Simple classical Lie algebras in characteristic 2 and their gradations, II.

A. N. Grishkov[1]
Rua do Matão, 1010 - Butantã
CEP 05508-900 - São Paulo-SP - Brazil
grishkov@ime.usp.br

M. Guerreiro[2]
Departamento de Matemática - UFV
CEP 36571-000 - Viçosa-MG - Brazil
marines@ufv.br

1 Introduction

This paper is a continuation of [GG]. Here we prove Conjecture 5.1 [GG]. Recall some notations and definition of [GG].

Definition 1.1. *Let* $I_n = \{1, \ldots, n\}$. *We call* $\mathfrak{a} \subset \mathcal{P}(I_n) = \{\sigma \,|\, \sigma \subseteq I_n\}$ *an* **even** *set if for all* $\sigma, \tau \in \mathfrak{a}$, *we have* $|\sigma| \equiv |\tau| \equiv 0$ *and* $|\sigma \cap \tau| \equiv 0 \mod 2$.

We note that $\mathcal{P}(I_n)$ is an elementary abelian group with the operation $\sigma \triangle \tau = (\sigma \setminus \tau) \cup (\tau \setminus \sigma)$. For $\mathfrak{a} \subseteq \mathcal{P}(I_n)$, $< \mathfrak{a} >$ denotes the group generated by $\mathfrak{a}$.

Definition 1.2. *A subset* H *of* $\mathcal{P}(I_n)$ *is* **connected** *if, for every partition* $I_n = I \cup J$, *there is* $\sigma \in H$ *such that* $\sigma \cap I \neq \emptyset$ *and* $\sigma \cap J \neq \emptyset$.

Definition 1.3. *A subset* $\sigma \subseteq I_n$ *is called* $\mathfrak{a}$**-even** *if* $|\mu \cap \tau| \equiv 0 \mod 2$ *for all* $\tau \in \mathfrak{a}$. *A subset* $B \subseteq \mathcal{P}(I_n)$ *is called an* $\mathfrak{a}$**-even set** *if all its elements are* $\mathfrak{a}$**-even**.

For an even set $\mathfrak{a} \subset \mathcal{P}(I_n)$, in [GG] we defined a commutative algebra $\tilde{S} = \tilde{S}(\mathfrak{a})$ with basis $\{\, e_i,\ h_i,\ f_i,\ h^\sigma \,|\, i \in I_n,\ \sigma \in < \mathfrak{a} > \setminus \emptyset \,\}$ and multiplication given

[1]Supported by FAPEMIG as visitor to the Depto. de Matemática - UFV in February 2001
[2]Supported by FAPESP as visitor to IME-USP in February 2000

by

$$
\begin{aligned}
& e_i f_i = h_i, \\
& e_i h^\sigma = e_i, \quad f_i h^\sigma = f_i, \qquad \text{for } i \in \sigma\,,
\end{aligned} \tag{1}
$$

and zero for all other cases. We denoted $h_\mu = \sum_{i\in\mu} h_i$ and defined $h_\emptyset = 0$, $h^\emptyset = 0$.

The algebra $\tilde{S}(\mathfrak{a})$ contains a central ideal I generated by $\{h^\sigma + h^\tau + h^{\sigma\triangle\tau} + h_{\sigma\cap\tau} \,|\, \sigma,\, \tau \in <\mathfrak{a}>\}$. We denote $S(\mathfrak{a}) = S = \tilde{S}(\mathfrak{a})/I$.

For every $\mathfrak{a}$-even set σ, we also defined an S-module Λ_σ whose basis is $\{(\sigma, \mu) \,|\, \mu \subseteq \sigma\}$ and the S-action is given by

$$
\begin{aligned}
(\sigma,\, \mu)\, e_i &= (\sigma,\, \mu \cup i), \quad i \in \sigma \setminus \mu; \\
(\sigma,\, \mu)\, f_i &= (\sigma,\, \mu \setminus i), \quad i \in \mu; \\
(\sigma,\, \mu)\, h_i &= (\sigma,\, \mu), \quad i \in \sigma; \\
(\sigma,\, \mu)\, h^\varphi &= \left(\frac{|\sigma\cap\varphi|}{2} + |\varphi\cap\mu|\right)(\sigma,\, \mu), \ \text{ for } \varphi \in \mathfrak{a},
\end{aligned} \tag{2}
$$

and for all other cases the action is zero.

Now let $\Delta = \{0\} \cup \mathfrak{a}$.

Definition 1.4. *An algebra A is called a* **Δ-algebra** *if $A = \sum_{\alpha\in\Delta} \oplus A_\alpha$ and, for every $\alpha \neq \beta \in \mathfrak{a}$, we have $A_\alpha A_\beta \subseteq A_{\alpha\triangle\beta}$, $A_0^2 \subseteq A_0$, $A_0 A_\alpha \subseteq A_\alpha$, $A_\alpha A_\alpha \subseteq A_0 + A_\emptyset$ and $A_0 A_\emptyset = 0$.*

Define a commutative Δ-graded algebra Λ as follows. As a k-space, Λ is

$$
\Lambda = \Lambda_0 \bigoplus \sum_{\sigma\in\mathfrak{a}} \oplus \Lambda_\sigma\,, \quad \text{where } \Lambda_0 = S(\mathfrak{a})\,. \tag{3}
$$

Moreover, $S = S(\mathfrak{a})$ is a subalgebra of Λ and, by (2), each Λ_σ is an S-module. For $\sigma \neq \tau \in \mathfrak{a}$, the multiplication is given by

$$
(\sigma,\, \mu)\,(\tau,\, \varphi) = (\sigma\triangle\tau,\, (\mu\setminus\tau) \cup (\varphi\setminus\sigma)), \ \text{ if } \mu\cap\varphi = \emptyset,\ \mu\cup\varphi \supset \sigma\cap\tau\,. \tag{4}
$$

$$(\sigma,\,\mu)\,(\sigma,\,\varphi)\;=\;\begin{cases} e_i, & \mu\cap\varphi=i,\ \mu\cup\varphi=\sigma,\\ f_i, & \mu\cap\varphi=\emptyset,\ \mu\cup\varphi=\sigma\setminus i,\\ h^\sigma\,+\,h_\varphi\,+\,(\emptyset,\emptyset), & \mu\cap\varphi=\emptyset,\ \mu\cup\varphi=\sigma\,,\end{cases} \tag{5}$$

and all other products are zero.

