# REPRESENTING IDEMPOTENTS AS A SUM OF TWO NILPOTENTS OF DEGREE FOUR

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Abstract. The freest minimal algebra $R$ over the field of rational numbers where an idempotent is a sum of two nilpotents of degree 4 is presented by $\mathbb{Q} < e, b | e^2 = e, a^4 = b^4 = 0, e = a + b >$. We produce a basis for $R$, show that $ReR$ is its unique non-zero minimal ideal. Moreover, we provide a faithful representation of $R$ as a 4-dimensional matrix algebra over a 3-generated, 4-related ring where the image of $e$ is a nonzero matrix with zero diagonal.

## 1. Introduction

The problem in ring theory of the representation of an idempotent as a sum of two nilpotent elements of respective degrees $m, n$ was initiated in [1]. The freest corresponding minimal ring is

$$\mathcal{Z}(m,n) =< e, a, b | e^2 = e, a^m = 0, b^n = 0, e = a + b > .$$

and the freest corresponding minimal algebra in characteristic zero is $\mathcal{A}(m,n) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{Z}(m,n)$. It may be assumed by symmetry that $m \leq n$.

By applying the trace function, it is easy to see that in any finite dimensional representation of $\mathcal{A}(m,n)$ over fields of characterisitic zero, the image of $e$ is the zero linear transformation. It was shown in [1] that the same conclusion holds in any representation of $\mathcal{A}(m,n)$ as a PI algebra of characteristic zero. Furthermore, it was proven that the ring $\mathcal{Z}(m,n)$ was finitely generated as a $\mathbb{Z}$-module for $m = 2, n$ arbitrary and for $m = 3, n = 2, 3, 4, 5$ and therefore, in this range of parameters, the ideal generated by $e$ is finite.

Matrix representations of $\mathcal{A}(4,4)$ in $M_{4\times 4}(D)$ over division rings $D$ in characteristic 0 was undertaken by Salwa in [3]. He showed that such a matrix ring contains a nonzero idempotent $E$ with zero diagonal if and only if $D$ contains a copy of the first Weyl algebra. Moreover, he obtained a representation for $\mathcal{A}(3,6)$ in characteristic zero where the image of $e$ is non-zero. Considering that $\mathcal{A}(m,n)$ maps onto $\mathcal{A}(k,l)$ whenever $m \geq k, n \geq l$, these results establish that the algebra $\mathcal{A}(m,n)$ is infinite dimensional if and only if the pair $(m,n) \geq (3,6)$ or $(4,4)$, under lexicographical ordering.

The purpose of this paper is to construct a relatively easy non-trivial representation of $\mathcal{Z}(4,4)$ and furthermore to prove that $\mathcal{A}(4,4)$ is minimal, in the sense that it has no proper non-commutative quotients.

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The new representation of $\mathcal{Z}(4,4)$ has the advantage of being an elementary application of the Diamond Lemma. We prove

**Theorem 1.** *Let $T$ be the ring with the presentation*

$$\begin{aligned} &< && x,y,z|xy+yx=z, yz+zy=x, \\ zx+xz &= && y, x^2+y^2+z^2=0>. \end{aligned}$$

*Then, $T$ has as $\mathbb{Z}$-basis the set*

$$\{x^i y^j z^k | i,j \geq 0, k=0,1\}.$$

*Furthermore, the element*

$$E = \begin{pmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{pmatrix}$$

*of $M_{4\times 4}(T)$ is an idempotent.*

Next, we provide an explicit $\mathbb{Q}$-basis for the algebra $R = \mathcal{A}(4,4)$ built from one for the subalgebra $eRe$. Having this basis we are able to prove

**Theorem 2.** *The ideal $ReR$ generated by $e$ is the unique minimal non-zero ideal of $R$.*

This theorem implies that our representation of $R$ into $M_{4\times 4}(T)$ and that of Salwa's into $M_{4\times 4}(D)$ are both faithful.

Our results raise the question about the ideal structure of $\mathcal{A}(m,n)$ in general and of $\mathcal{A}(3,6)$ in particular.

