One combinatorial function and its aplication

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Abstract

1 Introduction

Consider the following counting problem of selection. Suppose that we have the set of balls such that every ball a is labeled by a pair of numbers (a_1, a_2). For every finite set N of such balls and $a \in N$ we define the numbers

$$t_N(a) = |\{b \in N | b_1 = a_1\}|$$

$$\alpha_i(N) = |\{a \in N | t_N(a) = i\}|/i$$

and a vector $\alpha(N) = (\alpha_1(N), \alpha_2(N), ...) \in P = \{(\alpha_1, \alpha_2, ...) | \alpha_i \in \mathbf{Z}_+, m, \forall n > m : \alpha_n = 0\}$. We shall call the vector $\alpha(N)$ *the type of the set* N. Now consider n numerated boxes with our balls such that the i-th box contains only i balls: $(1, i), (2, i), ..., (i, i)$.

Definition 1 *Let $\alpha \in P$ and denote by K^n_α the number of ways to choose a set of balls of the type α from the given n boxes such that from one box we can take no more than one ball.*

In the present work we begin the study of the function K^n_α as the function of n and α. We shall find the recurrence relation for this function and prove some properties of it as the function of n and α. As application of this study we shall resolve the following differential equation over finite field of p elements:

$$\underbrace{(...(f'f)'f)'...f)'f}_{p-1} = 0,$$

where $f \in \mathbf{Z_p}[[x]]$.

2 Preliminaries

For every vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \in P$ we can define the following numbers:

$$|\alpha| = \sum_{i=1}^{m} \alpha_i, ||\alpha|| = \sum_{i=1}^{m} i\alpha_i, \langle\alpha\rangle = \prod_{i=1}^{m} \alpha_i!((i+1)!)^{\alpha_i}$$

and partial operators $\partial_i, i = 1, 2, ..$ such that

$$\partial_i \alpha = \begin{cases} (\alpha_1 - 1, \alpha_2, ...), & \text{if } i = 1 \text{ and } \alpha_1 > 0 \\ (\alpha_1, ..., \alpha_{i-1} + 1, \alpha_i - 1, \alpha_{i+1}, ..) & \text{if } i > 1 \text{ and } \alpha_i > 0 \end{cases} \quad (1)$$

For every x and positive integer k set:

$$x^{[k]} = x(x-1)...(x-k+1).$$

Let p be interger positive number,denote by $\mathbf{Q}_(p)$ the ring of the p-intergers numbres, recall that a number $a = n/m, n, m \in \mathbf{Z}$ is called p-interger if $(n, p) = 1$

Theorem 1 *Function K_α^n satisfies the following recurrence relation:*

$$K_\alpha^{n+1} = K_\alpha^n + \sum_{i=2}^{m} (\alpha_{i-1} + 1) K_{\partial_i \alpha}^n + (n - ||\alpha|| + 2) K_{\partial_1 \alpha}^n \quad (2)$$

where $\alpha \in \mathbf{Z}_+^{\mathbf{m}}$.

Moreover there exists a unitary polinomial $f_\alpha(x) \in \mathbf{Q}[x]$ of degree $|\alpha| - 1$ such that

$$K_\alpha^n = \frac{(n+1)^{[||\alpha||]+1} f_\alpha(n)}{\langle\alpha\rangle} \quad (3)$$

and $f_\alpha(x) \in \mathbf{Q}_p[x]$, if p is a prime number, $p > ||\alpha|| + 1$.

