New series of simple finite dimensional Lie algebras in characteristic 2 and 3.

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1 Introduction.

In this paper we have generalized the classical and Cartan simple Lie algebras in characteristic 2. By $\mathbf{F}_m$ we denote the finite field with 2^m elements and by $k\{X\}$ a vector $k-$space with a bases X, where k is an algebraicly closed field of characteristic 2.

2 The classical simple Lie algebras.

Let V be $k-$space of dimension n. Then $gl(V)$ is the Lie algebra of all linear endomorphisms of V over k. We can realize $gl(V)$ as $V \otimes_k V^*$, where $V^* = Hom_k(V, k)$ is the dual space to V. By definition, if $v, w \in V$, $\phi \in V^*$ then $(v \otimes \phi)(w) = \phi(w)v$. It is easy to see that

$$[v \otimes \phi, w \otimes \psi] = \phi(w)v \otimes \psi + \psi(v)w \otimes \phi. \qquad (1)$$

Consider V and V^* as $gl(V)-$modules, where $\psi \cdot (v \otimes \phi) = \psi(v)\phi$. Then an homomorphism $\pi : V \otimes_k V^* \to gl(V)$ such that $\pi(v \otimes \phi) = v \otimes \phi$ as above, is an isomorphism of $gl(V)-$modules, it follows from (1). By definition, $sl(V) = [gl(V), gl(V)]$ is the Lie algebra of $k-$morphisms with the trace 0. Note that

$$v \otimes \phi \in sl(V) \text{ if and only if} \phi(v) = 0. \qquad (2)$$

We consider $V \oplus gl(V) \oplus V^*$ as an algebra with multiplication law:

$$[V, V] = [V^*, V^*] = 0,\ [gl(V), gl(V)] \subseteq gl(V),\ [v, \phi] = v \otimes \phi, \qquad (3)$$

where $v \in V$, $\phi \in V^*$. This algebra with multiplication defined above is a simple Lie algebra of dimension $n^2 + 2n$ and it is isomorphic to $sl(V \oplus kv)$.

Let $f : V \times V \to k$ be a bilinear symmetric form on V. We denote by $so(f)$ a Lie algebra $\{\phi \in gl(V) | f(v^\phi, w) = f(v, w^\phi), \forall v, w \in V\}$ and by $o(V)$ a Lie algebra $\{\phi \in so(V) | f(v^\phi, v) = 0, \forall v \in V\}$.

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Now we construct some analogous of this classical Lie algebras. Let V be a $\mathbf{F}_m-$space of dimension n. Let $W = W(V)$ be a $k-$space with a bases $B(V) = \{v | v \in V\}$, then $dim_k W(V) = 2^{nm}$. By $\langle v, \phi\rangle$ where $\phi \in V^*$ we denote an element of $Hom_k(W, W)$ such that

$$w \cdot \langle v, \phi\rangle = \phi(w)(v + w),\ v, w \in V. \tag{4}$$

Note that $\langle v, \phi\rangle$ is linear by ϕ but is not linear by v.

Lemma 1 *In notation above we get:*

$$[\langle v, \phi\rangle, \langle w, \psi\rangle] = \langle (v + w), \phi(w)\psi + \psi(v)\phi\rangle.$$

Proof. We have for any $u \in V$ by (4):

$$\begin{aligned}
&u \cdot [\langle v, \phi\rangle, \langle w, \psi\rangle] = (u \cdot \langle v, \phi\rangle)\langle w, \psi\rangle + (u \cdot \langle w, \psi\rangle)\langle v, \phi\rangle = \\
&\phi(u)(u + v) \cdot \langle w, \psi\rangle + \psi(u)(u + w) \cdot \langle v, \phi\rangle = \\
&\phi(u)\psi(u + v)(u + v + w) + \psi(u)\phi(u + w)(u + v + w) = \\
&(\phi(u)\psi(v) + \psi(u)\phi(w))(v + w + u).
\end{aligned}$$

On the other hand by (4):

$$u \cdot \langle (v + w), \phi(w)\psi + \psi(v)\phi\rangle = (\phi(u)\psi(v) + \psi(u)\phi(w))(v + w + u).$$

□

By Lemma 1 the $k-$space

$$\mathcal{GL}(n, m) = k\{\langle v, \phi\rangle | v \in V, \phi \in V^*\}$$

is a Lie algebra. We call this algebra an **$\mathbf{F}_m$-analog** of $gl(V)$ where V is a $k-$space of dimension n.