Recall the definition of the product of two Δ-algebras. Let $A\;=\;\sum_{\alpha\in\Delta}\oplus A_\alpha$ and $B\;=\;\sum_{\alpha\in\Delta}\oplus B_\alpha$ be two Δ-algebras. Then $A\Box B\;=\;\sum_{\alpha\in\Delta}\oplus\,A_\alpha\otimes B_\alpha$ is a Δ-algebra with multiplication $[\,\cdot\,,\,\cdot\,]$ given by

$$[a_\alpha\otimes b_\alpha,\,a_\beta\otimes b_\beta]\;=\;\sum_{\gamma\in\Delta}c_\gamma\otimes d_\gamma\,,\qquad \text{if }\ a_\alpha a_\beta\;=\;\sum_{\gamma\in\Delta}c_\gamma\,,\ b_\alpha b_\beta\;=\;\sum_{\gamma\in\Delta}d_\gamma.$$

Proposition 1.1. *Let $\mathfrak{a}$ be an even set, $\Lambda\,=\,\Lambda(\mathfrak{a})$ and $\Delta\,=\,\{0\}\cup\mathfrak{a}$. Let $M\,=\,M_0\oplus\sum_{\sigma\in\mathfrak{a}}\oplus M_\sigma$ be a commutative Δ-algebra. Then the algebra $L\,=\,\Lambda\Box M$ is a Lie algebra if and only if M satisfies a list of Δ-identities given in Proposition 3.2 [GG].*

We recall some of the Δ-identities which will be used in this paper.

$$a_\sigma b_\tau\cdot c_\lambda\,+\,b_\tau c_\lambda\cdot a_\sigma\,+\,c_\lambda a_\sigma\cdot b_\tau=0, |\sigma\cap\tau\cap\lambda|=0, \sigma\neq\tau\neq\lambda\neq\sigma\neq\tau\triangle\lambda, \tag{6}$$

$$(a_\sigma b_\sigma)_\emptyset c_\tau\;=\;0,\ (a_\sigma b_\sigma)_0 c_\tau\;=\;a_\sigma c_\tau\cdot b_\sigma, \qquad \sigma\neq\tau,\ |\sigma\cap\tau|=2, \tag{7}$$

$$a_\sigma c_\tau\cdot b_\sigma\;=\;a_\sigma\cdot c_\tau b_\sigma\,+\,(a_\sigma b_\sigma)_\emptyset c_\tau\,, \qquad |\sigma\cap\tau|=0, \tag{8}$$

$$(a_\sigma b_\tau\cdot c_\lambda)_0\;=\;(a_\sigma\cdot b_\tau c_\lambda)_0, \qquad \lambda=\sigma\triangle\tau, \tag{9}$$

$$(ab)_0 c\;=\;(cb)_0 a,\ \ (ca)_\emptyset b\;=\;0, \qquad a,b,c\in M_\tau,\ |\tau|=4, \tag{10}$$

$$(ab)_0 c\,+\,(bc)_0 a\;=\;(ac)_\emptyset b, \qquad a,b,c\in M_\tau,\ |\tau|=2, \tag{11}$$

$$(a_\sigma b_\sigma)_0 c_\tau\;=\;0, \qquad |\sigma|>4, \tag{12}$$

$$(a_\emptyset b_\sigma\cdot c_\sigma)_\emptyset\,+\,(b_\sigma c_\sigma)_\emptyset\cdot a_\emptyset\,+\,(c_\sigma a_\emptyset\cdot b_\sigma)_\emptyset\;=\;0 \tag{13}$$

$$(a_\sigma b_\sigma)_0\cdot c_0\;=\;(a_\sigma c_0\cdot b_\sigma)_0\,, \qquad \sigma\neq\emptyset \tag{14}$$

We observe that if M is simple, then $L\,=\,\Lambda\Box M$ is not necessarily a simple algebra, but $L/Z(L)$ is simple, where $Z(L)$ is the center of L.

Let $\mathfrak{a}$ be an even connected set and $\Delta = \{0\} \cup \mathfrak{a}$. Let $\mathcal{M}$ be the variety of Δ-algebras satisfying the list of identities of Proposition 3.2 [GG]. Let $M = M_0 \oplus \sum_{\sigma\in\mathfrak{a}} \oplus M_\sigma \oplus M_\emptyset$ be a commutative Δ-algebra in $\mathcal{M}$. In [GG] (see Theorem 3.1 [GG]) we classified the simple Δ-algebras of the variety $\mathcal{M}$, for which $M_\emptyset = 0$. Now we consider the case when $M_\emptyset$ is abelian.

In the final section of [GG], we remarked that Theorem 3.1 [GG] is not true if we omit the condition $\emptyset \notin \mathfrak{a}$ and we formulated the following conjecture.

Conjecture 1.1. *Let M be an arbitrary simple finite dimensional $\triangle$-algebra which satisfies all the list of identities of Proposition 3.2 [GG] and $M_\emptyset^2 = 0$. Then the corresponding Lie algebra $L = M\square\Lambda$ is a simple Lie algebra of type $B_{2\ell}$, C_ℓ, $D_{2\ell+1}$, E_7 or E_8.*

2 Proof of Conjecture 1.1

In this section we prove Conjecture 1.1. For each $\emptyset \neq \sigma \in \mathfrak{a}$, define $M_\sigma^0 = \{x \in M_\sigma \,|\, xM_\sigma \subseteq M_\emptyset\} = \{x \in M_\sigma \,|\, (xM_\sigma)_0 = 0\}$.

Lemma 2.1. $I = \displaystyle\sum_{\sigma\in\mathfrak{a}\setminus\emptyset} \oplus M_\sigma^0 \bigoplus \sum_{\sigma\in\mathfrak{a}\setminus\emptyset} (M_\sigma M_\sigma^0)$ *is an ideal in M.*

Proof. (a) First we prove that $M_\tau M_\sigma^0 \subseteq M_{\sigma\triangle\tau}^0$, for all $\sigma \neq \tau \in \mathfrak{a}$. Indeed, by (9), for $a_\tau \in M_\tau,\ b_\sigma \in M_\sigma^0,\ c_\lambda \in M_{\sigma\triangle\tau}^0$, we have $(a_\tau\, b_\sigma \cdot c_\lambda)_0 = (b_\sigma \cdot a_\tau\, c_\lambda)_0 = 0$.