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## 2. Symmetric Matrix Representation of $\mathcal{Z}(4,4)$

We consider a generic symmetric matrix $E = (x_{ij})$ of dimension 4 with zero diagonal such that $E^2 = E$. The six entries of $E$ satisfy the following sixteen equations:

$$x_{ij}^2 = -x_{ik}^2 - x_{il}^2,\ i \neq j,k,l,$$
$$x_{ij} = \sum_{k\neq i,j} x_{ik}x_{kj},\ i \neq j.$$

These equations imply

$$2(x_{12}^2 + x_{13}^2 + x_{23}^2) = 0,\ 2x_{23}^2 = 2x_{14}^2,\ 2x_{24}^2 = 2x_{13}^2,\ 2x_{34}^2 = 2x_{12}^2.$$

On assuming the partial algebra of entries of our matrix to be torsion-free and on choosing

$$x_{23} = x_{14},\ x_{24} = x_{13},\ x_{34} = x_{12},$$

the conditions reduce to the four equations

$$\begin{aligned} x_{12}^2 + x_{13}^2 + x_{23}^2 &= 0, \\ x_{12} &= x_{13}x_{14} + x_{14}x_{13}, \\ x_{13} &= x_{12}x_{14} + x_{14}x_{12}, \\ x_{14} &= x_{12}x_{13} + x_{13}x_{12}. \end{aligned}$$

Rename the entries as $x_{12} = x, x_{13} = y, x_{14} = z$. Then the algebra of entries of our matrix is now the ring $T$ with the presentation

$$\begin{aligned} & < && x,y,z|xy+yx=z, yz+zy=x, \\ zx+xz & = && y, x^2+y^2+z^2=0> . \end{aligned}$$

**Proposition 1.** *The ring $T$ has as $\mathbb{Z}$-basis the set $\{x^i y^j z^k \mid i,j \geq 0, k=0,1\}$.*

*Proof.* We shall apply the Diamond Lemma where the relations of $S$ are interpreted as substitutions. The ambiguities to be resolved appear in calculating the following products $zzx, zzy, zyx$. First we compute the auxiliary equations:

$$\begin{aligned} (yx)\,x &= y-2xz+x^2y,\ (yx)\,y = x-yz-xy^2, \\ y\,(yx) &= -x+2yz+xy^2,\ y\,(zx) = x^2+2y^2+xyz, \\ (zx)\,z &= yz+x^3+xy^2,\ (zx)\,y = y^2-x^2+xyz, \\ (zy)\,z &= y-xz+x^2y+y^3. \end{aligned}$$

On using the above equations it is straightforward to check that the ambiguities are resolved. ∎

**Corollary 1.** *Let* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & z & 0 & 0 \\ z & y & x & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & x & y & z \\ 0 & 0 & z & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *be elements of* $M_{4\times 4}(T)$ *and let* $E = A+B$. *Then,* $E$ *is an idempotent and* $A^4 = B^4 = 0$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ xzx & 0 & 0 & 0 \end{bmatrix}$, *where* $xzx = xy - x^2z \neq 0$.

A direct consequence of the above is

**Corollary 2.** *The algebra $R = \mathcal{A}(4,4)$ has infinite $\mathbb{Q}$-dimension.*

## 3. A basis for the algebra $\mathcal{A}(4,4)$

We rewrite the presentation of the algebra $R = \mathcal{A}(4,4)$ as

$$\mathbb{Q} < e, b | e^2 = e, b^4 = (e+b)^4 = 0 > .$$

It is clear that $e$ is not a central idempotent in $R$. We extend our algebra by a unit, $P = R \oplus \mathbb{Q}1$. Then $f = 1-e$ is an idempotent and we have the Peirce decomposition

$$P = ePe \oplus ePf \oplus fPe \oplus fPf.$$

Define the subalgebras $S = eRe, T = fRf$. Then,

$$ePe = S, ePf = eRf, fPe = fRe, fPf = T.$$

The algebra $R$ decomposes as

$$R = \sum \{\mathbb{Q}b^i | i = 1,2,3\} + \sum \{b^i S b^j | 0 \leq i,j \leq 3\}.$$

Thus, in any representation, $R$ has finite $\mathbb{Q}$-dimension if and only if $S$ has finite $\mathbb{Q}$-dimension.

The monomials $w$ in $P$ have the form $w = e, f$, or $g^l b^{i_1} g ... b^{i_k} g^m$ where $g = e, f$, $l, m \in \{0, 1\}$, $k \geq 1$ and $1 \leq i_1, ..., i_k \leq 3$. Define the formal $b$-length of $w$ (that is when $w$ is seen as an element of the free semi-group generated by $e, f, b$) to be $|w| = 0$ if $w = e$ or $f$ and $|w| = i_1 + ... + i_k$, otherwise. If $W$ is a subspace of $P$, then $W_n$ denotes the $\mathbb{Q}-$space generated by all elements of $W$ represented as monomials of $b$-length at most $n$.

