IMAGINARY VERMA MODULES FOR QUANTUM AFFINE LIE ALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra and $M(\lambda)$ an imaginary Verma module for $\mathfrak{g}$ with S-highest weight $\lambda \in P$. We construct quantum imaginary Verma modules $M^q(\lambda)$ over the quantum group $U_q(\mathfrak{g})$, investigate their properties and show that $M^q(\lambda)$ is a true quantum deformation of $M(\lambda)$ in the sense that the weight structure is preserved under the deformation.

Introduction.

The representation theory of Kac-Moody algebras is much richer than that of finite-dimensional simple algebras. In particular, Kac-Moody algebras have modules containing both finite and infinite-dimensional weight spaces, something that cannot happen in the finite-dimensional setting [Le]. These representations of Kac-Moody algebras arise from taking non-standard partitions of the root system, partitions which are not Weyl-equivalent to the standard partition into positive and negative roots. For affine algebras, there is always a finite number of Weyl-equivalency classes of these nonstandard partitions. Corresponding to each partition is a Borel subalgebra and one can form representations induced from one-dimensional modules for these nonstandard Borel subalgebras. These induced modules are called modules of Verma-type, in

[1]Regular Associate of the ICTP.

[2]Research supported in part by a Faculty Research Grant from St. Lawrence University.

1991 Mathematics Subject Classifications: 17B37, 17B67, 81R50 analogy with standard Verma modules induced from a standard Borel subalgebra.

Verma-type modules were first studied and classified by Jakobsen and Kac [JK1, JK2], and by Futorny [Fu1, Fu2]. Further work elucidating their structure, including the construction of the appropriate categorical setting, determination of irreducibility criteria, BGG duality and BGG resolutions can be found in [Co1, Co2, CFM, Fu4]. In this paper, we consider what are, in a sense, the most extreme case of nonstandard partitions for untwisted Kac-Moody algebras, and the representations arising from them, the imaginary Verma modules. We will recall needed properties of these modules from [Fu3].

Since their introduction by Drinfeld [Dr] and Jimbo [Ji] in 1985, there has been a tremendous interest in studying quantized enveloping algebras for Kac-Moody algebras and their representations. Quantum groups have turned out to be extremely important objects with rich and diverse connections to an ever-increasing number of areas of mathematics and physics. A vigorous body of research is developing on determining their structure, their representations and their applications.

In many cases, the representation theory of quantum groups parallels that of the associated underlying classical algebras, although often with some subtle differences. The closest parallel comes when the classical and quantum representations have the same weight structure. In this paper, we construct quantum imaginary Verma modules over the quantum group $U_q(\mathfrak{g})$ associated to an untwisted affine Kac-Moody algebra $\mathfrak{g}$.

One of the problems to be faced when studying nonstandard representations of a Kac-Moody algebra $\mathfrak{g}$, is that the absence of a general PBW theorem for quantum groups means that we cannot lift the triangular decomposition of $\mathfrak{g}$ up to a triangular decomposition of $U_q(\mathfrak{g})$. In [CFKM], the authors studied quantum imaginary Verma modules for the algebra $U_q(A_1^{(1)})$. There, they had to construct an appropriate PBW basis to deal with that particular case (see Proposition 2.2). The techniques used there do not easily generalize to the case of all affine algebras.

In this paper we rely heavily on the work by Beck [Be1, Be2] and Beck and Kac [BK] on PBW bases for quantum groups of affine algebras, and we exploit a particularly convenient description of the nonstandard partition of the root system used to construct the quantum imaginary Verma modules. This approach allows us to determine a basis for the quantum imaginary Verma modules in a unified manner without giving a PBW basis for the algebra.

Having constructed the quantum representations we wish to check that they are true quantum deformations of the equivalent classical modules. That is, the quantum and classical modules have the same weight structure. To do this, we follow the $\mathbb{A}$-form technique introduced by Lusztig [Lu], and subsequently refined and developed by Kang and co-authors [Ka, CFKM, BKMe]. For an overview of this procedure and a summary of known quantum deformation results, see [M]. Our main result, generalizing that of [CFKM] is that any quantum imaginary Verma module with integral S-highest weight λ is a quantum deformation of the equivalent imaginary Verma module over $U(\mathfrak{g})$ for $\mathfrak{g}$ an untwisted affine Kac-Moody algebra.

Armed with the quantum deformation theorem, we study some of the structural properties of quantum imaginary Verma modules. In particular, we prove an irreducibility criterion and probe the structure of these modules when they are reducible. The results obtained are similar to those given for non-quantum imaginary Verma modules in [Fu3], showing that these quantum modules are closely related to their classical cousins.

The structure of the paper is as follows. First, we recall background information and establish notation in Section 1. In Section 2 we review imaginary Verma modules for affine algebras. In Section 3 we construct quantum imaginary Verma modules and provide them with a basis. Section 4 constructs $\mathbb{A}$-forms, and Section 5 gives the classical limits and quantum deformation theorem. Section 6 discusses the irreducibility results. For additional basic background material and notation on Kac-Moody algebras, see the book by Kac [K]; for background information on quantum groups, see the excellent texts by Chari and Pressley [CP]and Jantzen [Ja].

1. Preliminaries.

1.1. Let N be a positive integer. Fix index sets $\dot{I} = \{1, \ldots, N\}$ and $I = \{0, \ldots, N\}$. Let $\dot{\mathfrak{g}}$ be a finite-dimensional simple complex Lie algebra with

Cartan subalgebra $\dot{\mathfrak{h}}$, root system $\dot{\Delta} \subset \dot{\mathfrak{h}}^*$, and set of simple roots $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_N\}$. Denote by $\dot{\Delta}_+$ and $\dot{\Delta}_-$, the positive and negative roots of $\mathfrak{g}$. Let $\dot{Q} = \oplus_{i=1}^N \mathbb{Z}\alpha_i$ be the root lattice of $\dot{\mathfrak{g}}$, and let $\dot{A} = (a_{ij})_{1\le i,j\le N}$ be the Cartan matrix for $\dot{\mathfrak{g}}$. Define a basis $h_1, \ldots, h_N$ of $\dot{\mathfrak{h}}$ by $\alpha_i(h_j) = a_{ij}$. Let $P = \{\lambda \in \mathfrak{h} \mid \lambda(h_i) \in \mathbb{Z}, i = 1, \ldots, N\}$ be the weight lattice of $\dot{\mathfrak{g}}$. Let $(.|.)$ denote both the symmetric invariant bilinear form on $\dot{\mathfrak{g}}$ and the induced form on $\dot{\mathfrak{g}}^*$, normalized so that $(\alpha|\alpha) = 2$ for any short root α. For $i = 1, \ldots, N$, let $d_i = (\alpha_i|\alpha_i)/2$. Then each d_i is a positive integer, the d_i are relatively prime and the diagonal matrix $\dot{D} = \mathrm{diag}(d_1, \ldots, d_N)$ is such that $\dot{D}\dot{A}$ is symmetric.

1.2. Let $\mathfrak{g}$ denote the untwisted affine Kac-Moody algebra associated to $\dot{\mathfrak{g}}$. Then $\mathfrak{g}$ has the loop space realization

$$\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is central in $\mathfrak{g}$; d is the degree derivation, so that $[d, x \otimes t^n] = nx \otimes t^n$ for any $x \in \dot{\mathfrak{g}}$ and $n \in \mathbb{Z}$, and $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \delta_{n+m,0} n(x|y)c$ for all $x, y \in \dot{\mathfrak{g}}$, $n, m \in \mathbb{Z}$. We set $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$.