(b) Now we prove that $(M_\sigma M_\sigma^0)M_\tau \subseteq M_\tau^0$, for all $\tau \in \mathfrak{a}$. We need to prove that $(((b_\sigma\, c_\sigma)_\emptyset\, a_\tau)d_\tau)_0 = 0$ for all $a_\tau \in M_\tau,\ b_\sigma \in M_\sigma^0,\ c_\sigma \in M_\sigma,\ d_\tau \in M_\tau$. We have two cases:

(b.1) $\sigma \neq \tau$. If $|\,\sigma\cap\tau\,| = 2$, we have by (7) that $(((b_\sigma\, c_\sigma)_\emptyset\, a_\tau)d_\tau)_0 = 0$. If $|\,\sigma\cap\tau\,| = 0$ then, by (8) and (9), $(((b_\sigma\, c_\sigma)_\emptyset\, a_\tau)d_\tau)_0 = ((b_\sigma\, a_\tau \cdot c_\sigma)d_\tau)_0 + ((c_\sigma\, a_\tau \cdot b_\sigma)d_\tau)_0 = (b_\sigma\, a_\tau \cdot c_\sigma\, d_\tau)_0 + (b_\sigma \cdot (c_\sigma\, a_\tau \cdot d_\tau))_0 = (b_\sigma \cdot a_\tau(c_\sigma\, d_\tau))_0 = 0.$

(b.2) $\sigma = \tau$. If $\mid \sigma \mid = 2$, then by (11) we have $(((b_\sigma\, c_\sigma)_\emptyset\, a_\sigma)d_\sigma)_0 = ((b_\sigma\, a_\sigma)_0\, c_\sigma)d_\sigma)_0 + ((c_\sigma\, a_\sigma)_0\, b_\sigma)d_\sigma)_0 \subseteq k\,(b_\sigma\, d_\sigma)_0 = 0$, as $b_\sigma \in M_\sigma^0$. If $\mid \sigma \mid = 4$, then by (10) $(((b_\sigma\, c_\sigma)_\emptyset\, a_\sigma)d_\sigma)_0 = 0$. This proves the lemma. □

By Lemma 2.1, if M is simple then $I = 0$ and, for each $\sigma \neq \emptyset$, $M_\sigma^0 = 0$.

Lemma 2.2. *For a simple algebra M as defined above and $\sigma \in \mathfrak{a}$, we have*

0. $M_0 = k\,s$, for $s^2 = s$.

1. If $\mid \sigma \mid = 4$ then $M_\sigma = k\,a_\sigma$ where $a_\sigma^2 = s$.

2. If $\mid \sigma \mid = 2$ then

2.1) $M_\sigma = k\,a_\sigma$, where $(a_\sigma^2)_0 = s$, or

2.2) $M_\sigma = k\,a_\sigma \oplus k\,b_\sigma$, where $(a_\sigma^2)_0 = (b_\sigma^2)_0 = s$ and $(a_\sigma\, b_\sigma)_0 = 0$ or

2.3) $M_\sigma = k\,a_\sigma \oplus k\,b_\sigma$, where $(a_\sigma\, b_\sigma)_0 = s$ and $(a_\sigma^2)_0 = (b_\sigma^2)_0 = 0$.

Proof. The proof of item 0. is the same as in Lemma 3.1 of [GG].

Let $\mid \sigma \mid = 4$ and $a_\sigma \in M_\sigma$. By Lemma 2.1, there exists $b_\sigma \in M_\sigma$ such that $(a_\sigma\, b_\sigma)_0 = s$, then we have on the one hand $(a_\sigma\, b_\sigma)_0\, a_\sigma = a_\sigma$ and on the other hand, by (10), $(a_\sigma\, b_\sigma)_0\, a_\sigma = (a_\sigma\, a_\sigma)_0\, b_\sigma = \alpha\, b_\sigma$, if $(a_\sigma\, a_\sigma)_0 = \alpha\, s$. Hence $a_\sigma = \alpha\, b_\sigma$. If $c \in M_\sigma$ and $(b\,c)_0 = \gamma\, s$, then by (10), $c = (a\,b)_0\, c = (b\,c)_0\, a = \gamma\, a$. Hence $\dim M_\sigma = 1$ and so item 1. is proved.

Now let $\mid \sigma \mid = 2$.

(a) There exists $a \in M_\sigma$ such that $(a^2)_0 = s$. If $\dim M_\sigma = 1$, then we have case 2.1. Suppose that there exists $b \in M_\sigma \setminus k\,a$. If $(a\,b)_0 = \alpha s \neq 0$ then we can replace b by $b + \alpha a = \tilde{b}$ and we get $(\tilde{b}\,a\,)_0 = 0$. Hence we can suppose that b satisfies $(a\,b)_0 = 0$.

(a.1) Suppose that for all $a \in M_\sigma$ such that $(a\,b)_0 = 0$ we have $(b^2)_0 = 0$. By Lemma 2.1, there exists $c \in M_\sigma$ such that $(c\,b)_0 = s$. We can suppose that $(c\,a)_0 = 0$ (by replacing c by $c + \alpha a = \tilde{c}$ as before). Now, using identity (11), we get $(ab)_\emptyset\, c = (cb)_0\, a + (ac)_0\, b = a$, $(bc)_\emptyset\, a = (ba)_0\, c + (ca)_0\, b = 0$ and

$(bc)_\emptyset\, c \,=\, (bc)_0\, c \,+\, (cc)_0\, b \,=\, c$. Hence, $[(ab)_\emptyset,\, (bc)_\emptyset]\, c \,=\, a \neq 0$, contradicting the fact that $M_\emptyset$ is abelian.

(a.2) There exists $b \in M_\sigma$ such that $(b^2)_0 \,=\, s$ and $(a\, b)_0 \,=\, 0$. If $\dim M_\sigma \,=\, 2$ then item 2.2 is proved.

Suppose that $\dim M_\sigma \,>\, 2$. By Lemma 2.1, there exists $c \in M_\sigma$ such that $(a\, c)_0 \,=\, (b\, c)_0 \,=\, 0$.

(a.3) If $(c^2)_0 \,=\, s$ then by (11), $(ac)_\emptyset\, c \,=\, (cc)_0\, a \,=\, a$, $(ab)_\emptyset\, c \,=\, 0$ and $(ab)_\emptyset\, a \,=\, b$. Hence $[(ac)_\emptyset,\, (ab)_\emptyset]\, c \,=\, b \neq 0$, contradicting the fact that $M_\emptyset$ is abelian.