Proof. By definition, $K_\alpha^{n+1} = |M_\alpha^{n+1}|$, where M_α^{n+1} is the set of the selections of type α from the $(n+1)$ boxes as it was desribed in the Introduction. It is obvious that $M_\alpha^{n+1} = M_\alpha^n \cup \bar{M}_\alpha^{n+1}$, where $\bar{M}_\alpha^{n+1} = M_\alpha^{n+1} \setminus M_\alpha^n$. We have by construction that every selection $a \in \bar{M}_\alpha^{n+1}$ contains a ball δ_a with the second lable $(n+1)$ hence $\bar{M}_\alpha^{n+1} = \cup_{i=1}^{n+1} M_i$, where $M_i = \{a \in \bar{M}_\alpha^{n+1} | t_a(\delta_a) = i\}$. Note that $a \in M_i$ if and only if $\alpha(a \setminus \{\delta_a\}) = \partial_i \alpha$. But if a is a selection of the type $\partial_i \alpha$ then the $(n+1)-$th box contains exactly $(\alpha_{i-1} + 1)$,(respectivly $(n - |\alpha| + 2)$,) balls such that the selection a

with any of those balls has the type α, if $i > 1$ (respectivly $i = 1$). Hence $|M_i| = (\alpha_{i-1}+1)K^n_{\partial_i\alpha}$ if $i > 1$ and $|M_1| = (n - |\alpha| + 2)K^n_{\partial_1\alpha}$. From here the formula (2) follows.

To prove formula (3) note that $K^n_\alpha = 0$ if $0 \le n < ||\alpha||$ by definition. Hence formula (2) defines the function K^n_α for all n. We shall prove (3) by induction on $||\alpha||$. If we put together all equalities (2) for $0 \le n \le s$ then obtain:

$$K^{s+1}_\alpha = K^n_\alpha + \sum_{i=2}^{m}(\alpha_{i-1}+1)\sum_{n=0}^{s} K^n_{\partial_i\alpha} + \sum_{n=0}^{s}(n - ||\alpha|| + 2)K^n_{\partial_1\alpha} \tag{4}$$

Note that for intergers k and s there exist rational numbers $a_0, ..., a_k$ such that

$$\sum_{n=0}^{s} n^k = n^{k+1}/(k+1) + \sum_{i=0}^{k} a_i n^i \tag{5}$$

By induction from (4) and (5) follows that K^{s+1}_α is a polynomial on s of degree $|\alpha| + ||\alpha||$. Then formula (2) is valid for all $n \in \mathbf{Z}$.

As we note $K^n_\alpha = 0$ if $0 \le n < ||\alpha||$. Moreover from (2) we have

$$0 = K^0_\alpha = K^{-1}_\alpha + \sum_{i=2}^{m}(\alpha_{i-1}+1)K^{-1}_{\partial_i\alpha} + (1 - ||\alpha||)K^{-1}_{\partial_1\alpha} \tag{6}$$

If $\alpha = (1,0,0,..)$ then $||\alpha|| = 1$ and (6) implies that $K^{-1}_\alpha = 0$. If $\alpha \ne (1,0,...)$ then using induction on $||\alpha||$, one obtains from (6) that $K^{-1}_\alpha = 0$. From here we can obtain that

$$K^n_\alpha = \frac{n^{[||\alpha||]} f_\alpha(n)}{r_\alpha}, \tag{7}$$

where $r_\alpha \in \mathbf{Q}$ and $f_\alpha(x)$ is a unitary polynomial degree $|\alpha| - 1$. By induction on $||\alpha||$ we shall prove that $r_\alpha = \langle\alpha\rangle$. Substituting (7) in (2) and canceling $(n+1)n(n-1)...(n - ||\alpha|| + 2)$ we obtain:

$$\frac{(n+2)f_\alpha(n+1)}{r_\alpha} = \frac{(n - |\alpha| + 1)f_\alpha(n)}{r_\alpha} +$$

$$\sum_{i=2}^{m}\frac{(\alpha_{i-1}+1)f_{\partial_i\alpha}(n)}{\langle\partial_i\alpha\rangle} + \frac{(n - |\alpha| + 2)f_{\partial_1\alpha}(n)}{\langle\partial_1\alpha\rangle}.$$

