rank}(x) = r \,\}$$

for $0 \leq r \leq \min(m, n)$. The closure of $(\mathfrak{g}_1)_r$ is the determinantal variety

$$\overline{(\mathfrak{g}_1)_r} = \{\, x \in \mathfrak{g}_1 \mid \operatorname{rank}(x) \leq r \,\};$$

thus $(\mathfrak{g}_1)_r \subset \overline{(\mathfrak{g}_1)_s}$ if and only if $r \leq s$. The situation for $\mathfrak{g}_{-1}$ is analogous. Thus to test for Kac filtrations, dual Kac filtrations, and projectivity, it suffices to consider a single rank one element from $\mathfrak{g}_1$ and/or $\mathfrak{g}_{-1}$.

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The same holds true for $\mathfrak{sl}(m|n),\ \mathfrak{psl}(n|n)$, and the simple Lie superalgebras of type $C(n)$ and $P(n)$.

In fact, for $\mathfrak{g} = \mathfrak{gl}(m|n)$ one can say even more. Because the orbit closures form a simple chain, it follows that for any $M \in \mathcal{F}$,

$$\mathcal{V}_{\mathfrak{g}_1}(M) = \overline{(\mathfrak{g}_1)_r} \quad \text{for some } 0 \leq r \leq \min(m, n),$$

and similarly for $\mathcal{V}_{\mathfrak{g}_{-1}}(M)$. In particular, $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ is an irreducible variety.

When $M = L(\lambda)$ is an irreducible module, it follows from the work of Duflo-Serganova that

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(L(\lambda)) = \overline{(\mathfrak{g}_{\pm 1})_r} \quad \text{where } r = \operatorname{atyp}(\lambda).$$

Further Questions

1. By the self-injectivity result, given a simple module $S \in \mathcal{F}$, its projective cover is the injective envelope of some simple $T \in \mathcal{F}$. What is the relationship between S and T? (For finite-dimensional cocommutative Hopf algebras, the analogous answer is known.)

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2. Can one construct a "support variety" for a module $M \in \mathcal{F}$ whose dimension equals the complexity $c_{\mathcal{F}}(M)$? The $\mathfrak{gl}(1|1)$ example shows that we need something bigger than the relative support variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ defined in our earlier paper.