(a.4) Suppose that for all $c \in M_\sigma$ such that $(a\, c)_0 \,=\, (b\, c)_0 \,=\, 0$ we have $(c^2)_0 \,=\, 0$. By Lemma 2.1, there exists $d \in M_\sigma$ such that $(d^2)_0 \,=\, 0$ and $(c\, d)_0 \,=\, s$. Then, by identity (11), $(ab)_\emptyset\, a \,=\, b$, $(ab)_\emptyset\, d \,=\, 0$ and $(ac)_\emptyset\, d \,=\, a$. Hence $[(ab)_\emptyset,\, (ac)_\emptyset]\, d \neq 0$ and again the fact that $M_\emptyset$ is abelian is contradicted.

(b) For all $a \in M_\sigma$, $(a^2)_0 \,=\, 0$. By Lemma 2.1, there exist $a, b \in M_\sigma$ such that $(a^2)_0 \,=\, (b^2)_0 \,=\, 0$ and $(a\, b)_0 \,=\, s$. If $\dim M_\sigma \,=\, 2$, then we have case 2.3.

If $\dim M_\sigma \,>\, 2$, then by Lemma 2.1 there exist $c, d \,\in\, M_\sigma$ such that $(a\, c)_0 \,=\, (a\, d)_0 \,=\, (b\, c)_0 \,=\, (b\, d)_0 \,=\, (c^2)_0 \,=\, (d^2)_0 \,=\, 0$ and $(c\, d)_0 \,=\, s$. In this case, by (11), $(ab)_\emptyset\, a \,=\, a$ and $(ac)_\emptyset\, d \,=\, a$. Hence $[(ab)_\emptyset,\, (ac)_\emptyset]\, d \,=\, a \neq 0$, contradicting the fact that $M_\emptyset$ is abelian. This proves the lemma. $\square$

Lemma 2.3. *Let $\mathfrak{a} \subset \mathcal{P}(I_n)$ be an even set and $\Delta \,=\, \{0\} \cup \mathfrak{a}$. Let $\mathcal{M}$ be the variety of Δ-algebras satisfying the list of identities of Proposition 3.2 [GG]. If $M \in \mathcal{M}$ is a simple Δ-algebra (containing no graded ideals), then $\mathfrak{M} \,=\, \{\sigma \in \mathfrak{a} \,|\, M_\sigma \neq 0\}$ is one of the following sets:*

(i) $\{(2i-1,\, 2i,\, 2j-1,\, 2j) \,|\, 1 \leq i < j \leq \ell\} \,=\, \mathcal{C}_{2\ell}$,

(ii) $\{(2i-1,\, 2i,\, 2j-1,\, 2j),\ (2i-1,\, 2i) \,|\, 1 \leq i < j \leq \ell\} \,=\, \mathcal{B}_{2\ell}$,

(iii) $\{\, (1234),\, (1256),\, (1357),\, (3456),\, (2457),\, (2367),\, (1467) \,\} \,=\, \mathcal{E}_7$,

(iv) $\mathcal{E}_7 \cup \{\, \overline{\sigma} \,|\, \sigma \in \mathcal{E}_7,\ \overline{\sigma} \,=\, I_8 \setminus \sigma \,\} \,=\, \mathcal{E}_8$.

Proof. The proof of this lemma in [GG] is based on the following facts:

(1) for all $\sigma \in \mathfrak{M}$, we have $|\sigma| = 2$ or 4.

(2) If $\sigma \neq \tau \in \mathfrak{M}$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \triangle \tau \in \mathfrak{M}$.

The item (2) may be proved as in [GG]. Let us prove item (1). Suppose that $\sigma \in \mathfrak{M}$ and $|\,\sigma\,| > 4$. Thus, by (12), $(M_\sigma\, M_\sigma)_0\, M_\sigma = 0$, hence $(M_\sigma\, M_\sigma)_0 = 0$ and $M_\sigma = M_\sigma^0$. But by Lemma 2.1, $M_\sigma^0 = 0$. □

Theorem 2.1. *Let $M \in \mathcal{M}$ be a simple Δ-algebra such that $M_\emptyset \neq 0$ and $M_\emptyset^2 = 0$. Then $\mathfrak{M} = \mathcal{B}_{2\ell}$ and M has a basis*

$$\{\, s,\, d_{ij},\, a_i,\, b_i,\, \lambda \,|\, 1 \leq i < j \leq \ell \,\}$$

with one of the following set of multiplication rules:

$$\begin{aligned} d_{ij}\, d_{jk} &= d_{ik}, & d_{ij}\, a_j &= b_i, \\ d_{ij}\, b_j &= a_i, & a_i\, b_j &= d_{ij}, \\ (a_i\, b_i)_\emptyset &= \lambda, & \lambda\, a_i &= b_i, \\ \lambda\, b_i &= a_i, & (d_{ij}^2)_0 &= s, \\ (a_i^2)_0 &= (b_i^2)_0 = s \end{aligned} \tag{15}$$

or

$$\begin{aligned} d_{ij}\, d_{jk} &= d_{ik}, & d_{ij}\, a_i &= a_j, \\ d_{ij}\, b_j &= b_i, & a_i\, b_j &= d_{ij}, \\ a_i\, b_i &= s + \lambda, & (d_{ij}^2)_0 &= s, \\ \lambda\, a_i &= a_i, & \lambda\, b_i &= b_i, \end{aligned} \tag{16}$$

where $M_{(2i-1,2i,2j-1,2j)} = k\, d_{ij}$, $M_{(2i-1,2i)} = k\, a_i \oplus k\, b_i$ and $M_\emptyset = k\, \lambda$.

Proof. If $\mathfrak{M} = \mathcal{B}_{2\ell} = \{(2i-1,\, 2i,\, 2j-1,\, 2j),\ (2i-1,\, 2i) \,|\, 1 \leq i < j \leq \ell\}$, then by Lemma 2.2 for $\sigma = (2i-1,\, 2i) \in \mathfrak{M}$ we have three cases

(a) $M_\sigma = k\, a_\sigma$, where $(a_\sigma^2)_0 = s$, or

(b) $M_\sigma = k\,a_\sigma \oplus k\,b_\sigma$, where $(a_\sigma^2)_0 = (b_\sigma^2)_0 = s$ and $(a_\sigma\,b_\sigma)_0 = 0$ or

(c) $M_\sigma = k\,a_\sigma \oplus k\,b_\sigma$, where $(a_\sigma\,b_\sigma)_0 = s$ and $(a_\sigma^2)_0 = (b_\sigma^2)_0 = 0.$

Let us consider each case.