Definition 1 *A Lie algebra L is* **almost restricted** *if for some Cartan subalgebra $H \subseteq L$ we have that a Lie algebra $T + L$ is restricted. Here T is 2-envelope for some toroidal subalgebra of H.*

A restricted 2-algebra Lie is called of **toroidal type** *if it has some toroidal Cartan subalgebra.*

Recall that the classical Lie algebra $gl(V)$ is a restricted Lie algebra of toroidal type. The following theorem is an analogous of this fact.

Theorem 1 *1. An algebra $\mathcal{GL}(n, m)$ is a simple almost restricted Lie algebra of toroidal type, if $(n, m) \neq (1, 1)$.*

2. An algebra $\mathcal{GL}(n, m)$ is restricted if and only if $m = 1$.

3. $dim_k \mathcal{GL}(n, m) = n2^{nm}$.

4. Toroidal range of $\mathcal{GL}(n, m)$ is equal nm.

By (2) we can define an k-analog of $sl(V)$ as a subalgebra of $\mathcal{GL}(n,m)$ as follows:

$$\mathcal{SL}(n,m) = \{\langle v, \phi\rangle \in \mathcal{GL}(n,m) | \phi(v) = 0\}. \tag{5}$$

A flag or $V-$flag $\mathcal{F}$ is a chain of subspaces of V :

$$\{0\} \subset V_1 \subset V_2 \subset ... \subset V_s \subset V. \tag{6}$$

The main characteristic of $\mathcal{F}$ is a vector $\mathbf{n} = \mathbf{n}(\mathcal{F})$ =$(n_1, ..., n_s, n)$, where $n_i = dim_{\mathbf{F}_m} V_i$, $n = dim_{\mathbf{F}_m} V$. Note that $1 < n_1 < ..., n_s < n$. By denition $V-$coflag or coflag is a V^*-flag. For any $V-$flag $\mathcal{F}$ we can define the corresponding coflag $co\mathcal{F}$ as follows:

$$\{0\} \subset^* V_s \subset ... \subset^* V_1 \subset V^*, \tag{7}$$

where ${}^*V_i = \{\phi \in V^* | \phi(V_i) = 0\}$. Let $\mathcal{G}$ be a $V-$flag: $\{0\} \subset W_1 \subset ... \subset W_t \subset V$. We write $\mathcal{F} \leq \mathcal{G}$ if $s = t$ and $V_i \subseteq W_i$, $i = 1, ..., s$. Note that in this case $\mathbf{n} = \mathbf{n}(\mathcal{F}) \leq \mathbf{m} = \mathbf{n}(\mathcal{G})$, which means that $n_i \leq m_i$,$i = 1, ..., s$. Let $(\mathcal{F}, \mathcal{G})$ be a $V-$flag and coflag correspondingly such that $co\mathcal{F} \leq \mathcal{G}$. In notation above it means that $\phi(V_i) = 0$ for all $\phi \in W_{s-i+1}$, where $\mathcal{G} = \{\{0\} \subset W_1 \subset ... \subset W_s \subset V^*\}$ We define

$$\mathcal{O}(\mathcal{F}, \mathcal{G}) = \{\langle v, \phi\rangle \in \mathcal{GL}(V) | v \notin V_i \,\text{hence}\, \phi \in W_i\}. \tag{8}$$

Lemma 2 *$\mathcal{O}(\mathcal{F}, \mathcal{G})$ is a subalgebra of $\mathcal{GL}(V)$.*

Proof. Let $\langle v, \phi\rangle, \langle w, \psi\rangle \in \mathcal{O}(\mathcal{F}, \mathcal{G})$. By (1) $[\langle v, \phi\rangle, \langle w, \psi\rangle] = \langle (v + w), \xi\rangle$, where $\xi = \phi(w)\psi + \psi(v)\phi$. Suppose that $v + w \notin V_i$. We need to prove that $\xi \in W_i$. If $v \notin V_i$ and $w \notin V_i$ then by (8) it means that $\phi, \psi \in W_i$, hence $\xi \in W_i$. If $v \in V_i$, then $w \notin V_i$ and $\psi \in W_i$. As $co\mathcal{F} \leq \mathcal{G}$ then $\psi(V_i) = 0$, hence $\xi = \phi(w)\psi \in W_i$. $\square$