3.1. **Computations in R.** Define $x = ebe$, $y = eb^2e$, $z = eb^3e$, $U = \{x^i, x^i y x^j | i, j \geq 0\}$ in $S$.

Expand the equation $(e + b)^4 = 0$ and use $b^4 = 0$ to produce

$$\begin{gathered} b^3e + eb^3 + b^2eb + beb^2 + eb^2e + b^2e + eb^2 + \\ ebeb + bebebeb + 2ebe + be + eb + e = 0. \end{gathered} \tag{1}$$

The multiplication $e \times (1) \times e$ produces

$$z \equiv -\frac{1}{2} (yx + xy) \mod S_2; \tag{2}$$

$b^3 \times (1) \times b^3$ produces

$$b^3 (y + 2x + e) b^3 = 0; \tag{3}$$

$(1) \times b^3$ produces

$$b^3 e b^3 \equiv 0 \mod R_5; \tag{4}$$

$e \times (1)$ produces

$$eb^3 \equiv -xb^2 - yb + \frac{1}{2} (yx + xy) \mod R_2; \tag{5}$$

$(5) \times b$ produces

$$yb^2 \equiv x^2b^2 + \frac{1}{2} (yx + 3xy) b - \frac{1}{2} x (yx + xy) \mod R_3; \tag{6}$$

$(6) \times e$ produces

$$2y^2 \equiv x^2y + 2xyx + yx^2 \mod R_3; \tag{7}$$

$(1) \times e$ leads to

$$b^3e \equiv -b^2x - by + \frac{1}{2} (yx + xy) \mod R_2 \tag{8}$$

$b \times (8)$ leads to

$$b^2y \equiv b^2x^2 + \frac{1}{2} b (3yx + xy) - \frac{1}{2} (yx + xy) x \mod R_3. \qquad (9);$$

On substituting $b^3e$ and $eb^3$ in (1) we get

$$beb^2 \equiv -b^2eb + b^2x + xb^2 + by + yb - (yx + xy) \mod R_2. \tag{10}$$

The multiplication $b \times (10)$ produces

$$b^2eb^2 \equiv -\frac{1}{2} (yx + xy) b - \frac{1}{2} b (yx + xy) + 2byb + b^2xb + bxb^2 \mod R_3; \tag{11}$$

(6)$\times be$ produces

$$4yxy \equiv -\left(3x^2yx + 3xyx^2 + yx^3 + x^3y\right) \mod \mathrm{S}_4. \tag{12}$$

We have $z^2 = eb^3eb^3e = \left(eb^3e\right)^2 \equiv (xy+yx)^2 \equiv 0 \bmod S_5$ from which we derive, using (4) and (2),

$$yx^2y \equiv \frac{1}{2}xyx^3 + \frac{1}{4}x^4y + \frac{1}{2}x^2yx^2 + \frac{1}{4}yx^4 + \frac{1}{2}x^3yx \mod S_5. \tag{13}$$

3.2. **A basis for the subalgebra $S = eRe$.**

**Proposition 2.** *The following congruences hold in the algebra $S$,*

$$\begin{aligned} yx^{2n-3}y &\equiv -\frac{1}{2n}yx^{2n-1} - \frac{1}{2n}x^{2n-1}y - \frac{2n-1}{2n\,(n-1)}\sum_{i=1}^{2n-2} x^iyx^{2n-i-1} \ \mathit{mod}\ S_{2n}, \\ \textit{for } n &\geq 2\ ; \\ yx^{2n-2}y &\equiv \frac{1}{2n}yx^{2n} + \frac{1}{2n}x^{2n}y + \frac{1}{n}\sum_{i=1}^{2n-1} x^iyx^{2n-i} \ \mathit{mod}\ S_{2n+1}, \\ \textit{for } n &\geq 1. \end{aligned}$$

*Proof.* Congruences (7), (12) and (13) of the previous section are the first three cases of the proposition.