(a) For $\mid \sigma \mid = 2$, by identity (11), we have $(a_\sigma\,a_\sigma)_\emptyset\,a_\sigma = 2\,(a_\sigma^2)_0 a_\sigma = 0$ and by (7), for $\mid \sigma \mid = \mid \tau \mid = 2$ with $\sigma \cap \tau = \emptyset$ $(a_\sigma\,a_\sigma)_\emptyset\,a_\tau = 2\,a_\sigma\,a_\tau \cdot a_\sigma = 0$. If $\mid \sigma \mid = 2$ and $\mid \sigma \cap \tau \mid = 2$ with $\sigma \neq \tau$, then by (7) $(a_\sigma\,a_\sigma)_\emptyset\,c_\tau = 0$. Therefore, $(a_\sigma\,a_\sigma)_\emptyset \in Z(M) = 0$. Let $\mu \in \mathfrak{M}$ such that $M_\mu = k\,d$ and $\sigma \subseteq \mu$, $\tau = \mu \setminus \sigma$. Denote $b_\tau = d\,a_\sigma$. If $c \in M_\tau$ then, by (7), $d\,c \cdot d = c(dd)_0 = c$. But $d\,c \in M_\sigma = k\,a_\sigma$. Thus $c = d\,c \cdot d = \alpha\,a_\sigma\,d = \alpha\,b_\tau$ and $M_\tau = k\,b_\tau$. In this case, M is the algebra obtained in [GG].

(b) Let $d = d_{12} \in M_{(1234)}$ and denote

$$b_2 = d\,b_1, \qquad a_2 = d\,a_1, \qquad a_1\,a_2 = \alpha\,d, \qquad b_2\,b_1 = \beta\,d$$

By (7), $d\,b_2 = d \cdot d\,b_1 = (d\,d)_0 b_1 = b_1$ and $d\,a_2 = d \cdot d\,a_1 = (d\,d)_0 a_1 = a_1$.

Now, by (9), we have $(b_2\,b_2)_0 = (d\,b_1 \cdot b_2)_0 = (d \cdot b_1\,b_2)_0 = (d\,b_2 \cdot b_1)_0 = s$. Hence, $\beta = 1$. Moreover, $(a_2\,a_2)_0 = (d\,a_1 \cdot a_2)_0 = (d\,a_2 \cdot a_1)_0 = (d \cdot a_1\,a_2)_0 = s$. Hence, $\alpha = 1$.

Again by (9), $(b_2\,a_2)_0 = (d\,b_1 \cdot a_2)_0 = (d \cdot b_1\,a_2)_0 = (d\,a_2 \cdot b_1)_0 = 0$. Hence, $b_1\,a_2 = 0$. Analogously, $a_1\,b_2 = 0$.

Now denote $\tau = (a_1\,a_1)_\emptyset$, $\xi = (b_1\,b_1)_\emptyset$, $\lambda = (a_1\,b_1)_\emptyset$. For $c \in M_{(12)}$, by identity (11), we have $(a_1\,a_1)_\emptyset\,c = 2(a_1\,c)_0\,a_1 = 0$.

By (8), for $c \in M_{(2i-1,2i)}$, $i \neq 1$, we have $(a_1\,a_1)_\emptyset\,c = 2a_1\,c \cdot a_1 = 0$. Hence, $(a_1\,a_1)_\emptyset \in Z(M) = 0$ and analogously $(a_i\,a_i)_\emptyset = (b_i\,b_i)_\emptyset = 0$.

Moreover, by (11), $\lambda\,a_1 = (a_1\,b_1)_\emptyset\,a_1 = (a_1\,a_1)_0 b_1 = b_1$ and analogously $\lambda\,b_1 = a_1$. By (7), $\lambda\,a_2 = (a_1\,b_1)_\emptyset\,a_2 = b_1\,a_2 \cdot a_1 = d\,a_1 = b_2$ and in the same way $\lambda\,b_2 = a_2$.

Now we denote $b_i = d_{i1}\,b_1$ and $a_i = d_{i1}\,a_1$. As above, we can prove that $b_i\,b_j = a_i\,a_j = d_{ij}$, $a_i\,b_j = 0$ and $(a_i\,a_i)_\emptyset = (b_i\,b_i)_\emptyset = 0$ and $(a_i\,b_i)_\emptyset = \lambda$. In this case, we have the multiplication rules given by (15).

(c) Let $d = d_{12} \in M_{(1234)}$ be such that $d^2 = s$ and denote

$$\begin{array}{ccc} a_2 = d\, a_1, & b_2 = d\, b_1, & a_1\, b_2 = \alpha\, d, \\ a_1\, a_2 = \gamma\, d, & a_2\, b_1 = \beta\, d, & b_1\, b_2 = \tau\, d, \end{array}$$

As in case (b), by (7), we have $d\, a_2 = d \cdot d\, a_1 = (d\, d)_0 a_1 = a_1$ and $d\, b_2 = d \cdot d\, b_1 = (d\, d)_0 b_1 = b_1$.

Now, by (9), we have $(a_2\, a_2)_0 = (d\, a_1 \cdot a_2)_0 = (d\, a_2 \cdot a_1)_0 = (a_1\, a_1)_0 = 0$, $(b_2\, b_2)_0 = 0$. Again by (9), $(a_2\, b_2)_0 = (d\, a_1 \cdot b_2)_0 = (d\, b_2 \cdot a_1)_0 = (b_1\, a_1)_0 = s$.

Now for $c \in M_{(12)}$, by identity (11), we have $(a_1\, a_1)_\emptyset\, c = 2(a_1\, c)_0\, a_1 = 0$ and by (8), for $c \in M_{(2i-1,2i)}$, $i \neq 1$, we have $(a_1\, a_1)_\emptyset\, c = 2a_1\, c \cdot a_1 = 0$. Hence, $(a_i\, a_i)_\emptyset = (b_i\, b_i)_\emptyset \in Z(M) = 0$.