Since $||\alpha|| = ||\partial_i\alpha|| + 1, |\partial_i\alpha| = |\alpha|$, if $i > 1$ and $|\partial_1\alpha| = |\alpha| - 1$ then comparing the coefficients of $n^{||\alpha||-1}$ we have:

$$\frac{(1 + |\alpha|)}{r_\alpha} = \frac{1 - ||\alpha||}{r_\alpha} + \sum_{i=2}^{m}\frac{(\alpha_{i-1}+1)}{\langle\partial_i\alpha\rangle} + \frac{1}{\langle\partial_1\alpha\rangle}$$

or

$$\frac{||\alpha||+|\alpha|}{r_\alpha}=\frac{\sum_{i=2}^m (i+1)\alpha_i+2\alpha_1}{\langle\alpha\rangle}=\frac{\sum_{i=1}^m i\alpha_i+\sum_{i=1}^m \alpha_i}{\langle\alpha\rangle}$$

hence $r_\alpha = \langle\alpha\rangle$.

Let p be a prime number and $p > ||\alpha|| + 1$, we have to prove that $f_\alpha(x) \in \mathbf{Q}_{p()}[x]$. It is obvious that we can write

$$K_\alpha^n = \sum_{i=||\alpha||+1}^{||\alpha||+|\alpha|} a_\alpha^i (n+1)^{[i]}$$

and $f_\alpha(x) \in \mathbf{Q}_{(p)}[x]$ if and only if $a_\alpha^i \in \mathbf{Q}_(p), i = ||\alpha|| + 1, ..., |\alpha| + ||\alpha||$, because $\langle\alpha\rangle \in \mathbf{Q}_(p)$.

By induction on $|\alpha| + ||\alpha||$ we have $K_\beta^{p+j} \equiv 0$ for $|\beta| + ||\beta|| < |\alpha| + ||\alpha||$ and $j = -1, 0, 1, ..., ||\beta|| - 1$. Hence for $j = 0, 1, ..., ||\alpha|| - 2$ we have from (2):

$$K_\alpha^{p+j} - K_\alpha^{p+j-1} \equiv 0. \tag{8}$$

Here and above $a \equiv b$ means $a \equiv b(mod\ p)$.

Note that $pa_\alpha^i, i = ||\alpha|| + 1, ..., ||\alpha|| + |\alpha|$ and $a_\alpha^j, j = ||\alpha|| + 1, ..., p - 1$ are p-intergers. Indeed,

$$a_\alpha^i = (K_\alpha^{i-1} - \sum_{j=||\alpha||+1}^{i-1} a_\alpha^j i^{[j]})/i!$$

and now it is enougt to note that $|\alpha| + ||\alpha|| \le 2||\alpha|| < 2(p-1)$.

From here we have for $j = -1, 0, ..., |\alpha| + ||\alpha||$

$$\begin{array}{l} K_\alpha^{p+j} = \sum_{i=||\alpha||+1}^{||\alpha||+|\alpha|} a_\alpha^i (p+j+1)^{[i]} \\ \sum_{i=p}^{||\alpha||+|\alpha|} a_\alpha^i (p+j+1)^{[i]} \\ \sum_{i=p}^{p+j+1} a_\alpha^i (p+j+1)^{[i]} \\ \sum_{i=p}^{p+j+1} pa_\alpha^i [(p+j+1)^{[i]}/p]. \end{array} \tag{9}$$

From (8) and (9) hence that the numbers

$$A_j = \sum_{i=p}^{p+j+1} a_\alpha^i [(j+1)!(p-1)^{[i-j-2]} - j!(p-1)^{[i-j-1]}] \tag{10}$$

are p-intergers for $j = 0, ..., ||\alpha|| + |\alpha|$.