Suppose that for $p = 0, .., n+1$ we have established the congruences

$$yx^py \equiv \sum_{i=0}^{p+2} \alpha(i,p)x^iyx^{p-i+2} \mod S_{p+3}, \tag{1}$$

for some rational coefficients $\alpha(i,p)$ and where

$$\alpha(0,p) = \begin{cases} -\frac{1}{p+3} & \text{for } p \text{ odd} \\ \dfrac{1}{p+2} & \text{for } p \text{ even} \end{cases}.$$

Then, it follows from (1) that

$$\begin{aligned} (yx^n)\,y^2 \equiv\ & \tfrac{1}{2}\,(yx^ny)\,x^2 + \left(yx^{n+1}y\right)x + \tfrac{1}{2}yx^{n+2}y \equiv \\ & \frac{1}{2}\sum_{i=0}^{n+2}\alpha(i,n)x^iyx^{n-i+4} \ + \sum_{i=0}^{n+3}\alpha(i,n+1)x^iyx^{n-i+4} + \\ & \frac{1}{2}yx^{n+2}y \mod S_{n+5}. \end{aligned} \tag{2}$$

Also, from (1), we have

$$\begin{aligned} (yx^ny)\,y \equiv\ & \sum_{i=0}^{n+2}\alpha(i,n)x^i\left(yx^{n-i+2}y\right) \equiv \alpha(0,n)yx^{n+2}y + \\ & \sum_{i=1}^{n+2}\alpha(i,n)x^i\sum_{j=0}^{n-i+4}\alpha(j,n-i+2)x^jyx^{n-i-j+4} \mod S_{n+5}. \end{aligned} \tag{3}$$

Therefore, from (2) and (3), we obtain

$$yx^{n+2}y \equiv \alpha(0,n+2)yx^{n+4} + \sum_{i=1}^{n+4} \alpha(i,n+2)x^i yx^{n-i+4} \text{ modulo } S_{n+5}, \tag{4}$$

where

$$\begin{aligned} \alpha(0,n+2) &= \frac{2\alpha(0,n+1)+\alpha(0,n)}{1-2\alpha(0,n)} \text{ and} \\ \alpha(i,n+2) &= \frac{2\alpha(i,n+1)+\alpha(i,n)+2\sum_{j=1}^{i}\alpha(j,n)\alpha(i-j,n-j+2)}{1-2\alpha(0,n)} \\ \text{for } i &\geq 1. \end{aligned}$$

Given the values of $\alpha(0,n)$ and $\alpha(0,n+1)$ we find that

$$\alpha(0,n+2) = \begin{cases} \frac{2\alpha(0,n+1)+\alpha(0,n)}{1-2\alpha(0,n)} = -\frac{1}{n+5} \text{ for } n \text{ odd} \\ \dfrac{1}{n+4} \text{ for } n \text{ even.} \end{cases}$$

Let $V$ be the vector space generated by the set $U$. We have shown that $yx^n y \in V$ for all $n \geq 0$ and therefore, $V = S$. The precise form of the coefficients $\alpha(i,n+2)$ for $1 \leq i \leq n+4$ can be established in a straightforward, though lengthy manner. ∎

**Proposition 3.** *The set $U$ is a $\mathbb{Q}$-basis for the algebra $S$.*

*Proof.* Suppose $U$ is linearly dependent then there exist $m,n \geq 0$ such that

$$x^m yx^n = \sum_{(i,j)<(m,n)} \beta_{ij} x^i yx^j$$

where the order on the pairs $(i,j)$ is lexicographical and $\beta_{ij} \in \mathbb{Q}$. Let $K$ be the extension of $\mathbb{Q}$ by $x$. Then, $S$ is a finitely generated right $K$-module and we note that it is freely generated as a right $K$-module by $y, xy, x^2y, ..., x^l y$ for some $l$.

By Zorn's Lemma, there exists a 2-sided ideal $I$ in $R$, maximal with respect to not containing $e$. Let $\overline{R}$ be the quotient of the algebra $R$ by $I$. Then, easily, $\overline{R}$ is a prime ring.

Let $\overline{S}, \overline{K}$ be the respective images of $S, K$ in $\overline{R}$. Then, again, $\overline{S}$ is a free right $\overline{K}$-module of finite rank. As $\overline{e}$ is the identity element in $\overline{S}$, the representation of $\overline{S}$ on itself by multiplication on the left is faithful. Thus, $\overline{S}$ is identifiable with a subalgebra of $M_{n\times m}(\overline{K})$. Therefore, $\overline{S}$ is a PI-algebra and $\overline{R}$ is a GPI-algebra (it satisfies a polynomial identity with constant $\overline{e}$). By a Theorem of Martindale [2], there exists a field extension $F$ of $\mathbb{Q}$ such that $\overline{R}_F = F \otimes_{\mathbb{Q}} \overline{R}$ is primitive. Hence $\overline{e}\overline{R}_F\overline{e}$ is also primitive, but as this is a PI-algebra, it follows that $\overline{e}\overline{R}_F\overline{e}$ is isomorphic to $M_{p\times p}(F)$ for some $p$. Therefore $\overline{e} = 0$; a contradiction is reached. ∎