The algebra $\mathfrak{g}$ has a Cartan matrix $A = (a_{ij})_{0\le i,j\le N}$ which is an extension of $\dot{A}$. There exists an integer d_0 and a diagonal matrix $D = diag(d_0, \ldots, d_n)$ such that DA is symmetric. An alternative Chevalley-Serre presentation of $\mathfrak{g}$ is given by defining it as the Lie algebra with generators e_i, f_i, h_i $(i \in I)$ and d subject to the relations

$$\begin{aligned}
&[h_i, h_j] = 0, \qquad [d, h_i] = 0,\\
&[h_i, e_j] = a_{ij} e_j, \qquad [d, e_j] = \delta_{0,j} e_j,\\
&[h_i, f_j] = -a_{ij} f_j, \qquad [d, f_j] = -\delta_{0,j} f_j,\\
&[e_i, f_j] = \delta_{ij} h_i,\\
&(\mathrm{ad} e_i)^{1-a_{ij}}(e_j) = 0, \qquad (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0, \quad i \neq j.
\end{aligned}$$

1.3. We can define the root system of $\mathfrak{g}$ in the following way. Extend the root lattice $\dot{Q}$ of $\dot{\mathfrak{g}}$ to a lattice $Q = \dot{Q} \oplus \mathbb{Z}\delta$, and extend the form $(.|.)$ to Q by setting $(q|\delta) = 0$ for all $q \in \dot{Q}$ and $(\delta|\delta) = 0$. Then the root system Δ of $\mathfrak{g}$ is given by

$$\Delta = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}, k \neq 0\}.$$

The roots of the form $\alpha + n\delta$, $\alpha \in \dot{\Delta}, n \in \mathbb{Z}$ are called real roots, and those of the form $k\delta$, $k \in \mathbb{Z}, k \neq 0$ are called imaginary roots. We let Δ^{re} and Δ^{im} denote the sets of real and imaginary roots, respectively. The set of positive real roots of $\mathfrak{g}$ is $\Delta_+^{re} = \dot{\Delta}_+ \cup \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n > 0\}$ and the set of positive imaginary roots is $\Delta_+^{im} = \{k\delta \mid k > 0\}$. The the set of positive roots of $\mathfrak{g}$ is $\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$. Similarly, on the negative side, we have $\Delta_- = \Delta_-^{re} \cup \Delta_-^{im}$, where $\Delta_-^{re} = \dot{\Delta}_- \cup \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n < 0\}$ and $\Delta_-^{im} = \{k\delta \mid k < 0\}$. Further, if θ denotes the highest positive root of $\dot{\mathfrak{g}}$ and $\alpha_0 := \delta - \theta$, then $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$ is a set of simple roots for $\mathfrak{g}$. Let W denote the Weyl group of $\mathfrak{g}$ generated by the simple reflections $r_0, r_1, \ldots, r_N$ and B denote the associated braid group with generators $T_0, T_1, \ldots, T_N$.

1.4. Beck, [Be1, Be2] has introduced a total ordering of the root system leading to PBW bases for $\mathfrak{g}$ and its quantum analog, $U_q(\mathfrak{g})$. We state the construction here, partially following the more abstract notation developed by Damiani [Da] and Gavarini [Ga].

For any affine algebra $\mathfrak{g}$, there exists a map $\pi : \mathbb{Z} \mapsto I$ such that, if we define

$$\beta_k = \begin{cases} r_{\pi(0)} r_{\pi(-1)} \cdots r_{\pi(k+1)}(\alpha_{\pi(k)}) & \text{for all } k \leq 0 \\ r_{\pi(1)} r_{\pi(2)} \cdots r_{\pi(k-1)}(\alpha_{\pi(k)}) & \text{for all } k \geq 1, \end{cases}$$

then the map $\pi' : \mathbb{Z} \mapsto \Delta_+^{re}$ given by $\pi'(k) = \beta_k$ is a bijection. Further we can choose π so that, $\{\beta_k \mid k \leq 0\} = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\}$ and $\{\beta_k \mid k \geq 1\} = \{-\alpha + n\delta \mid \alpha \in \dot{\Delta}^+, n > 0\}$.

It will be convenient for us to invert Beck's original ordering of the positive roots (cf [BK 1.4.1] for the original order, and [Ga, §2.1] for this ordering). Thus, we set

$$\beta_0 > \beta_{-1} > \beta_{-2} > \cdots > \delta > 2\delta > \cdots > \beta_2 > \beta_1.$$

Clearly, if we say $-\alpha < -\beta$ iff $\beta > \alpha$ for all positive roots α, β, we obtain a corresponding ordering on Δ_-.

The following elementary observation on the ordering will play a crucial role later. Write $A < B$ for two sets A and B if $x < y$ for all $x \in A$ and $y \in B$. Then Beck's total ordering of the positive roots can be divided into three sets:

$$\{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\} > \{n\delta \mid n > 0\} > \{-\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n > 0\}.$$

Similarly, for the negative roots, we have,

$$\{-\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\} < \{-n\delta \mid n > 0\} < \{\alpha - n\delta \mid \alpha \in \dot{\Delta}_+, n > 0\}.$$

Note that the map π, and so the total ordering, is not unique. We assume a suitable π chosen and fixed now throughout the paper. Beck's original approach and proof is constructive, but the existential approach avoids some technicalities we do not need below.

1.5. The quantum group, or quantized universal enveloping algebra of $\mathfrak{g}$ is the associative algebra $U_q(\mathfrak{g})$ with 1 over $\mathbb{C}(q)$ with generators $E_i, F_i, K_i^{\pm 1}$ $(i \in I)$ and $D^{\pm 1}$ subject to the defining relations

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = DD^{-1} = D^{-1}D = 1,\\
&K_i K_j = K_j K_i, \qquad K_i D = D K_i,\\
&K_i E_j = q_i^{a_{ij}} E_j K_i, \qquad D E_j = q_0^{\delta_{j,0}} E_j D,\\
&K_i F_j = q_i^{-a_{ij}} F_i K_i, \qquad D F_j = q_0^{-\delta_{j,0}} F_j D,\\
&E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},\\
&\sum_{k=1}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \qquad i \neq j\\
&\sum_{k=1}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \qquad i \neq j,
\end{aligned}$$

where $q_i = q^{d_i}$ (we can choose d_i so that $d_0 = 1$ and $q_0 = q$), and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q!}, \qquad [m]_q! = \prod_{j=1}^{m} [j]_q, \qquad [j]_q = \frac{q^j - q^{-j}}{q - q^{-1}}$$

for all $i \in I$, $m, n \in \mathbb{Z}$, $m \geq n > 0$. For any $\mu \in Q$, we have $\mu = \sum_{i \in I} c_i \alpha_i$, for some integers c_i. Denote $K_\mu = \prod_{i \in I} K_i^{c_i}$. Then $K_\lambda K_\mu = K_{\lambda+\mu}$ for all $\lambda, \mu \in Q$. In particular, we have $K_{\pm\alpha_i} = K_i^{\pm 1}$. Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by E_i (resp. F_i), $i \in I$, and let $U_q^0(\mathfrak{g})$ denote the subalgebra generated by $K_i^{\pm}$ $(i \in I)$ and $D^{\pm}$.

The action of the braid group generators T_i on the generators of the quantum group $U_q(\mathfrak{g})$ is given by the following.

$$T_i(E_i) = -F_iK_i, \qquad T_i(F_i) = -K_i^{-1}E_i,$$

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} \frac{1}{[-a_{ij}-r]_{q_i}!}\frac{1}{[-r]_{q_i}!} q_i^{-r} E_i^{-a_{ij}-r} E_j E_i^r, \qquad \text{if } i \neq j,$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} \frac{1}{[-r]_{q_i}!}\frac{1}{[-a_{ij}-r]_{q_i}!} q_i^{r} F_i^r F_j F_i^{-a_{ij}-r}, \qquad \text{if } i \neq j,$$

$$T_i(K_j) = K_jK_i^{-a_{ij}}, \qquad T_i(K_j^{-1}) = K_j^{-1}K_i^{a_{ij}},$$

$$T_i(D) = DK_i^{-\delta_{i,0}}, \qquad T_i(D^{-1}) = D^{-1}K_i^{\delta_{i,0}}.$$

1.6. For each $\beta_k \in \Delta_+^{re}$, define the root vector E_{β_k} in $U_q(\mathfrak{g})$ by

$$E_{\beta_k} = \begin{cases} E_{\pi(0)} & , k=0 \\ T_{\pi(0)}^{-1}T_{\pi(-1)}^{-1}\cdots T_{\pi(k+1)}^{-1}(E_{\pi(k)}) & \text{for all } k<0 \\ E_{\pi(1)} & , k=1 \\ T_{\pi(1)}T_{\pi(2)}\cdots T_{\pi(k-1)}(E_{\pi(k)}) & \text{for all } k>1. \end{cases}$$

Each real root space is 1-dimensional, but each imaginary root space is N-dimensional. Hence, for each positive imaginary root $k\delta$ ($k>0$) we define N imaginary root vectors, $E_{k\delta}^{(i)}$ ($i \in \dot{I}$) by

$$\exp\left((q^i - q^{-i})\sum_{k=1}^{\infty} E_{k\delta}^{(i)} z^k\right) = 1 + (q^i - q^{-i})\sum_{k=1}^{\infty} K_i^{-1}[E_i, E_{-\alpha_i+k\delta}]z^k.$$

Then for each k, the $E_{k\delta}^{(i)}$ span the $k\delta$-root space and commute with each other. Further, the E_{β_k} ($k \in \mathbb{Z}$) and $E_{k\delta}^{(i)}$ ($k>0$) form a basis for U_q^+.