Now we shall prove the following usufull property of K_α^n :

Proposition 1

$$\sum_{\beta\leq\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n}K_{\alpha-\beta}^{-||\beta||-n-2}=0. \qquad (11)$$

Proof. We shall prove this identity by induction on $||\alpha||$ and with fixed $||\alpha||$ on n. It is obvious that it is enougt to prove (9) for all $n\geq -1$. For $n=-1$ it is obvious since $K_{\beta}^{||\beta||-1}=0$ and $K_{\alpha}^{-1}=0$. And now:

$$\sum_{\beta\leq\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n+1}K_{\alpha-\beta}^{-||\beta||-n-3}=\text{(from (2))}=$$

$(-1)^{||\alpha||}K_{\alpha}^{||\alpha||+n+1}+$

$\sum_{\beta<\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n+1}(K_{\alpha-\beta}^{-||\beta||-n-2}-$

$\sum_{i=2}^{m}(\alpha_{i-1}-\beta_{i-1}+1)K_{\partial_i(\alpha-\beta)}^{-||\beta||-n-3}-(-||\beta||-n-3-||\alpha||+||\beta||+2)K_{\partial_1(\alpha-\beta)}^{-||\beta||-n-3})=$

$(-1)^{||\alpha||}(K_{\alpha}^{||\alpha||+n}+\sum_{i=2}^{m}(\alpha_{i-1}+1)K_{\partial_i\alpha}^{||\alpha||+n}+(||\alpha||+n+1-||\alpha||+2)K_{\partial_1\alpha}^{||\alpha||+n})+$

$K_{\alpha}^{-||\alpha||-n-2}-\sum_{i=2}^{m}(\alpha_{i-1}+1)K_{\partial_i\alpha}^{-n-3}+(n+||\alpha||+1)K_{\partial_1\alpha}^{-n-3}+$

$\sum_{0<\beta<\alpha}(-1)^{||\beta||}(K_{\beta}^{||\beta||+n}+$

$\sum_{i=2}^{m}(\beta_{i-1}+1)K_{\partial_i\beta}^{||\beta||+n}+(||\beta||+n-||\beta||+2)K_{\partial_1\beta}^{||\beta||+n})K_{\alpha-\beta}^{-||\beta||-n-2}-$

$\sum_{i=2}^{m}\sum_{\beta<\alpha}(-1)^{||\beta||}(\alpha_{i-1}-\beta_{i-1}+1)K_{\beta}^{||\beta||+n+1}K_{\partial_i(\alpha-\beta)}^{-||\beta||-n-3}+$

$(n+||\alpha||+1)\sum_{\beta<\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n+1}K_{\partial_1(\alpha-\beta)}^{-||\beta||-n-3}+$

$(-1)^{||\alpha||}S_{\alpha}^{n}+\sum_{i=2}^{m}(\alpha_{i-1}+1)((-1)^{||\alpha||}K_{\partial_i\alpha}^{||\alpha||+n}-K_{\partial_i\alpha}^{-n-3})+$

$\sum_{i=2}^{m}\sum_{0<\beta<\alpha}(-1)^{||\beta||}(\beta_{i-1}+1)K_{\partial_i\beta}^{||\beta||+n}K_{\alpha-\beta}^{-||\beta||-n-2}-$

$\sum_{i=2}^{m}\sum_{\beta<\alpha}(-1)^{||\beta||}(\alpha_{i-1}-\beta_{i-1}+1)K_{\beta}^{||\beta||+n+1}K_{\partial_i(\alpha-\beta)}^{-||\beta||-n-3}+$

$(n+2)((-1)^{||\alpha||}K_{\partial_1\alpha}^{||\alpha||+n} + \sum_{0<\beta<\alpha}(-1)^{||\beta||}K_{\partial_1\beta}^{||\beta||+n}K_{\alpha-\beta}^{-||\beta||-n-2})+$

$$(n+||\alpha||+1)(K_{\partial_1\alpha}^{-n-3} + \sum_{\beta<\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n+1}K_{\partial_1(\alpha-\beta)}^{-||\beta||-n-3}) \qquad (12)$$

Note that:

$\sum_{i=2}^m \sum_{0<\beta<\alpha}(-1)^{||\beta||}(\beta_{i-1}+1)K_{\partial_i\beta}^{||\beta||+n}K_{\alpha-\beta}^{-||\beta||-n-2} =$
$\sum_{i=2}^m \sum_{0<\gamma<\partial_i\alpha}(-1)^{||\gamma||+1}(\gamma_{i-1})K_{\gamma}^{||\gamma||+n+1}K_{\partial_i\alpha-\partial_i\beta}^{-||\gamma||-n-3} =$

$$\sum_{i=2}^m \sum_{0<\beta<\partial_i\alpha}(-1)^{||\beta||+1}(\beta_{i-1})K_{\beta}^{||\beta||+n+1}K_{\partial_i\alpha-\partial_i\beta}^{-||\beta||-n-3} \qquad (13)$$

since $(-1)^{||\alpha||=-(-1)^{||\partial_i\alpha||}}$, if $\alpha_i > 0$, and $\alpha-\beta = \partial_i\alpha - \partial_i\beta, ||\beta|| = ||\partial_i\beta||+1$ if $\beta_i > 0$.
Moreover,

$$(-1)^{||\alpha||}K_{\partial_1\alpha}^{||\alpha||+n} + \sum_{0<\beta<\alpha}(-1)^{||\beta||}K_{\partial_1\beta}^{||\beta||+n}K_{\alpha-\beta}^{-||\beta||-n-2} =$$

$$\sum_{0\le\gamma\le\partial_1\alpha}(-1)^{||\gamma||+1}K_{\gamma}^{||\gamma||+n+1}K_{\partial_1\alpha-\gamma}^{-||\gamma||-n-3} = S_{\partial_1\alpha}^{n+1}, \qquad (14)$$

since $||\beta|| = ||\partial_1\beta||+1$ and $\partial_1\alpha - \partial_1\beta = \alpha-\beta$, if $\beta_1 > 0$.

By analogy
$K_{\partial_1\alpha}^{-n-3} + \sum_{\beta<\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n+1}K_{\partial_1(\alpha-\beta)}^{-||\beta||-n-3} =$

$$\sum_{\beta\le\partial_1\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n+1}K_{\partial_1\alpha-\beta)}^{-||\beta||-n-3} = S_{\partial_1\alpha}^{n+1}, \qquad (15)$$

since $\partial_1(\alpha-\beta) = \partial_1\alpha - \beta$.

Substituting (10),(11) and (12) in (9) we obtain

$$S_{\alpha}^{n+1} = (-1)^{||\alpha||}S_{\alpha}^{n} + \sum_{i=2}^m(\alpha_i+1)S_{\partial_i\alpha}^{n+1} + (||\alpha||-1)S_{\partial_1\alpha}^{n+1}.$$

From here by induction we have $S_{\alpha}^{n+1} = 0$. Proposition is proved.

Corollary 1 *Let p be a prime number, then*
$\sum_{\beta\leq\alpha}(-1)^{||\beta||}K_{\beta}^{||\beta||+n}K_{\alpha-\beta}^{p-||\beta||-n-2}=0.$

Consider the following interpretation of function K_{α}^{n}.

The symmetric group S_n acts naturally on the ring $k[x_1,...,x_n]:(x_i)^{\sigma}=x_{\sigma i}$. Denote by $\Delta(x_1,...,x_n)=\Delta_n=\Delta$ the set of all monomials from $x_1,...,x_n$. For $v,w\in\Delta$ we shall wright $v\sim w$ if $v^{\sigma}=w$ for some $\sigma\in S_n$. If N is a selection of balls as in the Introduction with the first lable $\leq n$ then we can construct the corresponding element $v(N)\in\Delta:$