Let $\mathbf{n} = \mathbf{n}(co\mathcal{F})$ and $\mathbf{m} = \mathbf{n}(\mathcal{G})$ then in the situation above we have $\mathbf{n} \leq \mathbf{m}$. Note that the par $(\mathbf{n}, \mathbf{m})$ defines the par of flag and coflag $(\mathcal{F}, \mathcal{G})$ uniquely. We will denote $\mathcal{O}(\mathcal{F}, \mathcal{G})$ by $\mathcal{O}(\mathbf{n}, \mathbf{m})$, where $\mathbf{n} = \mathbf{n}(\mathcal{F})$ and $\mathbf{m} = \mathbf{n}(\mathcal{G})$. Now define

$$\mathcal{SO}(\mathbf{n}, \mathbf{m}) = \mathcal{O}(\mathbf{n}, \mathbf{m}) \cap \mathcal{GL}(n, m). \tag{9}$$

The following series of simple Lie algebras are new (?) series in characteristic 2 and 3.

Let V be a 3-dimensional $k-$space and (v, w, u) be the unique (up to isomorphism) 3-linear antisymmetris $k-$form on V (form of determinant.) For any $v, w \in V$ we define $d_v^w \in V^*$(dual space), where $d_v^w(u) = (v, w, u)$.

We denote

$\mathcal{E}_1(m) = W(V) \oplus \mathcal{SL}(V)$, $\mathcal{E}_2(m) = W(V)_1 \oplus \mathcal{SL}(V) \oplus W(V)_{-1}$,

where $W(V)_1$ and $W(V)_2$ are two copies of $W(V)$.

We define the structurs of anticommutative algebras on $\mathcal{E}_1(V)$ and $\mathcal{E}_2(V)$ by the following formules:

$[v, w] = \langle v + w, d_{v+w}^w\rangle$, $v, w \in V$.

$[V_1, V_1] = [V_2, V_2] = 0$, $[v, w] = \langle v + w, d_{v+w}^w\rangle$, $v, \in V_1.$, $w \in V_2$.

Morever, consider $\mathcal{SL}(V)$ as a subalgebra and V, V_1, V_2 as $\mathcal{SL}(V)-$modules.

Theorem 2 *(i)* $\mathcal{E}_1(m)^2$ *is a simple Lie* $k-$*algebra if* k *has the characterstic 3.*
(ii) $\mathcal{E}_2(m)^2$ *is a simple Lie* $k-$*algebra if* k *has the characterstic 2.*

Proof. First consider the case $\mathcal{E}_1(V)$. For $v, w, u \in V$ we have
$[[v, w], u] + [[w, u], v] + [[u, v], w] =$
$[\langle v + w, d^v_{v+w}\rangle, u] + [\langle w + u, d^w_{w+u}\rangle, v] + [\langle u + v, d^u_{u+v}\rangle, w] =$
$((v, v + w, u) + (w, w + u, v) + (u, u + v, w))(v + w + u) = 3(v, w, u)(v + w + u) = 0.$

3 2-analogs of Lie algebras of Cartan type.

The Lie algebras of Cartan type appear as Lie algebras of derivations of graduate algebra of truncated polynomiais. Let $\mathbf{m} = (m_1, ..., m_n)$ be a vector with natural coordinates. Then $A(\mathbf{m})$ is an associative commutative $k-$algebra with generators $X_n = \{x^{(i)}_1, ..., x^{(i)}_n | i = 1, ...\}$ and relations:

$$x^{(i)} \cdot x^{(j)} = \left(\begin{array}{c} i+j \\ i \end{array} \right) x^{(i+j)},\ x_i^{(2^{m_i})} = 0, x \in \{x_1, ..., x_n\}. \tag{10}$$