Let ω denote the standard $\mathbb{C}$-linear antiautomorphism of $U_q(\mathfrak{g})$, and set $E_{-\alpha} = \omega(E_\alpha)$ for all $\alpha \in \Delta_+$. Then U_q has a basis of elements of the form E_-HE_+, where $E_\pm$ are ordered monomials in the E_α, $\alpha \in \Delta_\pm$ and H is a monomial in $K_i^{\pm 1}$, and $D^{\pm 1}$ (which all commute).

Furthermore, this basis is, in Beck's terminology, convex, meaning that, if $\alpha, \beta \in \Delta_+$ and $E_\beta > E_\alpha$, then

$$E_\beta E_\alpha - q^{(\alpha|\beta)}E_\alpha E_\beta = \sum_{\alpha<\gamma_1<\cdots<\gamma_r<\beta} c_\gamma E_{\gamma_1}^{a_1}\cdots E_{\gamma_r}^{a_r}$$

for some integers $a_1, \ldots, a_r$ and scalars $c_\gamma \in \mathbb{C}[q, q^{-1}]$, $\gamma = (\gamma_1, \ldots, \gamma_r)$ [BK, Proposition 1.7c], and similarly for the negative roots.

2. Imaginary Verma modules for affine algebras.

Consider the partition $\Delta = S \cup -S$ of the root system of $\mathfrak{g}$ where $S = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\}$ and $-S = \{\alpha \in \Delta \mid -a \in S\}$. This partition is closed in the sense that, if $\alpha, \beta \in S$ and $\alpha+\beta \in \Delta$, then $\alpha+\beta \in S$. Hence the space $\mathfrak{g}_S = \oplus_{\alpha \in S} \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{g}_{-S} \oplus \mathfrak{h} \oplus \mathfrak{g}_S$. This is an example of the non-standard triangular decompositions of affine algebras studied and classified by Jakobsen and Kac [JK1, JK2]and Futorny [Fu1, Fu2].

Let $U(\mathfrak{g}_S)$ (resp. $U(\mathfrak{g}_{-S})$) denote the universal enveloping algebra of $\mathfrak{g}_S$ (resp. $\mathfrak{g}_{-S}$). Then, from the PBW theorem, the triangular decomposition of $\mathfrak{g}$ determines a triangular decomposition of $U(\mathfrak{g})$ as $U(\mathfrak{g}) = U(\mathfrak{g}_{-S}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}_S)$.

Let $\lambda \in \mathfrak{h}^*$. Then λ extends to a map on $(U(\mathfrak{h}))^*$, also denoted by λ. A $U(\mathfrak{g})$-module V is called a weight module if $V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v$ for all $h \in U(\mathfrak{h})\}$. The non-zero subspaces V_μ are called weight spaces. Any submodule of a weight module is a weight module. A $U(\mathfrak{g})$-module V is called an S-highest weight module with highest weight λ if there is some nonzero vector $v \in V$ such that

(i) $u^+ \cdot v = 0$ for all $u^+ \in U(\mathfrak{g}_S)$;
(ii) $h \cdot v = \lambda(h)v$ for all $h \in U(\mathfrak{h})$;
(iii) $V = U(\mathfrak{g}) \cdot v$.

An S-highest weight module is a weight module.

Let $\lambda \in \mathfrak{h}^*$. We make $\mathbb{C}$ into a 1-dimensional $U(\mathfrak{g}_S \oplus \mathfrak{h})$-module by picking a generating vector v and determining the action by $(x+h) \cdot v = \lambda(h)v$, for all $x \in \mathfrak{g}_S, h \in \mathfrak{h}$. The induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_S \oplus \mathfrak{h})} \mathbb{C}v = U(\mathfrak{g}_{-S}) \otimes \mathbb{C}v$$

is called the imaginary Verma module with S-highest weight λ. Imaginary Verma modules are in many ways similar to ordinary Verma modules except they contain both finite and infinite-dimensional weight spaces. Their basic structural properties were studied in [Fu3], from which the relvant properties are summarized here (cf. [Fu3, Proposition 1, Theorem 1]).

Proposition 2.1. *Let $\lambda \in \mathfrak{h}^*$, and let $M(\lambda)$ be the imaginary Verma module of S-highest weight λ. Then $M(\lambda)$ has the following properties.*

(i) *The module $M(\lambda)$ is a free $U(\mathfrak{g}_{-S})$-module of rank 1 generated by the S-highest weight vector $1 \otimes 1$ of weight λ.*

(ii) *$M(\lambda)$ has a unique maximal submodule.*

(iii) *Let V be a $U(\mathfrak{g})$-module generated by some S-highest weight vector v of weight λ. Then there exists a unique surjective homomorphism $\phi : M(\lambda) \mapsto V$ such that $\phi(1 \otimes 1) = v$.*

(iv) $\dim M(\lambda)_\lambda = 1$. *For any $\mu = \lambda - k\delta$, k a positive integer, $0 < \dim M(\lambda)_\mu < \infty$. If $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and $\dim M(\lambda)_\mu \neq 0$, then $\dim M(\lambda)_\mu = \infty$.*

(v) *Let $\lambda, \mu \in \mathfrak{h}^*$. Any non-zero element of $Hom_{U(\mathfrak{g})}(M(\lambda), M(\mu))$ is injective.*

(vi) *The module $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

3. Imaginary Verma modules for quantum affine algebras.

As in the classical case, we use the partition of the root system $\Delta = S \cup -S$ where $S = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\}$, and $-S = \Delta \setminus S$. Let $U_q(\pm S)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta \mid \beta \in \pm S\}$, and let B_q denote the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta \mid \beta \in S\} \cup H$

A $U_q(\mathfrak{g})$-module V^q is called a quantum weight module if $V^q = \oplus_{\mu \in P} V^q_\mu$, where

$$V^q_\mu = \{v \in V \mid K_i^{\pm 1} \cdot v = q_i^{\pm \mu(h_i)}, D^{\pm 1} \cdot v = q_0^{\pm \mu(d)}\}.$$

Any submodule of a quantum weight module is a weight module. A $U_q(\mathfrak{g})$-module V^q is called an S-highest weight module with highest weight $\lambda \in P$ if there is a non-zero vector $v \in V^q$ such that:

(i) $u^+ \cdot v = 0$ for all $u^+ \in U_q(S) \setminus \mathbb{C}(q)^*$;

(ii) For each $i \in I$, $K_i^{\pm 1} \cdot v = q_i^{\pm \lambda(h_i)} v$, $D^{\pm 1} \cdot v = q_0^{\pm \lambda(d)} v$;

(iii) $V^q = U_q(\mathfrak{g}) \cdot v$.

Note that, in the absence of a general quantum PBW theorem for non-standard partitions, we cannot immediately claim that an S-highest weight module V^q

is generated by $U_q(-S)$. This is in contrast to the classical case, and the reason behind our next theorem.

Now we define a $U_q(\mathfrak{g})$-module as follows. Let $\mathbb{C}\cdot v$ be a 1-dimensional vector space. Let $\lambda \in \mathfrak{h}^*$, and set $E_\beta \cdot v = 0$ for all $\beta \in S$, $K_i^{\pm 1} \cdot v = q_i^{\pm\lambda(h_i)} v$ $(i \in I)$ and $D^{\pm 1} \cdot v = q_0^{\pm\lambda(d)} v$. Now define $M^q(\lambda) = U_q \otimes_{B_q} \mathbb{C}v$. Then $M^q(\lambda)$ is an S-highest weight U_q-module called the *quantum imaginary Verma module* with highest weight λ.

In order to show that the module $M^q(\lambda)$ is spanned by the "right" set of vectors, we must appeal to Beck's PBW basis of $U_q(\mathfrak{g})$ and to the very useful grading by degree introduced by Beck and Kac [BK, §1.8], which we reproduce here as our notation differs slightly from theirs.

Beck [Be2] has shown that $U_q(\mathfrak{g})$ has a basis comprising elements of the form $N_{(a_\beta)} K M_{(a'_\beta)}$, where the $M_{(a_\beta)}$ are ordered monomials in $E_\beta^{a_\beta}$, $\beta \in \Delta_+$, $a_\beta \in \mathbb{Z}_+$, $N_{(a_\beta)} = \omega(M_{(a_\beta)})$, and K is an ordered monomial in $K_i^\pm$ and $D^\pm$. The notation (a_β) indicates the sequence of powers a_β as β runs over Δ_+. Of course, almost all terms of the sequence are zero.