By (11) we have $(a_1\, b_1)_\emptyset\, a_1 = (a_1\, b_1)_0\, a_1 = a_1$ and, by (7), $(a_1\, b_1)_\emptyset\, a_2 = a_2\, b_1 \cdot a_1 = d\, a_1 = a_2$.

From this, analogously to the previous case, for $\lambda = (a_1\, b_1)_\emptyset$ we get $\lambda\, a_i = a_i$ and $\lambda\, b_i = b_i$.

Now for $d_{ij} \in M_{(2i-1,2i,2j-1,2j)}$, it is clear, by (7), (8) and the fact that $\mid \sigma \mid \leq 4$, that $\lambda\, d_{ij} = (a_1\, b_1)_\emptyset\, d_{ij} = 0$ $\square$

We will denote by $D_{2\ell+1}$ the Δ-algebra M with multiplication rules given by (15) and by $C_{2\ell+1}$ the one with multiplication rules given by (16).

3 Irreducible Representations of Δ-algebras

In this section we study the action of the Δ-algebras M of Theorem 2.1.

Theorem 3.1. *Let M be a Δ-algebra as in Theorem 2.1 and V be an irreducible M-module. Then*

1. *$M = D_{2\ell+1}$ and*

1.1. $V = < v_1, \ldots, v_\ell, \xi, \mu >$ *, where* $v_i \in V_{(2i-1,2i)}$ *,* $\xi, \mu \in V_\emptyset$ *and*

$$\begin{aligned} &v_i\, d_{ij} = v_j, \qquad && (v_i\, a_i)_\emptyset = \xi, \qquad && (v_i\, b_i)_\emptyset = \mu, \\ &\xi\, a_i = v_i, && \mu\, b_i = v_i, && \lambda\, \mu = \xi, \quad \lambda\, \xi = \mu \end{aligned} \tag{17}$$

and all the other products are zero.

1.2. V *is the adjoint module.*

2. $M = C_{2\ell+1}$

2.1. $V = < v_1, \ldots, v_\ell, \tau, \mu >$ *, where* $v_i \in V_{(2i-1,2i)}$ *,* $\tau, \mu \in V_\emptyset$ *and*

$$\begin{aligned} &v_i\, d_{ij} = v_j, \qquad && (v_i\, a_i)_\emptyset = \tau, \qquad && (v_i\, b_i)_\emptyset = \mu, \\ &\tau\, b_i = \mu\, a_i = v_i, && \lambda\, \tau = \tau, && \lambda\, \mu = \mu \end{aligned} \tag{18}$$

and all the other products are zero.

2.2. V *is the adjoint module.*

Proof. 1. Let $V_0 \neq 0$ and $v_0 \in V_0$. Define

$$v_{ij} = v_0\, d_{ij}, \quad v_i = v_0\, a_i, \quad w_i = v_0\, b_i, \quad (v_i\, b_i)_\emptyset = \mu_i \quad (w_i\, a_i)_\emptyset = \xi_i\,. \tag{19}$$

By (14) we have $(v_i\, a_i)_0 = (v_0\, a_i \cdot a_i)_0 = v_0\,(a_i a_i)_0 = v_0$. Thus, by (8) and (11), we have that $\mu = \mu_1 = \cdots = \mu_\ell = \xi_1 = \cdots = \xi_\ell$ and $\mu_i\, a_i = (v_i\, b_i)_\emptyset\, a_i = (v_i\, a_i)_0\, b_i + (a_i\, b_i)_0\, v_i = v_0\, b_i = w_i$ and analogously $\mu\, b_i = v_i$.

Hence V has a basis $\{v_0, v_{ij}, v_i, w_i, \mu \,|\, i \leq i, j \leq \ell\}$ and V is the adjoint M-module.

Now suppose that $V_0 = 0$ and take V_μ, $\mu \neq 0$. As $|\ \mu \cap (2i-1, 2i)\ | = 0$ or 2, then $(2i-1, 2i) \subseteq \mu$ or $\mu \cap (2i-1, 2i) = \emptyset$ for all $1 \leq i \leq \ell$. Suppose $(1, 2) \subseteq \mu$. If $(2i-1, 2i) \subseteq \mu$, $i > 2$, then $\sigma_{1i} = (1, 2, 2i-1, 2i) \subseteq \mu$ and, by (12), $V_\mu = s\, V_\mu = (d_{1i}\, d_{1i})_0\, V_\mu = 0$, a contradiction. Hence, $\mu = (12)$. Let $0 \neq v_1 \in V_{(12)}$ and denote

$$v_i = v_1\, d_{1i}, \qquad (v_i\, b_i)_\emptyset = \tau_i \qquad (v_i\, a_i)_\emptyset = \mu_i\,. \tag{20}$$

Now by (6), we have $v_i\, d_{ij} = v_1\, d_{1i} \cdot d_{ij} = v_1\, d_{ij} \cdot d_{1i} + v_1 \cdot d_{1i} d_{ij} = v_1\, d_{1j} = v_j$.

By (11), $\mu_i\, a_i = (v_i\, a_i)_\emptyset\, a_i = (v_i\, a_i)_0\, a_i + (a_i\, a_i)_0\, v_i = v_i$ and $\mu_i\, b_i = (v_i\, a_i)_\emptyset\, b_i = (v_i\, b_i)_0\, a_i + (a_i\, b_i)_0\, v_i = 0$. Analogously, $\tau_i\, a_i = 0$ and $\tau_i\, b_i = w_i$.

Moreover, by (8), $\mu_i\, a_j = (v_i\, a_i)_\emptyset\, a_j = v_i\, a_j \cdot a_i + v_i \cdot a_i\, a_j = v_i\, d_{ij} = v_j$.

Analogously, we prove that $\mu_i\, b_j = 0$, $\tau_i\, b_j = w_i$, $\tau_i\, a_j = 0$. Hence $\mu = \mu_1 = \cdots = \mu_\ell$ and $\tau = \tau_1 = \cdots = \tau_\ell$. Furthermore, by (13), $\lambda\mu = (a_1\, b_1)_\emptyset\, \mu = (a_1\, \mu \cdot b_1)_\emptyset + (a_1 \cdot \mu\, b_1)_\emptyset = (v_1\, b_1)_\emptyset = \tau$ and, analogously, $\lambda\tau = \mu$. Hence $V = < v_1, \ldots, v_\ell, \tau, \mu >$, is the standard M-module.