Then $W(\mathbf{m}) = DerA(\mathbf{m})$ is a generalizade Jacobson-Witt Lie algebra(see [**?**]). Note that $A(\mathbf{m}) = \sum_{i=0} \oplus A(\mathbf{m})_i$ and $W(\mathbf{m}) = \sum_{i=-1} \oplus W(\mathbf{m})_i$ are $\mathbf{Z}-$graded algebras where
$A(\mathbf{m})_i = \{x^{(a)} | a = (a_1, ..., a_n) \in \mathbf{N}^n, |a| = a_1 + ... + a_n = i, x^{(a)} = x^{(a_1)}_1 ... x^{(a_n)}_n\}$,
$W(\mathbf{m})_i = \{a\frac{\partial}{\partial x_i}, i = 1, ..., n, a \in A(\mathbf{m})_{i+1}\}$.
It is clear that a set $B(\mathbf{m})_i = \{x^{(a)} || a| = i\}$ is a bases of $A(\mathbf{m})_i$, $i = 0, ...$
We construct an 2-analog of $W(\mathbf{m})$ in the following way. As above we fix a $\mathbf{F}_m-$space V with a bases $v_1, ..., v_n$. For any $v = (\alpha_1, ..., \alpha_n) \in V$ we define an derivation ∂_v of $A(\mathbf{m})$ such that $\partial_v(x^{(j)}_i) = \alpha_i x^{(j-1)}_i$. By definition

$$\mathcal{W}(m, \mathbf{m}) = \sum_{i=-1}^{r} \oplus \mathcal{W}(m, \mathbf{m})_i,$$

where $\mathcal{W}(m, \mathbf{m})_i$ is a $k-$space with a bases $\{\langle\partial_v, a\rangle | v \in V, a \in B(\mathbf{m})_{i+1}\}$. Note that $B_0(\mathbf{m}) = \{1\}$ hence $\mathcal{W}(m, \mathbf{m})_{-1} = k\{\partial_v | v \in V\}$. A multiplication in $\mathcal{W}(m, \mathbf{m})$ is given by the following formulae:

$$[\langle\partial_v, a\rangle, \langle\partial_w, b\rangle] = \langle\partial_{v+w}, \partial_v(b)a + \partial_w(a)b\rangle. \tag{11}$$

Lemma 3 *The algebra* $\mathcal{W}(m, \mathbf{m})$ *with the multiplicatiom law (11) is a simple Lie algebra.*

Proof. We have by (11):

$$\begin{array}{l} p_1 = [[\langle\partial_v, a\rangle, \langle\partial_w, b\rangle], \langle\partial_u, c\rangle] = [\langle\partial_{v+w}, \partial_v(b)a + \partial_w(a)b\rangle, \langle\partial_u, c\rangle] = \\ \langle\partial_{v+w+u}, \partial_{v+w}(c)(\partial_v(b)a + \partial_w(a)b) + \partial_u(\partial_v(b)a + \partial_w(a)b)c\rangle = \\ \langle\partial_{v+w+u}, \partial_v(c)\partial_v(b)a + \partial_w(c)\partial_v(b)a + \partial_v(c)\partial_w(a)b + \partial_w(c)\partial_w(a)b + \\ \partial_u(\partial_v(b))ac + \partial_v(b)\partial_u(a)c + \partial_u(\partial_w(a))bc + \partial_w(a)\partial_u(b)c\rangle \end{array}$$

Analogously,

$$p_2 = [[\langle\partial_w, b\rangle, \langle\partial_u, c\rangle], \langle\partial_v, a\rangle] =$$
$$\langle\partial_{v+w+u}, \partial_w(a)\partial_w(c)b + \partial_u(a)\partial_w(c)b + \partial_w(a)\partial_u(b)c + \partial_u(a)\partial_u(b)c +$$
$$\partial_v(\partial_w(c))ab + \partial_w(c)\partial_v(b)a + \partial_v(\partial_u(b))ac + \partial_u(b)\partial_v(c)a\rangle$$

$$p_3 = [[\langle\partial_u, c\rangle, \langle\partial_v, a\rangle], \langle\partial_w, b\rangle] =$$
$$\langle\partial_{v+w+u}, \partial_u(b)\partial_u(a)c + \partial_v(b)\partial_u(a)c + \partial_u(b)\partial_v(c)a + \partial_v(b)\partial_v(c)a +$$
$$\partial_w(\partial_u(a))bc + \partial_u(a)\partial_w(c)b + \partial_w(\partial_v(c))ab + \partial_v(c)\partial_w(a)b\rangle$$

It is obviously that $p_1 + p_2 + p_3 = 0$. □ **Note.** The algebra $\mathcal{W}(m, \mathbf{m})$ is not a new simple Lie algebra, really it is a Cartan algebra of Hamilton type.