In [BK, §1.8], Beck and Kac define the *total height* of such a basis element by

$$d_0(N_{(a_\beta)} K M_{(a'_\beta)}) = \sum_{\beta \in \Delta_+} (a_\beta + a'_\beta)\mathrm{ht}\beta,$$

where $\mathrm{ht}\beta$ is the usual height of a root. Next, they set the *total degree* of a basis element to be

$$d(N_{(a_\beta)} K M_{(a'_\beta)}) = (d_0(N_{(a_\beta)} K M_{(a'_\beta)}), (a_\beta), (a'_\beta)) \in \mathbb{Z}_+^{2\Delta_+ + 1}.$$

Considering $\mathbb{Z}_+^{2\Delta_+ + 1}$ as a totally ordered semigroup with the usual lexicographical ordering, Beck and Kac introduce a filtration of $U_q(\mathfrak{g})$ by defining U_s, for any $s \in \mathbb{Z}_+^{2\Delta_+ + 1}$, to be the span of the basis monomials $N_{(a_\beta)} K M_{(a'_\beta)}$ with degree $d(N_{(a_\beta)} K M_{(a'_\beta)}) \leq s$. Finally, they obtain the following proposition.

Proposition 3.1 [BK, Proposition 1.8]**.** *The associated graded algebra $GrU_q(\mathfrak{g})$ of the $\mathbb{Z}_+^{2\Delta_+ + 1}$-filtered algebra $U_q(\mathfrak{g})$ is algebra over $\mathbb{C}(q)$ generated by E_α,*

$\alpha \in \Delta$, counting multiplicities, $K_i^{\pm 1}$ $(i \in I)$ and $D^{\pm}$ subject to the relations

$$K_iK_i^{-1} = K_i^{-1}K_i = DD^{-1} = D^{-1}D = 1,$$
$$K_iK_j = K_jK_i, \qquad K_iD = DK_i,$$
$$K_iE_\alpha = q^{(\alpha_i|\alpha)}E_\alpha K_i, \qquad DE_\alpha = q^n E_\alpha D, \text{ for } \alpha = \gamma + n\delta, \gamma \in \dot{\Delta}.$$
$$E_\alpha E_{-\beta} = E_{-\beta}E_\alpha \text{ if } \alpha, \beta \in \Delta_+,$$
$$E_\alpha E_\beta = q^{(\alpha|\beta)}E_\beta E_\alpha, \qquad E_{-\alpha}E_{-\beta} = q^{(\alpha|\beta)}E_{-\beta}E_{-\alpha}, \text{ if } \alpha, \beta \in \Delta_+ \text{ and } \beta < \alpha.$$

Next we need the following technical result, which we state in an abstract manner. Let $\mathcal{I}$ be a totally ordered set without infinite decreasing chains (i.e., $\mathcal{I}$ has a minimal element with respect to the ordering). Let A be an associative algebra over a field K with generators $\{a_i \mid i \in \mathcal{I}\}$. Let $O = \{(n_i|i \in \mathcal{I}, n_i \in \mathbb{Z}_+, n_i = 0 \text{ for all but finite number of indices })\}$. The set O has a total lexicographical order such that $(n_i \mid i \in \mathcal{I}) > (m_i \mid i \in \mathcal{I})$ if and only if there exists some $j \in \mathcal{I}$ such that $n_j > m_j$ and $n_k = m_k$ for all $k > j$.

Suppose that A has a basis $B = \{v = a_{i_1}^{k_1} \dots a_{i_s}^{k_s} \mid i_1 > \dots > i_s\}$. Then the orderings of $\mathcal{I}$ and O impose an ordering on B. For any word $v = a_{j_1} ... a_{j_k} \in A$, denote by $\overline{v}$ the unique element in B such that $v = \lambda \overline{v} +$ (terms lower in the ordering), for some $\lambda \in K$. If $\overline{v} = a_{i_1}^{k_{i_1}} \dots a_{i_s}^{k_{i_s}}$ as an element of B, set $|v| = (k_{i_1}, \dots, k_{i_s}) \in O$.

Proposition 3.2. *Suppose that for all i and j in $\mathcal{I}$ we have: (*) $a_ia_j = \xi_{ij}a_ja_i + \sum_{v \in B} \xi_v v$, where $\xi_{ij} \neq 0$ and $|a_ia_j| > |v|$ for all v such that $\xi_v \neq 0$. Then $|vw| = |v| + |w|$ for all $v, w \in B$.*

Proof. We proceed by induction on $|w|$ and, for fixed $|w|$, by induction on $|v|$. By our assumption the set $\mathcal{I}$ has a minimal element i_0. If $|w| = \epsilon_{i_0} = (n_i | n_i = 0, i \neq i_0, n_{i_0} = 1)$ then w is the minimal element in O. Hence, for any $v \in B$, we have $\overline{vw} = vw$ and so $|vw| = |\overline{vw}| = |v| + |w|$.

Let $w = a_i$ and $v = v_1a_j$. If $j \geq i$, then $\overline{vw} = vw$ and $|vw| = |v| + |w|$. If $j < i$, then by (*), $vw = v_1a_ja_i = \xi_{ij}v_1a_ia_j + \sum_{u \in B} \xi_u v_1 u$. We will prove that $|v_1u| < |v_1a_ia_j|$. By induction on $|v|$ we have $|v_1a_i| = |v_1| + |a_i|$, while by induction on $|w|$ we get $|v_1a_ia_j| = |v_1a_i| + |a_j| = |v_1| + |a_i| + |a_j|$. Let $\xi_u \neq 0$ and $u = a_ku_1$. If $k < i$ then by induction on $|w|$ we have $|v_1u| = |v_1| + |u|$, since $|u| < |a_i|$. If $k = i$ then $u_1 = a_su_2$ and $s < j < i$ since $|u| = |a_ia_su_2| < |a_ia_j|$.

Then $|v_1u| = |v_1a_ia_su_2| = |v_1a_i| + |a_su_2|$ and, as $|a_i| > |a_su_2|$, we also have $|v_1a_i| + |a_su_2| = |v_1| + |a_i| + |a_su_2| < |v_1| + |a_i| + |a_j|$, since $s < j$. Hence we have $|v_1u| < |v_1a_ia_j|$ if $\xi_u \neq 0$ and $|vw| = |v_1a_ia_j| = |v_1|+|a_i|+|a_j| = |v|+|a_i|$.

Now assume that $w = w_1a_i$ with $|w| > |w_1| > |a_i|$. By induction, $|vw| = |vw_1a_i| = |vw_1| + |a_i| = |v| + |w_1| + |a_i| = |v| + |w|$. Hence, the proposition is proved. □

Let u be an arbitrary element of $U_q(\mathfrak{g})$. We may write u uniquely as a sum of basis monomials and define the total degree of u to be the largest total degree of these basis elements. We make the following observation.

Proposition 3.3. *Let $u \in U_q(\mathfrak{g})$ be an arbitrary element and u', u'' two basis monomials with $d(u') < d(u'')$. Then $d(uu') < d(uu'')$ and $d(u'u) < d(u''u)$.*

Proof. The result follows from Propositions 3.1 and 3.2. □

Now we are ready to the main result of this section.

Theorem 3.4. *As a vector space, $M^q(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta} \dots E_{-k\delta} \dots E_{-\alpha+k\delta}$, $\alpha \in \dot{\Delta}_+$, $n \geq 0$, $k > 0$.*

Proof. First, we introduce some notation to clarify the argument. Consider the following subsets of Δ:

$$A_1 = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\},$$
$$A_2 = \{k\delta \mid k > 0\},$$
$$A_3 = \{-\alpha + k\delta \mid \alpha \in \dot{\Delta}_+, k > 0\},$$
$$B_1 = \{-\alpha - n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\},$$
$$B_2 = \{-k\delta \mid k > 0\},$$
$$B_3 = \{\alpha - k\delta \mid \alpha \in \dot{\Delta}_+, k > 0\}.$$

Then $\Delta_+ = A_1 \cup A_2 \cup A_3$ and $\Delta_- = B_1 \cup B_2 \cup B_3$, while $S = A_1 \cup A_2 \cup B_3$ and $-S = A_3 \cup B_1 \cup B_2$. Note that, in our ordering of the root system, we have

$$B_1 < B_2 < B_3 < A_3 < A_2 < A_1.$$

For $i = 1, 2, 3$, let X_i denote an ordered monomial in elements E_β, $\beta \in A_i$, and similarly, let Y_i denote an ordered monomial of elements E_β, $\beta \in B_i$.

Then, utilizing Beck's PBW basis, any element $u \in U_q(\mathfrak{g})$ can be written in the form

$$u = \sum Y_1 Y_2 Y_3 Z X_3 X_2 X_1,$$

where $Z \in U_q^0(\mathfrak{g})$.

Now, let v be the canonical generator of $M(\lambda)$. Suppose $w \in M(\lambda)$. Then, since $M(\lambda) = U_q(\mathfrak{g}) \cdot v$, we have $w = u \cdot v$ for some $u \in U_q(\mathfrak{g})$. In view of the discussion above, we may write $w = \sum Y_1 Y_2 Y_3 Z X_3 X_2 X_1 \cdot v$, for suitable monomials X_i, Y_i, Z.

