On the toral rank of simple Lie algebra over a field of characteristic 2

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1 Abstract

We give a short proof of the following result of S.Skryabin [1]: a finite dimensional Lie algebra over a field of characteristic 2 and absolute toral rank one is solvable.

2 Introduction.

In this note we give a short proof of the following theorem of S.Skryabin [1].

Theorem 1 *Finite dimensional simple Lie algebra over a field of characteristic 2 has absolute toral rank at least 2.*

Proof. Let F be an algebraically closed field of characteristic 2, let $\tilde{L}$ be a finite dimensional simple Lie algebra over F of toral rank one, let L be a 2-envelope of $\tilde{L}$. Then $L = L_0 \oplus L_1$, where L_0 is a Cartan subalgebra of L, $[L_0, L_1] \subseteq L_1$ and $[L_1, L_1] \subseteq L_0$. By the definition of absolute toral rank, L_0 contains a unique toroidal element $h = h^{[2]}$. We denote

$$\mathcal{N} = \{a \in L | \exists n : a^{[2^n]} = 0\},\ \mathcal{T} = \{t \in L | t^{[2]} = t \neq 0\}.$$

We need the following simple result.

Lemma 1 *Let $n \in \mathcal{N}$, $t, s \in \mathcal{T}$.*

For any $a \in L$, if $a + [a, n] = 0$ then $a = 0$. If $[t, s] = 0$ then $t = s$.

Note that $L_1 \subseteq \mathcal{N}$. Indeed, if $a \in L_1$, then $a^{[2]} \in L_0$ and if $a^{[2]}$ is not nil then $a^{[2]} = h+n$, where n is a nil element. In this case $[a, a^{[2]}] = a+[a, n] = 0$, hence $a = 0$ by Lemma 1.

If $H = [L_1, L_1]$ then $\tilde{L} = H \oplus L_1$ and H is not a nil-subalgebra. Indeed, if H is a nil subalgebra then by the Jacobson-Engel theorem [2] $\tilde{L}$ is nilpotent since all elements of L_1 are nil. Then $h \in H$.

Let N be a maximal nil-ideal of L_0 and $N_0 = Fh \oplus N$. If $L_0 = N_0$, then for some $a, b \in L_1$ we have $[a, b] = h + n$, $n \in N$, since $h \in [L_1, L_1]$. As N is a nil-ideal of L_0 we can suppose that $[[a, N], b], [[b, N], a] \in N$. Hence

$[a^{[2]}, b^{[2]}] = [[[a, b], b], a] = h + n + [[b, n], a] = h + m$, $m \in N$.

On the other hand $a^{[2]}, b^{[2]} \in N$ as we proved above. Hence, $[a^{[2]}, b^{[2]}] \in N$, it is proved that $L_0 \neq N_0$. Let $F\bar{e}$ be an ideal of the factor-algebra L_0/N_0 and e be some preimage of $\bar{e}$. It is clear that $e^{[2]} \in N_0$. For some $\gamma \in k$ we have $(e + \gamma h)^{[2]} \in N$. Hence we can suppose that $e^{[2]} \in N$. Denote $L_0' = \{x \in L_0 | [x, e] \in N\}$. If $L_0' = L_0$ then $Fe \oplus N$ is an ideal of L_0 with nil-radical $Fe \oplus N$, contradicting the maximality of N. Hence $L_0 = L_0' \oplus Ff$ for some $f \in L_0$. Let $X = \{a^{[2]} | a \in L_1\} \subseteq L_0$. Since $[L_1, L_1] = H$ and $L_0 = H + F\{x^{[2^i]}, i = 1, 2, ..., x \in H\}$, it follows that the $F-$space $F\{X\}$ generated by X is equal to L_0. This means that there exists $c \in L_1$ such that $c^{[2]} \notin L_0'$, hence $[c^2, e] = \alpha h + m$, where $\alpha \in F^*$, $m \in N_0$. Now we can choose $a = c$, $b = \alpha^{-1}[c, e]$ and in this case we have

$[a, b] = \alpha^{-1}[c, [c, e]] = \alpha^{-1}[c^{[2]}, e] = h + n$, $n \in N$.