We define $\mathcal{SW}(m, n) = \sum_{i=-1}^{r} \oplus \mathcal{SW}(m, n)_i$ where

$$\mathcal{SW}(m, n)_i = \{\langle\partial_v, a\rangle \in \mathcal{W}(m, \mathbf{1}_n)_i | \partial_v(a) = \partial_w(\partial_w(a)) = 0, \ \forall w \in V\}. \tag{12}$$

Lemma 4 *$\mathcal{SW}(m, n)$ is a simple Lie subalgebra of $\mathcal{W}(m, \mathbf{1}_n)$. Moreover, $\mathcal{SW}(m, \mathbf{1}_n)_{-1} = \mathcal{W}(m, \mathbf{1}_n)_{-1}$, $\mathcal{SW}(m, \mathbf{1}_n)_0 = \mathcal{SL}(m, n)$ and $dim_k \mathcal{SW}(m, \mathbf{1}_n) = (n-1)(2^{nm} - 1)$.*

Proof. Let $\langle\partial_v, a\rangle, \langle\partial_w, b\rangle \in \mathcal{SW}(m, \mathbf{m})$. By (11) and (12) we need to prove that $\partial_{(v+w)}(\partial_v(b)a + \partial_w(a)b) = \partial_u(\partial_u(\partial_v(b)a + \partial_w(a)b)) = 0$ for all $u \in V$.

We get

$$\partial_{(v+w)}(\partial_v(b)a + \partial_w(a)b) = \partial_v(\partial_v(b))a + \partial_v(b)\partial_v(a) + \partial_w(\partial_v(b))a + \partial_v(b)\partial_w(a) +$$
$$\partial_v(\partial_w(a))b + \partial_w(a)\partial_v(b) + \partial_w(\partial_w(a))b + \partial_w(b)\partial_w(a) = 0.$$

Analogously,

$$\partial_u(\partial_u(\partial_v(b)a + \partial_w(a)b)) = \partial_v(\partial_u(b))\partial_u(a) + \partial_u(\partial_v(b))\partial_u(a) +$$
$$\partial_w(\partial_u(a))\partial_u(b) + \partial_u(\partial_w(a))\partial_u(b) = 0.$$

□

Theorem 3 *The simple Lie algebras $\mathcal{SW}(m, n)$ are new simple Lie algebras over an algebraicly closed field of characteristic 2 if $n > 2$, moreover, $\mathcal{SW}(m, 3) = \mathcal{E}_2(m)$.*

Note. All constructions above we can generalise in the following way. Let $\mathbf{s}(\mathbf{m}) = (s_0, s_1, ..., s_{n-1}), s_i \leq s_{i+1}, s_{n-1} = m$ and let

$$V_n = \{0\} \subset V_{n-1} \subset ... \subset V_1 \subset V, dim_k V = n.$$

Let $v_0, ..., v_{n-1}$ be a bases of V such that $v_i, ..., v_{n-1}$ is a bases of V_i. There exists unique chain of finite subfields

$$\mathbf{F}_{s_0} \subseteq \mathbf{F}_{s_1} \subseteq ... \subseteq \mathbf{F}_m$$

Denote $V(\mathbf{s}(\mathbf{m})) = \{v \in V | v = \alpha_0 v_0 + ... + \alpha_{n-1} v_{n-1}, \alpha_i \in \mathbf{F}_i.\}$ Then by definition $\mathcal{W}(\mathbf{s}(\mathbf{m}), \mathbf{m}) = \{\langle\partial_v, a\rangle | v \in V(s(m)), a \in A(\mathbf{m}).\}$