By definition of $M(\lambda)$, monomials of the form X_1 and X_2 act as 0 on v, and Z commutes with X_3 up to scalar in $\mathbb{C}(q)$. Hence, we can write $w = \sum Y_1 Y_2 Y_3 X_3 \cdot v$. The theorem asserts that $M(\lambda)$ is spanned by monomials of the form $Y_1 Y_2 X_3$, so we must determine how to commute monomials of the forms Y_3 and X_3. Let

$$X_3 = E_{-\alpha_1 + k_1 \delta} \dots E_{-\alpha_r + k_r \delta}, \text{ and}$$
$$Y_3 = E_{\beta_1 - m_1 \delta} \dots E_{\beta_s - m_s \delta},$$

for suitable roots $\alpha_i, \beta_j \in \dot{\Delta}_+$ and positive integers k_i, m_j, $i = 1, \dots, r$, $j = 1, \dots, s$. Then

$$\begin{aligned} Y_3 X_3 &= E_{\beta_1 - m_1 \delta} \dots E_{\beta_s - m_s \delta} E_{-\alpha_1 + k_1 \delta} \dots E_{-\alpha_r + k_r \delta} \\ &= E_{\beta_1 - m_1 \delta} \dots E_{\beta_{s-1} - m_{s-1} \delta} E_{-\alpha_1 + k_1 \delta} E_{\beta_s - m_s \delta} E_{-\alpha_2 + k_2 \delta} \dots E_{-\alpha_r + k_r \delta} \\ &\qquad + (\text{terms of lower total degree}) \end{aligned}$$

by using the grading of Proposition 3.1. Repeating this process we get that

$$Y_3 X_3 \cdot v = X_3 Y_3 \cdot v + (\text{ terms of lower degree}) \cdot v.$$

By induction on the total degree, we may order the terms of lower degree (as they act on v). The base of the induction is trivial as, if $d_0 = 1$, there is only one simple root involved and nothing to do. Since, by definition of $M_q(\lambda)$, we have $Y_3 \cdot v = 0$, we are done. $\square$

4. $\mathbb{A}$-forms of imaginary Verma modules.

In the previous section, we constructed quantum imaginary Verma modules. Now we wish to show that these quantum imaginary Verma modules are quantum deformations of imaginary Verma modules defined over the affine algebra. That is, we wish to show that the weight-space structure of a given module $M_q(\lambda)$ is the same as that of its classical counterpart $M(\lambda)$ for any $\lambda \in P$. To do this, we construct an intermediate module, called an $\mathbb{A}$-form.

Following [Lu], for each $i \in I$, $s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we define the *Lusztig numbers* in $U_q(\mathfrak{g})$:

$$\begin{bmatrix} K_i \ ; \ s \\ n \end{bmatrix} = \prod_{r=1}^{n} \frac{K_i q_i^{s-r+1} - K_i^{-1} q_i^{-(s-r+1)}}{q_i^r - q_i^{-r}},$$
$$\begin{bmatrix} D \ ; \ s \\ n \end{bmatrix} = \prod_{r=1}^{n} \frac{D q_0^{s-r+1} - D^{-1} q_0^{-(s-r+1)}}{q_0^r - q_0^{-r}}.$$

Let $\mathbb{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]_{q_i}}, i \in I, n > 0]$. Define the $\mathbb{A}$-form, $U_{\mathbb{A}}(\mathfrak{g})$, of $U_q(\mathfrak{g})$ to be the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $E_i, F_i, K_i^{\pm 1}$, $\begin{bmatrix} K_i \ ; \ 0 \\ 1 \end{bmatrix}$, $i \in I$, $D^{\pm 1}$, $\begin{bmatrix} D \ ; \ 0 \\ 1 \end{bmatrix}$. Let $U_{\mathbb{A}}^+$ (resp. $U_{\mathbb{A}}^-$) denote the subalgebra of $U_{\mathbb{A}}$ generated by the E_i, (resp. F_i), $i \in I$, and let $U_{\mathbb{A}}^0$ denote the subalgebra of $U_{\mathbb{A}}$ generated by the elements $K_i^{\pm}$, $\begin{bmatrix} K_i \ ; \ 0 \\ 1 \end{bmatrix}$, $i \in I$, $D^{\pm 1}$, $\begin{bmatrix} D \ ; \ 0 \\ 1 \end{bmatrix}$.

For any $i \in I$, $s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have the following identity

$$\begin{bmatrix} K_i \ ; \ s \\ n \end{bmatrix} = \prod_{i=1}^{n} \frac{1}{[r]_{q_i}} \left(\begin{bmatrix} K_i \ ; \ 0 \\ 1 \end{bmatrix} + [s-r+1]_{q_i} K_i^{-1} \right) \qquad \text{(cf. [BKMe, eq. 3.8]).}$$

Hence, all $\begin{bmatrix} K_i \ ; \ s \\ n \end{bmatrix}$ are in $U_{\mathbb{A}}$. Similarly, all $\begin{bmatrix} D \ ; \ s \\ n \end{bmatrix}$ are also in $U_{\mathbb{A}}$.

Proposition 4.1. *The following commutation relations hold between the gen-*

erators of $U_{\mathbb{A}}$. *For* $i, j \in I$, $s \in \mathbb{Z}$, $n \in \mathbb{Z}_+$,

$$E_i \begin{bmatrix} K_j \ ; \ s \\ n \end{bmatrix} = \begin{bmatrix} K_j \ ; \ s - a_{ij} \\ n \end{bmatrix} E_i,$$

$$\begin{bmatrix} K_j \ ; \ s \\ n \end{bmatrix} F_i = F_i \begin{bmatrix} K_j \ ; \ s - a_{ij} \\ n \end{bmatrix},$$

$$E_i \begin{bmatrix} D \ ; \ s \\ n \end{bmatrix} = \begin{bmatrix} D \ ; \ s - \delta_{i,0} \\ n \end{bmatrix} E_i,$$

$$\begin{bmatrix} D \ ; \ s \\ n \end{bmatrix} F_i = F_i \begin{bmatrix} D \ ; \ s - \delta_{i,0} \\ n \end{bmatrix},$$

$$E_i F_j = F_j E_i, \qquad \textit{for } i \neq j,$$

$$E_i F_i^n = F_i^n E_i + F_i^{n-1} \sum_{r=0}^{n-1} \begin{bmatrix} K_i \ ; \ -2r \\ 1 \end{bmatrix}.$$

Proof. The first five equalities follow from the defining relations of $U_q(\mathfrak{g})$ and the definition of the Lusztig numbers, while the last equality is proved by induction. □

An immediate consequence of Proposition 4.1 is that $U_{\mathbb{A}}$ inherits the standard triangular decomposition of $U_q(\mathfrak{g})$. In particular, any element u of $U_{\mathbb{A}}$ can be written as a sum of monomials of the form $u^- u^0 u^+$ where $u^{\pm} \in U_{\mathbb{A}}^{\pm}$ and $u^0 \in U_{\mathbb{A}}^0$. In fact, we can say rather more. For each positive real root β, the root vector E_β in Beck's basis is defined via the action of the braid group on the generators E_i. But the coefficients of this action are all in the ring $\mathbb{A}$. Consequently, the real root vectors are in $U_{\mathbb{A}}$. Next, consider the definition of the positive imaginary root vectors $E_{k\delta}^{(i)}$, $i \in I$, $k > 0$. These are given in terms of an exponential generating function containing commutators of the form $[E_i, E_{-\alpha_i + k\delta}]$, and these will also be in $U_{\mathbb{A}}$ since all the E_i and $E_{-\alpha_i + k\delta}$ are. The $\mathbb{C}(q)$ coefficients of the generating function are all in $\mathbb{A}$, and so the imaginary root vectors are all in $U_{\mathbb{A}}$. Thus, $U_{\mathbb{A}}$ inherits from $U_q(\mathfrak{g})$ a basis of monomials of the form $N_{(a_\beta)} K M_{(a'_\beta)}$, where $M_{(a'_\beta)}$ and $N_{(a_\beta)}$ are as before, and K is now an (ordered) monomial in the generators $K_i^{\pm}$, $\begin{bmatrix} K_i \ ; \ 0 \\ 1 \end{bmatrix}$, $i \in I$, $D^{\pm}$ and $\begin{bmatrix} D \ ; \ 0 \\ 1 \end{bmatrix}$ of $U_{\mathbb{A}}^0$.