$v(N)=\prod_{a\in N}x_{a_1}.$

If M is another selection of balls then we have:

$$v(N)\sim v(M)\Longleftrightarrow\alpha(N)=\alpha(M). \tag{16}$$

For every $v\in\Delta$ denote by $\{v\}=\{v\}_n$ equivalence class: $\{v\}=\{w\in\Delta_n|w\sim v\}$. It is obvious that $\{v\}$ contains a unique monomial of the form: $x_1...x_{\alpha_1}x_{\alpha_1+1}^2...x_{\alpha_1+\alpha_2}^2...x_{\alpha_1+...+\alpha_{k-1}+1}^k...x_{\alpha_1+...+\alpha_k}^k$, where $\alpha_i\in\mathbf{Z}_+$. In this case equivalence class $\{v\}_n$ we denote by $\{\alpha\}_n=\{\alpha\}$ and $\sum_{w\in\{v\}_n}w$ denote by v_{α}^n, where $\alpha=(\alpha_1,...,\alpha_m)$. Note that for a selection of balls $N:$

$v(N)\in\{\alpha\}$ if and only if $\alpha=\alpha(N)$.

Denote by $\sigma_n^i(x_1,...,x_n)$ the standart symmetric function:

$$(x-x_1)(x-x_2)...(x-x_n)=\sum_{i=0}^{n}(-1)^i\sigma_n^i(x_1,...,x_n)x^{n-i}. \tag{17}$$

Definition 2 *Let $f(x_1,...,x_n)\in\mathbf{Q}[x_1,...,x_n]$ and $\alpha\in P$. If $f(x_1,...,x_n)=\sum_{v\in\Delta}f_v v$ then by definition*
$\alpha(f(x_1,...,x_n))=\sum_{v\in\{\alpha\}}f_v.$

And now we can prove:

Proposition 2 $K_{\alpha}^{n}=\alpha(\sigma_n^{||\alpha||}(x_1,x_1+x_2,...,x_1+...+x_n))$

Proof. First we note that for $\alpha\in\mathbf{Z}_{+}^{\mathbf{m}}$ and $v\in\{\alpha\}_n$, $f_v\neq 0$ implies $||\alpha||=\deg v$. By definition, K_{α}^{n} is a number of ways to choose the α-type selections from n boxes if a $i-$th box contains the balls $(1,i),(2,i),...,(i,i)$ or $K_{\alpha}^{n}=|M_{\alpha}^{n}|$, where

$M_{\alpha}^{n}=\{a=[(i_1,j_1),...,(i_t,j_t)]|1\leq i_1<i_2<...<i_t\leq n,1\leq j_s\leq i_s,\alpha(a)=\alpha\}.$

Define on the set M_{α}^{n} an equivalence $\sim$:

$\forall a,b\in M_{\alpha}^{n}:a\sim b$ if and only if $v(a)=v(b)$.

Let $M_1, ..., M_r$ be equivalence classes of M_α^n. Then $K_\alpha^n = \sum_{i=1}^r |M_i|$. Note that $f_v = |M_i|$ if $v = v(a), a \in M_i$, which proves Proposition.

Corollary 2 $\sum_{\sigma \in S_n} \sigma_n^t(x_{\sigma 1}, ..., x_{\sigma 1} + ... + x_{\sigma n})) = \sum_{||\alpha||=t} K_\alpha^n v_\alpha^n$.

Theorem 2 *Let $R = \mathbf{Z_p}[[x]]$ be a ring of power series of x over the field of p elements for prime p. Then for every $f \in R$ we have*

$$\underbrace{(...(f'f)'f)'...f)'f}_{p-1} + (f^{p-1})^{(p-2)} = 0. \tag{18}$$

Prove. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ be an arbitrary element from R. A straightforward calculation shows that the coefficient of x^n in the left part of (14) is the following:

$$\sum_{(i_1,...,i_{p-1}) \in \Gamma_n} c_{i_1 ... i_{p-1}} a_{i_1} ... a_{i_{p-1}}$$

$$[i_1(i_1 + i_2 - 1)(i_1 + i_2 + i_3 - 2)...(i_1 + ... + i_{p-2} - p + 2) -$$
$$(i_1 + ... + i_{p-1})(i_1 + ... + i_{p-1} - 1)...(i_1 + ... + i_{p-1} - p + 2)],$$