Let $a^{[2^{k+1}]} = b^{[2^{s+1}]} = 0$ and define

$$x = h + a + a^{[2]} + ... + a^{[2^k]},\ y = h + b + b^{[2]} + ... + b^{[2^s]}.$$

Then $x, y \in \mathcal{T}$ and if $z = [x, y]$ then $[x, z] = z$, $[y, z] = z$. We prove that z is nil. Othewise, $z^{[2]} = p + n$, where $p \in \mathcal{T}$, $[p, n] = 0$ and $n \in \mathcal{N}$. We have

$[z^{[2]}, x] = 0 = [p, x] + [n, x]$. Hence

$[p, [p, x]] + [p, [n, x]] = [p, x] + [[p, n], x] + [n, [p, x]] = [p, x] + [n, [p, x]]$.

But n is nil, hence $[p, x] = 0$ by Lemma 1. But in this case $[z^{[2]}, z] = [x + n, z] = z + [n, z] = 0$, hence $z = 0$, by Lemma 1, since n is nil. Hence $0 = z = [x, y]$ and $x = y$ by Lemma 1, a contradiction.

Then $z^{[2^{r+1}]} = 0$ for some r. Denote $u = x + y + z + z^{[2]} + ... + z^{[2^r]}$. It is easy to see that $u^{[2]} = u$. We have $[u, x] = [y, x] + z = 0$, hence $u = x$ or $u = 0$, by Lemma 1. If $u \neq 0$, then $u = x$, and we can prove analogously that $u = y = x$, a contradiction. Hence $u = 0$.

Let $V = L_1 N$ be an $N-$submodule of L_1. Then $P = V \oplus L_0$ is a proper 2-subalgebra of L. We can suppose that $a \notin V$. Indeed, if $a \in V$ and $c \notin V.c \in L_1$, then for $\gamma \in k$ we have $f(\gamma) = [(a+\gamma c)^{[2]}, e] = [a^{[2]}, e] + \gamma[[a, c], e] + \gamma^2[c^{[2]}, e]$. If $f(\gamma) \in L'_0$ for all $\gamma \in k^*$ then $[a^{[2]}, e] \in L'_0$, contradiction. Then $f(\gamma) \notin L'_0$ for some $\gamma \in k^*$ and we can choose $a + \gamma c$ instead of a. But $a + \gamma c \notin V$.

Let us prove that $u \equiv (a + b)(mod(P))$. We have by definition,

$$z = a + b + [a, b^{[2]}] + [b, a^{[2]}] + \textstyle\sum_{i>0}([a, b^{[2^i]}] + [b, a^{[2^i]}] + z_0, \qquad (1)$$

where $z_0 \in L_0$. But

$$[a, b] = h + n,\ [a, b^{[2]}] = [h + n, b] = b + [b, n],\ [b, a^{[2]}] = a + [a, n],$$

$$[a, b^{[4]}] = [[b, n], b^{[2]}] = [b, [n, b^{[2]}]] \in V,$$

by induction, for $i > 1$,

$$[a, b^{[2^i]}] = [[n, b^{[2^{i-1}]}], b^{[2^{i-1}]}] \in V.$$

Hence, by (1), $z \in P$.

We have, $u = a + b + z + ... \equiv (a + b)(modP)$. But, if $u = 0$, it means that $a \equiv b(modP)$. By choice of a, b we have that $b = [a, e]$ and $[b, e] = [a, e^{[2]}] \in V$, since $e^{[2]} \in N$. As $[a, e] = b \equiv [b, e] \equiv 0(modP)$ we have $a \equiv 0(modP)$, a contradiction.

□

Note that S.Skryabin [1] obtained a much deeper result. He described all simple finite dimensional Lie algebras over a field F that contain some Cartan subalgebra of toral rank 1.

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References

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