Let $\lambda \in P$, and let $M^q(\lambda)$ be the imaginary Verma module over $U_q(\mathfrak{g})$ with S-highest weight λ and highest weight vector v_λ. The $\mathbb{A}$-form of $M^q(\lambda)$,

$M^{\mathbb{A}}(\lambda)$, is defined to be the $U_{\mathbb{A}}$ submodule of $M^q(\lambda)$ generated by v_λ. That is, we set

$$M^{\mathbb{A}}(\lambda) = U_{\mathbb{A}} \cdot v_\lambda.$$

Proposition 4.2. *As a vector space, $M^{\mathbb{A}}(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta} \dots E_{-k\delta} \dots E_{-\alpha+k\delta}$, $\alpha \in \dot{\Delta}_+$, $n \geq 0$, $k > 0$.*

Proof. As in the proof of Theorem 3.4, we note that any element u in $U_{\mathbb{A}}$ can be written as a sum of monomials of the form $Y_1Y_2Y_3ZX_3X_2X_1$, where the X_i and Y_i are as in the theorem and Z is now in $U^0_{\mathbb{A}}$. Let $w \in M^{\mathbb{A}}(\lambda)$. Then $w = u \cdot v_\lambda$ for some $u \in U_{\mathbb{A}}$. Write $w = \sum Y_1Y_2Y_3ZX_3X_2X_1 \cdot v_\lambda$. As before, we have $X_1 \cdot v_\lambda = 0$, and $X_2 \cdot v_\lambda = 0$. Also, Z commutes with X_3, up to a scalar in $\mathbb{A}$, by Proposition 4.1.

Now we must check the action of Z on v_λ. First, we have $K_i^{\pm} \cdot v_\lambda = q_i^{\pm\lambda(h_i)} v_\lambda \in \mathbb{A}v_\lambda$, and $D^{\pm} \cdot v_\lambda = q_0^{\pm\lambda(d)} v_\lambda \in \mathbb{A}v_\lambda$. It remains only to check the action of the Lusztig numbers. For $i \in I$, $s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\begin{bmatrix} K_i\ ;\ s \\ n \end{bmatrix} \cdot v_\lambda = \begin{bmatrix} \lambda(h_i) + s \\ n \end{bmatrix}_{q_i} v_\lambda.$$

The quantum binomials $\begin{bmatrix} \lambda(h_i) + s \\ n \end{bmatrix}_{q_i}$ are in the ring $\mathbb{A}$, and so it follows that $\begin{bmatrix} K_i\ ;\ s \\ n \end{bmatrix} \cdot v_\lambda \in \mathbb{A}v_\lambda$. Similarly, $\begin{bmatrix} D\ ;\ s \\ n \end{bmatrix} \cdot v_\lambda \in \mathbb{A}v_\lambda$. Hence, $Z \cdot v_\lambda \in \mathbb{A}v_\lambda$. We are left needing to commute the monomials Y_3 and X_3, but this commutator must, by definition, be in $U_{\mathbb{A}}$. The result then follows from Theorem 3.4. $\square$

Now that we have a vector space basis for the $\mathbb{A}$-form $M^{\mathbb{A}}(\lambda)$ of $M^q(\lambda)$, we can begin comparing the two modules, first as vector spaces.

Proposition 4.3. *For any $\lambda \in P$, as $\mathbb{C}(q)$-vector spaces, $\mathbb{C}(q) \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda) \cong M^q(\lambda)$.*

Proof. This proof is fairly standard. The $\mathbb{C}(q)$-linear map $\zeta : \mathbb{C}(q) \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda) \to M^q(\lambda)$ defined by $\zeta(f \otimes v) = fv$ for $f \in \mathbb{C}(q)$ and $v \in M^{\mathbb{A}}(\lambda)$ is clearly surjective. Let $\{F_\omega \cdot v_\lambda \mid \omega \in \Omega\}$ be the basis of $M^q(\lambda)$ determined by Theorem 3.4. Let $\xi : M^q(\lambda) \to \mathbb{C}(q) \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)$ be a $\mathbb{C}(q)$-linear map defined by

$$\xi(F_\omega \cdot v_\lambda) = 1 \otimes F_\omega \cdot v_\lambda.$$

Then, by Proposition 4.2, ξ is well-defined and the maps ζ and ξ are inverses. $\square$

We define a weight structure on $M^{\mathbb{A}}(\lambda)$ by setting $M^{\mathbb{A}}(\lambda)_\mu = M^{\mathbb{A}}(\lambda) \cap M^q(\lambda)_\mu$ for each $\mu \in P$.

Proposition 4.4. *$M^{\mathbb{A}}(\lambda)$ is a weight module with the weight decomposition $M^{\mathbb{A}}(\lambda) = \oplus_{\mu \in P} M^{\mathbb{A}}(\lambda)_\mu$.*

Proof. The proof is quite standard, as in [BKMe, Proposition 3.23]. $\square$

The vector-space isomorphism given above restricts to each weight space and we obtain the following result.

Proposition 4.5. *For each $\mu \in P$, $M^{\mathbb{A}}(\lambda)_\mu$ is a free $\mathbb{A}$-module and rank$_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu =$ $\dim_{\mathbb{C}(q)} M^q(\lambda)_\mu$.*

5. Classical limits.

In this section we take the classical limits of the $\mathbb{A}$-forms of the quantum imaginary Verma modules, and show that they are isomorphic to the imaginary Verma modules of $U(\mathfrak{g})$.

Recall that $\mathbb{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]_{q_i}}, i \in I, n > 0]$. Let $\mathbb{J}$ be the ideal of $\mathbb{A}$ generated by $q - 1$. Then there is an isomorphism of fields $\mathbb{A}/\mathbb{J} \cong \mathbb{C}$ given by $f + \mathbb{J} \mapsto f(1)$ for any $f \in \mathbb{A}$. For any untwisted affine Kac-Moody algebra $\mathfrak{g}$, let $U_{\mathbb{A}} = U_{\mathbb{A}}(\mathfrak{g})$, and set $U' = (\mathbb{A}/\mathbb{J}) \otimes_{\mathbb{A}} U_{\mathbb{A}}$. Then $U' \cong U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}}$. Denote by u' the image in U' of an element $u \in U_{\mathbb{A}}$. It was shown by Lusztig [Lu] and DeConcini and Kac [DK] that $(D')^2 = 1$ and $(K_i')^2 = 1$ for all $i \in I$. If we let K' denote the ideal of U' generated by $D' - 1$ and $\{K_i' - 1 \mid i \in I\}$, then $\overline{U} = U'/K' \cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

Note that, under the natural map $U_{\mathbb{A}} \to U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}} \cong U'$, we have $q \mapsto 1$. The composition of natural maps

$$U_{\mathbb{A}} \to U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}} \cong U' \to \overline{U} = U'/K' \cong U(\mathfrak{g}),$$

is called taking the classical limit of $U_{\mathbb{A}}$.

Let $\overline{u} \in \overline{U}$ denote the image of an element $u \in U_{\mathbb{A}}$. Then $\overline{U}$ is generated by the elements $\overline{E_i}, \overline{F_i}, \overline{D} := \overline{\begin{bmatrix} D \,;\, 0 \\ 1 \end{bmatrix}}$ and $\overline{H_i} := \overline{\begin{bmatrix} K_i \,;\, 0 \\ 1 \end{bmatrix}}$, $i \in I$, and, under the isomorphism between $U(\mathfrak{g})$ and $\overline{U}$, the elements e_i, f_i, d and h_i may be identified with $\overline{E_i}, \overline{F_i}, \overline{D}$ and $\overline{H_i}$, respectively. Further, Beck [Be1, Section 6] showed that we may identify the $\overline{E_\beta}$ with a PBW basis of $U(\mathfrak{g})$, with elements denoted e_β.