**Corollary 3.** *Let $I$ be an ideal of $R$ such that $e \notin I$. Then, $I \cap S = 0$.*

*Proof.* If $I \cap S \neq 0$ then $S/I \cap S$ is finite dimensional and therefore so is $R/I$. But then $e \in I$; a contradiction. ∎

3.3. **Bases for $fRf$ and $fRe$.** Define the subalgebra $T = fRf$ and subspace $W = fRe$ of $P$. Define in $T$ the elements $p = fbf, q = fb^2f$, the subset $U' = \{p^i, p^iqp^j | i, j \geq 0\}$ and in $W$ the subset $U'' = \{fbx^i, fb^2x^i, fbx^iyx^j | i, j \geq 0\}$.

It can be established following a similar routine as in the case of $S = eRe$ that $U'$ is a basis for $T$. For example, consider the congruence (10) from Section 3.1. Then the multiplication $f \times (10) \times f$ produces

$$fb^2ebf + fbeb^2f \equiv 0 \text{ mod } P_2.$$

Therefore, on substituting $e = 1 - f$, we obtain

$$fb^3f \equiv \frac{1}{2}(qp + pq) \text{ mod } T_2,$$

and so, $T$ is generated as an algebra by $p, q$.

More concretely, we have

**Proposition 4.** *The following congruences hold in the algebra $T$,*

$$qp^{2n-3}q \equiv \frac{1}{2n}qp^{2n-1} + \frac{1}{2n}p^{2n-1}q + \frac{2n-1}{2n(n-1)}\sum_{i=1}^{2n-2} p^iqp^{2n-i-1} \ \textit{mod}\ T_{2n},$$

*for* $n \geq 2$ *;*

$$qp^{2n-2}q \equiv -\frac{1}{2n}qp^{2n} - \frac{1}{2n}p^{2n}q - \frac{1}{n}\sum_{i=1}^{2n-1} p^iqp^{2n-i} \ \textit{mod}\ T_{2n+1},$$

*for* $n \geq 1$.

*Moreover, $U'$ is a $\mathbb{Q}$-basis for $T$.*

Again, similarly, we have

**Proposition 5.** *The following congruences hold in the subspace $W = fRe$,*

$$fb^2x^{2n-1}y \equiv -\frac{1}{2n+1}fb^2x^{2n+1} - \frac{1}{2n+1}fbx^{2n}y - \frac{4n+1}{2n(2n+1)}\sum_{i=0}^{2n-1} fbx^iyx^{2n-i}$$

*mod $T_{2n+2}$, for $n \geq 1$,*

$$fb^2x^{2n}y \equiv \frac{1}{2n+1}fb^2x^{2n+2} + \frac{1}{2(n+1)}fbx^{2n+1}y + \frac{4n+3}{(2n+1)(2n+2)}\sum_{i=0}^{2n} fbx^iyx^{2n-i+1}$$

*mod $T_{2n+1}$, for $n \geq 1$.*

*Moreover, $U''$ is a $\mathbb{Q}$-basis of $W$.*

*Proof.* We will only prove that $U''$ is linearly independent. Suppose we have a nontrivial dependence equation

$$\sum \alpha_i fb^2x^i + \sum \beta_i fbx^{i+1} + \sum \gamma_{ij} fbx^iyx^j = 0.$$

Suppose that the maximum $b$-degree of the monomials in the sum is $m + 2$, then we will work modulo $P_{m+1}$; thus we have

(*)

$$\alpha f b^2 x^m + \beta f b x^{m+1} + \sum_{i=0}^{m-1} \gamma_i f b x^i y x^{m-i-1} \equiv 0 \bmod P_{m+1}.$$

We multiply (*) on the left by $eb$ and make the substitution $f = 1 - e$.
This multiplication produces:
$\alpha\left(\left(z\right)x^m - xyx^m\right) + \beta\left(yx^{m+1} - x^{m+3}\right) +$
$\sum \gamma_i \left(\left(yx^i y\right)x^{m-i-1} - x^{i+2} y x^{m-i-1}\right) \equiv 0 \bmod S_{m+2}$.
Then, on substituting in the above

$$\begin{aligned} z &\equiv -\frac{1}{2}yx - \frac{1}{2}xy \bmod R_2, \\ yx^i y &\equiv \varepsilon_i\left(yx^{i+2} + x^{i+2}y\right) + \delta_i \sum_{1\le l\le i+1} x^l y x^{i+2-l} \bmod R_{i+3} \end{aligned}$$

we get

$$\alpha\left(-\frac{1}{2}yx^{m+1} - \frac{3}{2}xyx^m\right) + \beta\left(yx^{m+1} - x^{m+3}\right) +$$

$$\sum_i \gamma_i \left(\left(\varepsilon_i\left(yx^{i+2} + x^{i+2}y\right) + \delta_i \sum_{1\le l\le i+1} x^l y x^{i+2-l}\right) x^{m-i-1} - x^{i+2} y x^{m-i-1}\right)$$