2. Suppose $V_0 \neq 0$. As in case 1, we can prove that V is the adjoint M-module. Thus, let $V_0 = 0$. Again as in case 1 we can prove that there exists $\mu = (12)$ such that $V_\mu \neq 0$. Denote, as in the previous case,

$$v_i = v_1\, d_{1i}, \qquad (v_i\, a_i)_\emptyset = \mu_i \qquad (v_i\, b_i)_\emptyset = \tau_i \qquad v_i\, d_{ij} = v_j. \tag{21}$$

By (11), $\mu_i\, a_i = (v_i\, a_i)_\emptyset\, a_i = (v_i\, a_i)_0\, a_i + (a_i\, a_i)_0\, v_i = 0$ and $\mu_i\, b_i = v_i$, $\tau_i\, a_i = v_i$, $\tau_i\, b_i = 0$.

By (8), $\mu_i\, a_j = (v_i\, a_i)_\emptyset\, a_j = v_i\, a_j \cdot a_i + v_i \cdot a_i\, a_j = 0$, $\mu_i\, b_j = (v_i\, a_i)_\emptyset\, b_j = v_i\, b_j \cdot a_i + v_i \cdot a_i\, b_j = v_i\, d_{ij} = v_j$. Analogously, $\tau_i\, a_j = v_i$, $\tau_i\, b_j = 0$. Hence $\mu = \mu_1 = \cdots = \mu_\ell$ and $\tau = \tau_1 = \cdots = \tau_\ell$. Furthermore, by (13), $\lambda\mu = (a_1\, b_1)_\emptyset\, \mu = (a_1\, \mu \cdot b_1)_\emptyset + (a_1 \cdot \mu\, b_1)_\emptyset = (a_1\, v_1)_\emptyset = \mu$ and, analogously, $\lambda\tau = \tau$. $\square$

4 Lie algebras from Δ-algebras

First recall some well known facts about quadratic forms over an algebraically closed field of characteristic 2 and its corresponding Lie algebras.

Let V be a n-dimensional k-space and $f : V \times V \longrightarrow k$ be a non-degenerate symmetric bilinear form. This means that $f(x, y) = f(y, x)$, for all $x, y \in V$ and $f(x, V) = 0$ implies $x = 0$. A non-degenerate symmetric bilinear form f is called *symplectic* if $f(x, x) = 0$ and *orthogonal* otherwise. A vector space V

has a unique orthogonal form f and in some basis $\{v_1, \ldots, v_n\}$ the form can be written as

$$f(v,w) = \sum_{i=1}^{n} x_i\, y_i$$

where $v = \sum_{i=1}^{n} x_i\, v_i$ and $w = \sum_{i=1}^{n} y_i\, v_i$.

If $\dim V$ is odd, then the vector space V does not have a symplectic form and if $\dim V = 2\ell$ then it has a unique symplectic form. In this last case, the form can be written, in an appropriate basis $\{v_1, \ldots, v_\ell, w_1, \ldots, w_\ell\}$, as follows

$$f(v,w) = \sum_{i=1}^{\ell} (x_i t_i + y_i z_i)$$

where $v = \sum_{i=1}^{\ell} (x_i\, v_i + y_i\, w_i)$ and $w = \sum_{i=1}^{\ell} (z_i\, v_i + t_i w_i)$.

Let $End(V)$ be the associative algebra of all linear transformations of V. Consider the following sets

$$S(f) = \{a \in End(V) \,|\, f(va,w) = f(v,wa),\ \forall\, v,w \in V\}\,,$$

$$O(f) = \{a \in End(V) \,|\, f(va,v) = 0,\ \forall\, v \in V\}.$$

It is clear that $O(f) \subseteq S(f)$. For f orthogonal, we denote $D_\ell = O(f)$ when $\dim V = 2\ell$ and $B_\ell = O(f)$ when $\dim V = 2\ell + 1$. For f symplectic, $C_\ell = O(f)$.

Theorem 4.1. *In the notation above we have*

1. $[S(f), S(f)] = O(f)$.
2. $Z(S(f)) = 0$ *if* $\dim V = 2\ell + 1$ *and* $Z(O(f)) = 1$ *if* $\dim V = 2\ell$.
3. $O(f)/Z(O(f))$ *is simple if* $\dim V > 2$ *and* $\dim V \neq 4$.
4. C_ℓ *is a* 2*-algebra.*
5. B_ℓ *and* D_ℓ *are not* 2*-algebras and* $S(f)$ *is the* 2*-envelope of* B_ℓ *(resp.,* D_ℓ *) in* $End(V)$.
6. $\dim C_\ell = 2\ell^2 - \ell$, $\dim B_\ell = 2\ell^2 + \ell$ *and* $\dim D_\ell = 2\ell^2 - \ell$.

Theorem 4.2. *Let M be a simple Δ-algebra in a Δ-variety $\mathcal{M}$ as described above and $L = M \square \Lambda$ be the corresponding Lie algebra. Then*

1. *$L = C_{2\ell}$ if M has a basis $\{s, a_{ij} \,|\, 1 \leq i < j \leq \ell\}$, where $a_{ij} \in M_{(2i-1,2i,2j-1,2j)}$.*

2. *$L = B_{2\ell}$ if M has a basis $\{s, a_{ij}, a_i \,|\, 1 \leq i < j \leq \ell\}$, where $a_{ij} \in M_{(2i-1,2i,2j-1,2j)}$, $a_i \in M_{(2i-1,2i)}$.*

3. *$L = D_{2\ell+1}$ or $C_{2\ell+1}$ if M has a basis $\{s, a_{ij}, a_i, b_i, \lambda \,|\, 1 \leq i < j \leq \ell\}$, where $a_{ij} \in M_{(2i-1,2i,2j-1,2j)}$ and the multiplication rules are given by* (15) *or* (16).