For $\lambda \in P$, let $M'(\lambda) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)$. Then $M'(\lambda) \cong M^{\mathbb{A}}(\lambda)/\mathbb{J}M^{\mathbb{A}}(\lambda)$ and $M'(\lambda)$ is a U'-module. For $\mu \in P$, let $M'(\lambda)_\mu = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu$. Since $M^{\mathbb{A}}(\lambda) = \oplus_{\mu \in P} M^{\mathbb{A}}(\lambda)_\mu$, we must have $M'(\lambda) = \oplus_{\mu \in P} M'(\lambda)_\mu$. We also have the following standard result

Proposition 5.1. *For $\mu \in P$, $\dim_{\mathbb{A}/\mathbb{J}} M'(\lambda)_\mu = rank_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu$.*

Proof. By Proposition 4.5, each weight space $M^{\mathbb{A}}(\lambda)_\mu$, $\mu \in P$, is a free $\mathbb{A}$-module. Let $\{v_j \mid j \in \Omega\}$ be a basis for $M^{\mathbb{A}}(\lambda)_\mu$. Then every element $v' \in M'(\lambda)_\mu = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)_\mu$ can be written uniquely as $v' = \sum_{j \in \Omega} a_j \otimes v_j$ for some scalars $a_j \in \mathbb{A}/\mathbb{J}$. (see [Hu, Chapter 4, Theorem 5.11]). Hence, $\{1 \otimes v_j \mid j \in \Omega\}$ is a basis for $M'(\lambda)_\mu$. □

Proposition 5.2. *The elements D' and K_i' $(i \in I)$ in U' act as the identity on the U' module $M'(\lambda) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M^{\mathbb{A}}(\lambda)$.*

Proof. Let $\mu \in P$ and $\{v_j \mid j \in \Omega\}$ be a basis of $M^{\mathbb{A}}(\lambda)_\mu$. Then by Proposition 5.1, $\{v_j' = 1 \otimes v_j \mid j \in \Omega\}$ is an $\mathbb{A}/\mathbb{J}$-basis for $M'(\lambda)_\mu$. Let $i \in I$. For each $j \in \Omega$, we have $K_i \cdot v_j = q_i^{\mu(h_i)} v_j$. Letting $q \mapsto 1$, we see $K_i' \cdot v_j' = v_j'$. Thus, K_i' acts on the identity on each weight space of $M'(\lambda)$ and, since $M'(\lambda)$ is a weight module, each K_i' acts as the identity on the whole space. Similarly, $D \cdot v_j = q_0^{\mu(d)} v_j$ implies that $D' \cdot v_j' = v_j$, and that D' acts as a scalar on $M'(\lambda)$. □

Since $M'(\lambda)$ is a U'-module, $\overline{M}(\lambda) = M'(\lambda)/K'M'(\lambda)$ is a $\overline{U} = U'/K'$-module. But K' was the ideal generated by $D' - 1$ and the $K_i' - 1$, and D' and each K_i' acts as the identity on $M'(\lambda)$, so $\overline{M}(\lambda) = M'$. Since $\overline{U} \cong U(\mathfrak{g})$, this means $\overline{M}(\lambda)$ has a $U(\mathfrak{g})$-structure. The module $\overline{M}(\lambda)$ is called the classical limit of $M^{\mathbb{A}}(\lambda)$. For $v \in M^{\mathbb{A}}(\lambda)$, let $\overline{v}$ denote the image of v in $\overline{M}(\lambda)$.

Proposition 5.3. *Let v_λ be the generating vector for $M^{\mathbb{A}}(\lambda)$. Then as a $U(\mathfrak{g})$-module, $\overline{M}(\lambda)$ is a weight module generated by $\overline{v_\lambda}$ and such that, for any $\mu \in P$, $\overline{M}(\lambda)_\mu$ is the μ-weight space of $\overline{M}(\lambda)$.*

Proof. Let v_λ generate $M^{\mathbb{A}}(\lambda)$, so that $M^{\mathbb{A}}(\lambda) = U_{\mathbb{A}} \cdot v_\lambda$. Then $\overline{M}(\lambda) = \overline{U} \cdot \overline{v_\lambda}$, so $\overline{v_\lambda}$ generates $\overline{M}(\lambda)$. As noted above, $M'(\lambda)$ is a U'-weight module and since $\overline{M}(\lambda) = M'(\lambda)$, $\overline{M}(\lambda)$ is also a weight module. Hence, $\overline{M}(\lambda) = \oplus_{\mu \in P} \overline{M}(\lambda)_\mu$. It remains to show that the vector space $\overline{M}(\lambda)_\mu$ is actually the μ-weight space of $\overline{M}(\lambda)$. That is, we have to show that $h_i \cdot \overline{v_\mu} = \mu(h_i)\overline{v_\mu}$ and $d \cdot \overline{v_\mu} = \mu(d)\overline{v_\mu}$ for all $i \in I$ and $\overline{v_\mu} \in \overline{M}(\lambda)_\mu$.

For $v_\mu \in M^{\mathbb{A}}(\lambda)_\mu$ and $i \in I$, we have

$$\begin{bmatrix} K_i \ ; \ 0 \\ 1 \end{bmatrix} = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \cdot v_\mu = \frac{q_i^{\mu(h_i)} - q_i^{-\mu(h_i)}}{q_i - q_i^{-1}} v_\mu .$$

Passing to the classical limit, we obtain

$$h_i \cdot \overline{v_\mu} = \overline{H_i} \cdot \overline{v_\mu} = \mu(h_i)\overline{v_\mu}.$$

Similarly, we have $d \cdot \overline{v_\mu} = \overline{D} \cdot \overline{v_\mu} = \mu(d)\overline{v_\mu}$. □

Proposition 5.4. *The $U(\mathfrak{g})$-module $\overline{M}(\lambda)$ is an S-highest weight, $U(\mathfrak{g}_{-S})$-free module.*

Proof. Let v_λ be a generating vector for $M^q(\lambda)$. By definition, $E_\beta \cdot v_\lambda = 0$ for all $\beta \in S$. Hence, $E_\beta \cdot v_\lambda = 0$ in the $\mathbb{A}$-form $M^{\mathbb{A}}(\lambda)$. Thus we have $\overline{E_\beta} \cdot \overline{v_\lambda} = 0$ in $\overline{M}(\lambda)$. By Proposition 5.3, $\overline{M}(\lambda)$ is a weight module generated by $\overline{v_\lambda}$ and $U(\mathfrak{g}_S)$ is spanned by the $\overline{E_\beta}$, $\beta \in S$, so $\overline{M}(\lambda)$ is an S-highest weight module.

It remains to show that $\overline{M}(\lambda)$ is a free $U(\mathfrak{g}_{-S})$-module. ¿¿From Proposition 4.2, we know that $M^{\mathbb{A}}(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta} \dots E_{-k\delta} \dots E_{-\alpha+k\delta}$, $\alpha \in \dot{\Delta}_+$, $n \geq 0$, $k > 0$. Hence, $\overline{M}(\lambda)$ is isomorphic to the space spanned by the ordered monomials $\overline{E_{-\alpha-n\delta}} \dots \overline{E_{-k\delta}} \dots \overline{E_{-\alpha+k\delta}}$. But monomials in the images $\overline{E_{-\alpha-n\delta}}$, $\overline{E_{-k\delta}}$ and $\overline{E_{-\alpha+k\delta}}$ form a basis for $U(\mathfrak{g}_{-S})$ and so $\overline{M}(\lambda)$ is a $U(\mathfrak{g}_{-S})$-free module. □

We have shown that, for any $\lambda \in P$, if we start with a quantum imaginary Verma module $M^q(\lambda)$, construct the $\mathbb{A}$-form $M^{\mathbb{A}}(\lambda)$ and take the classical limit, then the resulting module $\overline{M}(\lambda)$ is a $U(\mathfrak{g})$-module isomorphic to the imaginary Verma module $M(\lambda)$. We have also seen that the weight space structure is preserved under these operations, and so $M^q(\lambda)$ is a true quantum deformation of $M(\lambda)$.

Proposition 5.5. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Let $\lambda \in P$. Then the imaginary Verma module $M(\lambda)$ admits a quantum deformation to the quantum imaginary Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ in such a way that the weight space decomposition is preserved.*

6. Properties of quantum imaginary Verma modules.

Using the quantum deformation results obtained above, we are now in a position to prove some structural results about quantum imaginary Verma modules. Let G_q be the subalgebra of $U_q(\mathfrak{g})$ generated by the imaginary root vectors $E_{k\delta}^{(i)}$, $i \in \dot{I}$, $k \in \mathbb{Z} \setminus \{0\}$. Then G_q has a 1-dimensional center, which we will denote by C and a triangular decomposition $G_q = G_q^- C G_q^+$ where G_q^+ (resp. G_q^-) comprises ordered monomials in $E_{k\delta}^{(i)}$ $i \in \dot{I}$, $k \in \mathbb{Z}_+$ (resp. $-k \in \mathbb{Z}_+$). Let G be the subalgebra of $U(\mathfrak{g})$ generated by the imaginary root vectors $e_{k\delta}^{(i)}$, $i \in \dot{I}$, $k \in \mathbb{Z} \setminus \{0\}$. Then G is isomorphic to $\overline{G_q}$ and we will identify the two algebras.

Let $M^q(\lambda)$ be the quantum imaginary Verma module over $U_q(\mathfrak{g})$ with S-highest weight $\lambda \in P$ and generating vector v_λ. Consider the G_q-submodule of $M^q(\lambda)$ generated by v_λ, $H^q(\lambda) = G_q \cdot v_\lambda$.

Proposition 6.1. *The G_q-module $H^q(\lambda)$ is irreducible iff $\lambda(c) \neq 0$.*

Proof. The proof is similar to the affine case.