$\equiv 0 \bmod S_{m+2}$.
Therefore,
$\alpha\left(-\frac{1}{2}yx^{m+1} - \frac{3}{2}xyx^m\right) + \beta\left(yx^{m+1} - x^{m+3}\right) +$

$$\left(\sum_i \gamma_i \varepsilon_i\right) yx^{m+1} + \sum_i \gamma_i \varepsilon_i x^{i+2} y x^{m-i-1} + \sum_i \gamma_i \delta_i \left(\sum_{1\le l\le i+1} x^l y x^{m-l+1}\right)$$

$-\sum_i \gamma_i x^{i+2} y x^{m-i-1} \equiv 0 \bmod S_{m+2}$.
Hence,

$$-\beta x^{m+3} + \left(-\frac{\alpha}{2} + \beta + \sum_{i\ge 0} \gamma_i \varepsilon_i\right) yx^{m+1} +$$

$$\left(-\frac{3}{2}\alpha + \sum_{i\ge 0} \gamma_i \delta_i\right) xyx^m + \sum_{0\le i\le m-2} \left(\gamma_i\left(\varepsilon_i - 1\right) + \sum_{i+1\le k} \gamma_k \delta_k\right) x^{i+2} y x^{m-i-1}$$

$+\gamma_{m-1}\left(\varepsilon_{m-1} - 1\right) x^{m+1} y \equiv 0 \bmod S_{m+2}$.
We conclude

$$\begin{aligned} \beta &= 0, \\ \alpha &= 2\sum_{i\ge 0} \gamma_i \varepsilon_i = \frac{2}{3}\sum_{i\ge 0} \gamma_i \delta_i, \\ \gamma_i\left(\varepsilon_i - 1\right) + \sum_{k\ge i+1} \gamma_k \delta_k &= 0, \text{ for } 0 \le i \le m-2, \\ \gamma_{m-1} &= 0 \end{aligned}$$

Since $\varepsilon_i \neq 1$ for all $i$, this system easily leads to $\gamma_i = 0$ for all $i$ and to $\alpha = 0$. A contradiction is reached. ∎

**Corollary 4.** *The set*

$$\{x^i, x^i y x^j, p^i, p^i q p^j, fbx^i, fbx^i y x^j, fb^2 x^i, x^i bf, x^i y x^j bf, x^i b^2 f | i, j \ge 0\},$$

*is a basis of* $P$*, where* $x^0 = e, p^0 = f$*. Furthermore, the set*

$$\{b^i | i = 1, 2, 3\} \cup \{b^k x^i b^l, b^k x^i y x^j b^l | i, j \geq 0, k, l = 0, 1\}$$

*is a basis for* $R$*, where* $b^0 = 1$*.*

## 4. Ideal Structure of $R$

The ideal generated by $e$ is $J = ReR$ and $R/J$ is isomorphic to $\mathbb{Q}[b|b^4 = 0]$. The ideal structure of $R$ is determined by

**Theorem 3.** *The ideal* $J$ *is the unique minimal non-zero ideal of the algebra* $R$*.*

*Proof.* Let $I$ be a minimal non-zero ideal of $P$ not containing $e$. Then, $I \cap ePe = 0$. Suppose that $f \in I$. Then, since $e + b = -f + (1 + b)$, we get $0 = (e + b)^4 = u + (1 + b)^4$ for some $u \in I$. We have a contradiction since $1 + b$ is invertible. Therefore $f \notin I$ and $I \cap fPf = 0$. From the Peirce decomposition, we obtain $I = eIf \oplus fIe$. Suppose $a$ is a non-zero element of $fIe$ of $b$-degree $m$. Then

$$\alpha f b^2 x^m + \beta f b x^{m+1} + \sum_{i=0}^{m-1} \gamma_i f b x^i y x^{m-i-1} \equiv 0 \bmod P_{m+1},$$

and a repetition of the argument in the previous proposition leads to a contradiction. ∎

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