4. *L is a Lie algebra of type E_7 or E_8, if $\mathfrak{M} = \{\sigma \,|\, M_\sigma \neq 0\} = \mathcal{E}_7$ or $\mathcal{E}_8$.*

Proof. 1. By Theorem [GG], a Δ-algebra M has a module V with a basis $\{v_1, \ldots, v_\ell\}$, $v_i \in V_{(2i-1,2i)}$. This M-module admits an M-invariant bilinear form given by $(v_i, v_j) = \delta_{ij}$. Note that if a M-module $V = V_0 \oplus \sum \oplus V_\sigma$ admits an M-invariant symmetric bilinear form f, then the corresponding L-module $W = V \square \Lambda$ admits a L-invariant symmetric bilinear form as follows:

$$\tilde{f}(v \otimes x, w \otimes y) = f(v, w)\,(x, y), \qquad v, w \in V, \; x, y \in \Lambda$$

Moreover, $\tilde{f}$ is symplectic (orthogonal) if and only if the restriction of f to $V_0 \oplus V_\emptyset$ is symplectic (orthogonal). In our case, $V_0 \oplus V_\emptyset = 0$ hence this form is non-degenerate and symplectic. As $\dim W = 4\ell$ and $\dim L = 8\ell^2 - 2\ell$, we have that $L = C_{2\ell}$.

2. and 3. In all this cases M has a module V with a basis $\{v_1, \ldots, v_\ell, \mu, \tau\}$ described in Theorem 3.1, with $v_i \in V_{(2i-1,2i)}$. The M-module V admits an M-invariant bilinear form given by $(v_i, v_j) = \delta_{ij}$ and $(\lambda, \mu) = 0$, $(\lambda, \lambda) = (\mu, \mu) = 1$, if M has multiplication rules defined by (15) or $(v_i, v_j) = \delta_{ij}$ and $(\lambda, \mu) = 1$, $(\lambda, \lambda) = (\mu, \mu) = 0$, if M has multiplication rules defined by (16).

In the first case, the corresponding L-invariant bilinear form on the L-module $W = V \square \Lambda$ is orthogonal and, in the second case, it is symplectic. As $\dim L = 8\ell^2 + 6\ell + 1$, then $L = D_{2\ell+1}$ in the first and $L = C_{2\ell+1}$, in the second case.

4. We prove this statement in the case $\mathcal{E}_8$. The case $\mathcal{E}_7$ is a corollary of this.

By definition, a Lie algebra L over a field k of characteristic 2 is a Lie algebra of type E_8 if there exists a $\mathbf{Z}$-form $\mathcal{L}_{\mathbf{Z}}$ of the Lie algebra $\mathcal{L}$ over the field $\mathbf{C}$ of all complex numbers such that $L = \mathcal{L}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$.

Let $\mathcal{L}$ be the Lie algebra of type E_8 over the complex field $\mathbf{C}$ constructed in [?] with a basis

$$\{ e_1,\ f_1, \ldots,\ e_8,\ f_8,\ h_1, \ldots,\ h_8,\ (\sigma,\mu),\ \mu \subseteq \sigma \in \mathcal{E}_8 \}$$

and multiplication rules stated by Theorem 1 [?].

Let $\mathcal{L}_{\mathbf{Z}}$ be a $\mathbf{Z}$- module with generators

$$\{ e_i,\ f_i,\ h_i,\ i=1,\ldots,8,\ (\sigma,\mu),\ h^\sigma \ = \ \frac{1}{2}(\sum_{i\in\sigma} h_i),\ \mu \subseteq \sigma \in \mathcal{E}_8\}.$$

Note that $[\mathcal{L}_{\mathbf{Z}}, \mathcal{L}_{\mathbf{Z}}] \subseteq \mathcal{L}_{\mathbf{Z}}$, since for $\varphi\cap\psi=\emptyset, \varphi\cup\psi=\sigma$ we have, by Theorem 1 [?], that

$$(\sigma,\varphi)(\sigma,\psi) \ = \ (-1)^{|\psi|+1}(\sum_{i\in\psi} h_i - \sum_{j\in\varphi} h_j)/2 \ = \ (-1)^{|\psi|+1}(h^\sigma - \sum_{j\in\varphi} h_j)\,,$$

$$(\sigma,\mu)h^\tau \ = \ \frac{1}{2}(|\ \mu\cap\tau\ | \ - \ |\ \overline{\mu}\cap\tau\ |)(\sigma,\mu), \text{ where } \overline{\mu} = \sigma\setminus\mu. \tag{22}$$

But $(|\ \mu\cap\tau\ | \ - \ |\ \overline{\mu}\cap\tau\ |) \ = \ (|\ \sigma\cap\tau\ | \ -2\ |\ \overline{\mu}\cap\tau\ |) \equiv |\ \sigma\cap\tau\ | \equiv 0 (mod\ 2)$. Hence $(\sigma,\mu)h^\tau \in \mathcal{L}_{\mathbf{Z}}$.

Now we prove that $L \cong \mathcal{L}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$. Define $\xi : L \longrightarrow \mathcal{L}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$ given by $\xi(e_i) = e_i$, $\xi(f_i) = f_i$, $\xi(\sigma,\mu) = (\sigma,\mu)$, $\xi(h_i) = h_i$ $\xi(h^\sigma) = h^\sigma$. (Note that although the notation for the elements is the same, they are in two diferent algebras.)

To prove that ξ is an algebra isomorphism, it is enough to prove that

$$\xi((\sigma,\mu)h^\varphi) = [\xi(\sigma,\mu), \xi(h^\varphi)] \quad (*).$$

By (2),

$$\xi((\sigma,\,\mu)\,h^\varphi) = \left(\frac{|\sigma\cap\varphi|}{2} \ + \ |\varphi\cap\mu|\right)\,\xi((\sigma,\,\mu))$$

and by (22)

$$\xi((\sigma,\mu))\,\xi(h^{\varphi}) \;=\; \frac{1}{2}(|\;\mu\cap\varphi\;| \;-\; |\;\overline{\mu}\cap\varphi\;|)\xi((\sigma,\mu)).$$

Now since $\frac{1}{2}(|\;\mu\cap\varphi\;| \;-\; |\;\overline{\mu}\cap\varphi\;|) \;=\; -\frac{1}{2}(|\;\mu\cap\varphi\;| \;+\; |\;\overline{\mu}\cap\varphi\;|)+\;|\;\mu\cap\varphi\;|= -\frac{1}{2}(|\;\sigma\cap\varphi\;| \;+\; |\;\mu\cap\varphi\;|\equiv \frac{|\sigma\cap\varphi|}{2} \;+\; |\varphi\cap\mu|$, the equality (*) holds.

$\square$

References

[GG] Grishkov, A.N., Guerreiro, M., *Simple classical Lie algebras in characteristic* 2 *and their gradations, I.* International Journal of Mathematics, Game Theory, and Algebra,(to appear.).