where $\Gamma_n = \{(i_1, ..., i_{p-1}) | i_j \in \mathbf{Z}_+, i_1 + ... + i_{p-1} = n + p - 2\}$, $c(i_1, ..., i_{p-1}) = c(i_{\sigma 1}, ..., i_{\sigma(p-1)}), \forall \sigma \in S_{p-1}$ and $c(i_1, ..., i_{p-1}) = 1/j_1!...j_s!$, if $i_1 = ... = i_{j_1} < i_{j_1+1} = ... = i_{j_1+j_2} < ... < i_{j_1+...+j_{s-1}+1} = ... i_{j_1+...+j_s} = i_{p-1}$.

Now the Theorem follows from the following

Proposition 3 *In the ring of polynomials from $x_1, ..., x_{p-1}$ with intergers coefficients we have the following comparison modulo p:*

$(x_1 + ... + x_{p-1})(x_1 + ... + x_{p-1} - 1)...(x_1 + ... + x_{p-1} - p + 2) \equiv$

$$\sum_{\sigma \in S_{p-1}} x_{\sigma 1}(x_{\sigma 1} + x_{\sigma 2} - 1)...(x_{\sigma 1} + ... + x_{\sigma(p-1)} - p + 2)(mod\ \ p). \tag{19}$$

Proof. Introduce the new variables:$y = y_1 + ... + y_{p-1}$, $y_i = x_i - 1, i = 1, ..., p-1$. Then (15) we can rewrite in the following form:

$(y-1)(y-2)...(y-(p-1)) \equiv$

$$\sum_{\sigma \in S_{p-1}} (y_{\sigma_1} + 1)(y_{\sigma_1} + y_{\sigma_2} + 1)...(y_{\sigma_1} + ... + y_{\sigma_{p-1}} + 1)(\mathrm{mod}\, p). \tag{20}$$

But

$$(y-1)(y-2)...(y-(p-1)) \equiv y^{p-1} - 1, \tag{21}$$

And from (15) with $x=-1$ we have:
$(y_1+1)(y_1+y_2+1)...(y_i+...+y_{p-1}+1) =$

$$\sum_{i=o}^{p-1} \sigma^i_{p-1}(y_1, y_1+y_2, ..., y_1+...+y_{p-1}), \tag{22}$$

hence from (19),(20) and Corollary 3 we can rerite the equality (18):

$$y_{p-1}-1 \equiv (p-1)! + \sum_{\sigma\in S_{p-1}} \prod_{i=1}^{p-1}(y_{\sigma 1}+...+y_{\sigma i}) +$$

$$\sum_{\sigma\in S_{p-1}} \sum_{i=1}^{p-2} \sigma^i_{p-1}(y_{\sigma 1}, ..., y_{\sigma 1}+...+y_{\sigma(p-1)}) \equiv 0. \tag{23}$$

We prove that

$$S = \sum_{\sigma\in S_{p-1}} \sum_{i=1}^{p-2} \sigma^i_{p-1}(y_{\sigma 1}, ..., y_{\sigma 1}+...+y_{\sigma(p-1)}) \equiv 0. \tag{24}$$

Lemma 1 *If $g \in \mathbf{Q}[x_1, ..., x_n], h \in \mathbf{Q}[x_{n+1}, ..., x_m]$ and $f = gh$ then for $\alpha \in P$ we have*
$\alpha(f) = \sum_{\beta\leq\alpha} \beta(g)(\alpha-\beta)(h)$.