Proposition 6.2. *Let $x \in U_q(\mathfrak{g})$ be such that $x \cdot v_\lambda \neq 0$. Then $U_q(\mathfrak{g})x \cdot v_\lambda \cap H^q(\lambda) \neq 0$.*

Proof. We recall the basis of $M^q(\lambda)$ constructed in Theorem 3.4 and write $x \cdot v_\lambda \in M^q(\lambda)$ in this basis. It is enough to consider homogeneous x. Set $N(x \cdot v_\lambda)$ to be minus the sum of the heights of the finite roots in the decomposition of $x \cdot v_\lambda$ (i.e., each $-\alpha + k\delta$ contributes $\mathrm{ht}(\alpha)$). Then it is clear that $N(x \cdot v_\lambda) = 0$ if and only if $x \in G_q$.

It is enough to show that there exists $y \in U_q(\mathfrak{g})$ such that $yx \cdot v_\lambda \neq 0$ and $N(yx \cdot v_\lambda) < N(x \cdot v_\lambda)$. We will find a n element $y = E_{\alpha - K\delta}$ where K is sufficiently large and α is some suitable root.

Let $w = x \cdot v_\lambda \in M^q(\lambda)$. Then, as in Proposition 4.3, we may write $w = fw'$ for some $f \in \mathbb{C}(q)$ and $w' \in M^{\mathbb{A}}(\lambda)$. Then $w' = f^{-1}w = f^{-1}x \cdot v_\lambda \in U_q \cdot w$.

Furthermore, suppose $w' = (q-1)^k w''$, with $k > 0$ and $w'' \in M^{\mathbb{A}}(\lambda)$. Then $w'' = (q-1)^{-k} w' \in U_q \cdot w$. Hence, without loss of generality, we may assume $w = x \cdot v_\lambda \in M^{\mathbb{A}}(\lambda)$ and $q-1$ is not a factor of w in $M^{\mathbb{A}}(\lambda)$. Taking the classical limit, we then have $\overline{w} = \overline{x \cdot v_\lambda} \neq 0$.

Suppose $\overline{x}$ is in G. Since x is homogeneous, the grading of $M^q(\lambda)$ ensures that $x \cdot v_\lambda$ is in $H^q(\lambda)$. Suppose that $\overline{x}$ is not in G. Then by [Fu3, Lemma 1] there exists a root α and nonnegative integer K such that $e_{\alpha - K\delta}\overline{x \cdot v_\lambda} \neq 0$. Note that $e_{\alpha-K\delta}$ is the image of $E_{\alpha-K\delta}$. Hence, $E_{\alpha-K\delta} x \cdot v_\lambda \neq 0$ and $N(E_{\alpha-K\delta} x \cdot v_\lambda) < N(x \cdot v_\lambda)$. We complete the proof by induction. □

Corollary 6.3. *$M^q(\lambda)$ is irreducible iff $\lambda(c) \neq 0$.*

Proof. Follows immediately from Propositions 6.1 and 6.2.

¿¿From now on we will assume that $\lambda(c) = 0$. In this case $H^q(\lambda)$ is a reducible G_q-module with maximal submodule $H_0^q(\lambda)$ consisting of all spaces except $\mathbb{C}v_\lambda$.

Denote $M_0^q(\lambda) = U_q(\mathfrak{g}) H_0^q(\lambda)$. We remark that $M_0^q(\lambda) \neq M^q(\lambda)$.

Set $\widetilde{M^q(\lambda)} = M^q(\lambda)/M_0^q(\lambda)$.

Theorem 6.4. *The $U_q(\mathfrak{g})$-module $\widetilde{M^q(\lambda)}$ is irreducible if and only if $\lambda(h_i) \neq 0$, for all $i = 1, \ldots, N$.*

Proof. Let $\widetilde{M^q(\lambda)}$ be irreducible and suppose there exists some $i \in \{1, \ldots, N\}$ such that $\lambda(h_i) = 0$. Set $\alpha = \alpha_i$ and $E = E_{-\alpha}$. We have that $\widetilde{M^q(\lambda)} = U_q(\mathfrak{g}) \cdot v_\lambda$. Consider $W = U_q E \cdot v_\lambda$. We show W is a proper submodule of $\widetilde{M^q(\lambda)}$.

Since $E \cdot v_\lambda \neq 0$, $W \neq (0)$. Suppose $W = \widetilde{M^q(\lambda)}$. Then there must exist elements m_j in U_q such that $v_\lambda = \sum_j m_j E \cdot v_\lambda$, and for each m_j, we have $\mathrm{ht}(m_j) = \alpha$. (Recall the height of a monomial is the sum of heights of finite roots involved.) Using Beck's ordering and the notation introduced in the proof of Theorem 3.4, we may write each m_j in the form $m_j = Y_1 Y_2 Y_3 Z X_3 X_2 X_1$. We consider the actions of the m_j on $E \cdot v_\lambda$. We need the following lemma.

Lemma 6.5. *For any $k \in \mathbb{Z}$ and $\beta \in \dot{\Delta}_+$, $E_{-\beta+k\delta} E \cdot v_\lambda = 0$.*

Proof. We divide the proof into cases.

1. $k \in \mathbb{Z}$, $\beta \neq \alpha$.

Using Beck's basis, we write $E_{\beta+k\delta}E\cdot v_\lambda = \sum_i n_i^- n_i^+ v_\lambda$ where n_i^+ are ordered monomials of the form $X_3X_2X_1$ and n_i^- are ordered monomials of the form $Y_1Y_2Y_3$. Since $\beta \neq \alpha$ and α is simple, then, due to the convexity of the basis, this ordered expression must contain on the right an element of the form X_1 or Y_3 (depending on the sign of k). But $X_1 \cdot v_\lambda = Y_3 \cdot v_\lambda = 0$. Hence $E_{\beta+k\delta}E \cdot v_\lambda = 0$.

2. $k \in \mathbb{Z} \setminus \{0\}$, $\beta = \alpha$.

By [BK, (1.6.5d)], $E_{\alpha+k\delta}E \cdot v_\lambda = X_{k\delta} \cdot v_\lambda = 0$ for some suitable vector $X_{k\delta}$ of weight $k\delta$, and this has a trivial action in $\widetilde{M^q(\lambda)}$.

3. $k = 0$, $\beta = \alpha$.

In this case $E_\alpha E_{-\alpha} = 0$ because $\lambda(h_i) = 0$.

In all cases, we have $E_{-\beta+k\delta}E \cdot v_\lambda = 0$. □

Return to Proof of Theorem.

Consider the action of an element $Y_1Y_2Y_3ZX_3X_2X_1E \cdot v_\lambda$. By Lemma 6.5, $X_1E \cdot v_\lambda = 0$. Now $X_2 \cdot v_\lambda = 0$, and so, [BK, (1.6.5b)], $X_2E \cdot v_\lambda$ is in the space of elements of the form $X_3 \cdot v_\lambda$. Elements of the form Z commute with elements of the form X_3 and act as scalars on v_λ. As shown in Theorem 3.4, we may commute elements of the form Y_3 with elements of the form X_3 and then $Y_3 \cdot v_\lambda = 0$. Hence we are left considering elements of the form $Y_1Y_2X_3 \cdot v_\lambda$. The monomial is non-zero as it contains E and has height 0 as we supposed $\sum m_jE \cdot v_\lambda = v_\lambda$. This is not possible and we have a contradiction. Hence W is a proper submodule of $\widetilde{M^q(\lambda)}$ and $\widetilde{M^q(\lambda)}$ is reducible.

Now we prove the converse statement of the theorem. Let $\lambda(h_i) \neq 0$, $i = 1, \ldots, N$. Let $\widetilde{M^{\mathbb{A}}(\lambda)} = \widetilde{M^q(\lambda)} \cap M^\alpha(\lambda)$ denote the $\mathbb{A}$-form of $\widetilde{M^q(\lambda)}$, and let $\widetilde{M(\lambda)}$ denote its classical limit. Suppose N^q is a proper submodule of $\widetilde{M^q(\lambda)}$. Then $N^{\mathbb{A}} = N^q \cap \widetilde{M^{\mathbb{A}}(\lambda)}$ is a submodule of $\widetilde{M^{\mathbb{A}}(\lambda)}$ and the classical limit of $N^{\mathbb{A}}$ gives a proper submodule of $\widetilde{M(\lambda)}$. But this last module is irreducible by [Fu3]. The Theorem is proved. □

Proposition 6.6. *Let $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$ for all i. Then $M^q(\lambda)$ has an infinite filtration with irreducible quotients $\widetilde{M^q(\lambda + k\delta)}$, $k \geq 0$.*

Proof. Follows from theorem above.

Corollary 6.7. *Let $\lambda(c) = 0$, $\lambda(h_i) \neq 0$, $i = 1, \ldots, N$ and N^q be a submodule of $M^q(\lambda)$. Then N^q is generated by $N^q \cap H^q(\lambda)$.*

Proof. Obvious from Proposition 6.6.

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