4 Automorphism Groups of 2-analogs.

Let $L = \mathcal{GL}(n,m)$ be Lie algebra of dimension $n2^{nm}$. By Theorem 1 it is a 2-algebra with toroidal Cartan subalgebra $H = k\{\langle\phi_i\rangle | i = 1,...,n\}$ and $\phi_i \in V^*$ such that $\phi_i(v_j) = 0, i \neq j$, $\phi_i(v_i) = 1$ for some $\mathbf{F}_m-$basis $\{v_1,...,v_n\}$ of V. We identify H and V^*. Let $G = Aut_k(L)$ be an algebraic group of automorphism of L. Our purpers is to define the structure of G. Let $T = Stab_G(H)$ and $Z = \{\xi \in T | h^\xi = h, \forall h \in H\}$. Note that $\mathcal{H} = \{h^{[2^i]} | h \in H, i = 0,1,...\}$ is a toroidal subalgebra of 2-envelope of L and 2-envelope of L isomorphic to $L_1 = \mathcal{H} + L$. Let $0 \neq v \in V$ and $L_v = L_\psi = \{\langle v,\phi\rangle | \phi \in V^*\} = \{x \in L | [x,t] = \psi(t)x,\ forallt \in H = V^*\}$, where $\psi(t) = t(v) \in \mathbf{F}_m$. By definition $L_v^0 = \{x \in L_v | v^{[2]} = 0\}$.

Lemma 5 $L_v^0 = \{t(??)\}$

Let $m = 1$ then L is a 2-algebra with toroidal Cartan subalgebra $H_0 = H$ defined above. For any $i \in I_n = \{1,...,n\}$ we define an other toroidal Cartan subalgebra $H_j = \{\langle\phi_i\rangle + \langle v_i,\phi_i\rangle,\ \langle\phi_k\rangle | i = 1,...,j,\ k = j+1,...,n\}$.

Proposition 1 *Any toroidal Cartan subalgebra of* $\mathcal{GL}(n,1)$ *is conjugated to one of* $H_0,...,H_n$. *Moreover, the Cartan subalgebras* H_i *and* H_j *are not conjugated if* $i \neq j$.

Let $H = H_n$ and $L = H \oplus \sum_{v\in V\setminus\{0\}} \oplus L_v$ the corresponding Cartan decomposition. We can identify $\mathbf{F}_1-$space V with a set of all subsets of I_n such that for $v = a_1v_1 + ... + a_nv_n \in V$ the corresponding subset $\sigma = \sigma(v) = \{i | a_i = 1\}$. If $\sigma \subseteq I_n$ and $i \in \sigma$, $j \notin \sigma$ then by definition

$$\sigma_i = \sum_{i\in\mu\subseteq\sigma} \langle\mu,\phi_i\rangle,\ \sigma^j = \sum_{\mu\subseteq\sigma\cup j} \langle\mu,\phi_j\rangle.$$

Lemma 6 $L_\sigma = k\{\sigma_i,\ \sigma^j | i \in \sigma,\ j \notin \sigma\}$.

Proof. It is clear that the elements $\{\sigma_i,\ \sigma^j | i \in \sigma,\ j \notin \sigma\}$ are liner independence. For $s_i = \langle\phi_i\rangle + \langle i,\phi_i\rangle$ we have, if $j,i \in \sigma$:

$$[\sigma_j, s_i] = \textstyle\sum_{i,j\in\mu\subseteq\sigma}(\langle\mu,\phi_j\rangle + \langle\mu\setminus i,\phi_j\rangle) + \delta_{ij}\sum_{i\in\mu\subseteq\sigma}\langle\mu\setminus i,\phi_i\rangle.$$

If $i \neq j$ hence $[\sigma_j, s_i] = \sum_{j\in\mu\subseteq\sigma}\langle\mu,\phi_j\rangle = \sigma_j$. If $i = j$ hence $[\sigma_j, s_i] = \sum_{i,j\in\mu\subseteq\sigma}\langle\mu,\phi_j\rangle = \sigma_i$.

Analogously, if $j \notin \sigma$, $i \in \sigma$ then

$$\begin{array}{l}[\sigma^j, s_i] = \sum_{i\in\mu\subseteq\sigma\cup j}(\langle\mu,\phi_j\rangle + \langle\mu\setminus i,\phi_j\rangle) + \\ \delta_{ij}\sum_{\mu\subseteq\sigma}\langle\mu \bigtriangleup i,\phi_i\rangle = \sigma^j.\end{array}$$

The equalities $[\sigma_j, s_i] = [\sigma^j, s_i] = 0$ if $i \notin \sigma$ are obviously. $\square$