Proof. Let Δ_1 and Δ_2 be the sets of monomials from $x_1, ..., x_n$ and $x_{n+1}, ..., x_m$ respectivly. Note that for $v \in \Delta_1$ and $w \in \Delta_2$ if $v \in \{b\}$ and $w \in \{g\}$ we have that $vw \in \{\beta+\gamma\}$. And now if $g = \sum_{\beta\in\Delta_1} g_v v, h = \sum_{w\in\Delta_2} h_w w$ then

$$\alpha(f) = \alpha(gh) = \alpha(\sum_u \sum_{vw=u} g_v h_w u) = \sum_{vw=u\in\{\alpha\}} g_v h_w =$$

$$\sum_{\beta\leq\alpha} \sum_{v\in\{\beta\}} \sum_{w\in\{\alpha-\beta\}} g_v h_w = \sum_{\beta\leq\alpha} (\sum_{v\in\{\beta\}} g_v)(\sum_{w\in\{\alpha-\beta\}} h_w) = \alpha(g)\alpha(h).$$

Lemma is proved. Note that $y_{p-1} = y - y_1 - ... - y_{p-2}$ and substitute y_{p-1}

in (22) we receive:
$S = \sum_{j=1}^{p-1} \sum_{\sigma\in S_{(j)}} \sum_{i=1}^{p-1}$
$\sigma^i_{p-1}(y_{\sigma 1}, ..., y_{\sigma 1}+...+y_{\sigma(j-1)}, y - y_{\sigma(j+1)} - ... - y_{\sigma(p-1)}, ..., y) =$

$\sum_{k=0}^{p-1} L_k y^k$,
where $S_{(i)} = \{\sigma \in S_{p-1} | \sigma(i) = p-1\}$. Using the fact that that

$$\sigma_n^i(x_1, ..., x_{j-1}, y - x_j, ..., y - x_{n-1}, y) =$$

$$\sum_{k=0}^{n-j+1} y^k [\sum_{l=0}^{j-1} (-1)^{i-l-k} \sigma_{j-1}^l (x_1, ..., x_{j-1}) \sigma_{n-j}^{i-l-k} (x_j, ..., x_{n-1})]$$

and $\sigma_n^m(...) = 0$, if $m > n$ we calculate L_k :

$$L_k = \sum_{j=1}^{p-1} \sum_{\sigma \in S_{(j)}} \sum_{i=1}^{p-2} \sum_{l=0}^{j-1} (-1)^{i-l-k}$$

$$\sigma_{j-1}^l (y_{\sigma 1}, ..., y_{\sigma 1} + ... + y_{\sigma(j-1)}) \sigma_{p-j-1}^{i-l-k} (y_{\sigma(p-1)}, ..., y_{\sigma(j+1)} - ... - y_{\sigma(p-1)}).$$

Hence for $\alpha \in P$ we have from Lemma 1:

$$\alpha(L_k) = \sum_{j=1}^{p-1} \sum_{\sigma \in S_{(j)}} \sum_{i=1}^{p-2} \sum_{l=0}^{j-1} \sum_{\beta \leq \alpha, ||\beta|| = l} (-1)^{i-l-k})$$

$$\beta(\sigma_{j-1}^l (y_{\sigma 1}, ..., y_{\sigma 1} + ... + y_{\sigma(j-1)}))(\alpha - \beta)(\sigma_{p-j-1}^{i-l-k} (y_{\sigma(p-1)}, ..., y_{\sigma(j+1)} - ... - y_{\sigma(p-1)})).$$

Note that every terms in the last sum does not depend of $\sigma \in S_j$, $|S_j| = (p-2)! \equiv 1$, $||\alpha|| = i$. We have from Proposition 2 and Corollary 2:
$\alpha(L_k) = (-1)^{i-k} \sum_{\beta \leq \alpha} K_\beta^{j-1} K_{\alpha-\beta}^{p-j-2} \equiv 0.$
Finally, one can prove similarly that

$$\sum_{\sigma \in S_{p-1}} \sigma_{p-1}^{p-1} (y_{\sigma 1}, ..., y_{\sigma 1} + ... + \sigma(p-1)) \equiv y^{p-1}$$

Proposition and, hence Theorem 2, is proved.

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