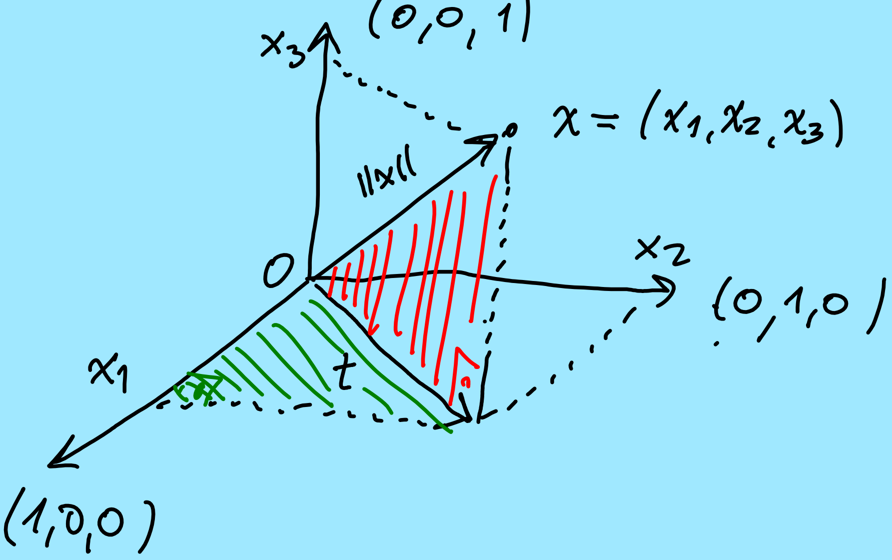
ESPAÇOS COM PRODUTO INTERNO

(CAP. 6)

PRODUTO ESCALAR EM $\mathbb{R}^n$



$\|x\|^2 = x_3^2 + t^2$ $\qquad$ $t^2 = x_1^2 + x_2^2$

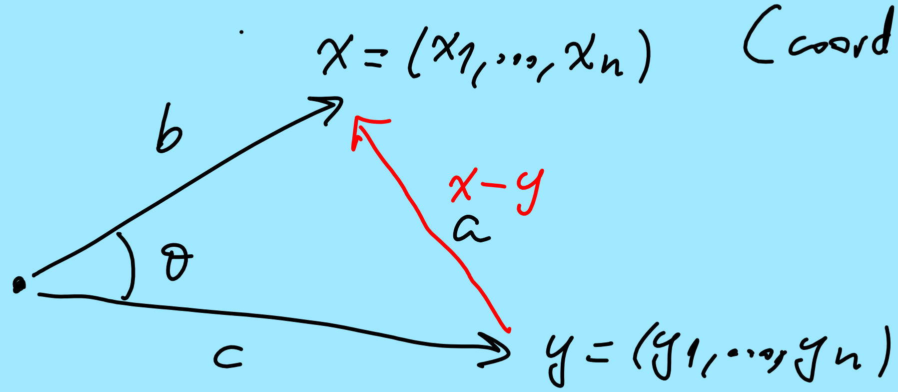
$$\therefore \quad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

comprimento ou norma ou módulo do vetor x

$$\mathbb{R}^n: \quad \|x\| = \sqrt{x_1^2 + \dots + x_n^2}$$

MEDIDA DE UM ÂNGULO EM $\mathbb{R}^n$

(coords rel base can)



$0 \leq \theta = \measuredangle(x,y) \leq \pi \qquad y + (x-y) = x$

Lei dos Cossenos: $a^2 = b^2 + c^2 - 2bc\cos\theta$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$$

$$(x_1-y_1)^2 + \dots + (x_n-y_n)^2 = \cancel{x_1^2} + \dots + \cancel{x_n^2} + \cancel{y_1^2} + \dots + \cancel{y_n^2} - 2\sqrt{x_1^2+\dots+x_n^2}\sqrt{y_1^2+\dots+y_n^2}\cos\theta$$

$$\cancel{x_1^2} - 2x_1y_1 + \cancel{y_1^2} + \dots + \cancel{x_n^2} - 2x_ny_n + \cancel{y_n^2}$$

$$-2(x_1y_1 + \dots + x_ny_n) = -2\sqrt{x_1^2+\dots+x_n^2}\sqrt{y_1^2+\dots+y_n^2}\cos\theta$$

$$x_1y_1 + \dots + x_ny_n = \underbrace{\sqrt{x_1^2+\dots+x_n^2}}_{\|x\|}\underbrace{\sqrt{y_1^2+\dots+y_n^2}}_{\|y\|}\cos\theta$$

Produto escalar de x e com y, denotado com $x \cdot y$

$$\therefore \quad x\cdot y = \|x\|\cdot\|y\|\cos\theta$$

Se $x, y \neq 0$ então

$$\boxed{\cos\measuredangle(x,y) \doteq \frac{x\cdot y}{\|x\|\,\|y\|}}$$

Notemos que $\|x\|^2 = x_1^2 + \dots + x_n^2 = x \cdot x$

Logo: ângulos & comprimentos reduzem-se ao produto escalar.

Em part: $x \perp y \iff \measuredangle(x,y) = \frac{\pi}{2}$ ou 90°

(x é <u>ortogonal</u> a y) $\qquad \Updownarrow$

$x, y \neq 0 \qquad\qquad x \cdot y = 0$

<u>Def</u> $x \perp y \iff x \cdot y = 0$

Em part, $0 \perp x, \ \forall x \in \mathbb{R}^n$

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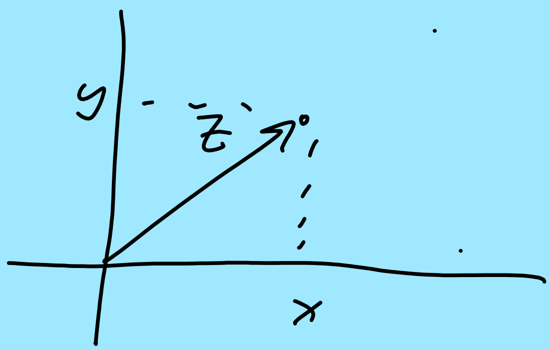
<u>O CASO COMPLEXO</u>



$\mathbb{C} \cong \mathbb{R}^2$

$z = x + iy \leftrightarrow (x,y)$

$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}_{\geq 0}$

$\mathbb{C}^n \ni z = (z_1, \dots, z_n)$

$\|z\| := \sqrt{|z_1|^2 + \dots + |z_n|^2} \in \mathbb{R}_{\geq 0}$

Produto (escalar) Hermiteano (Hermite)

$w = (w_1, \dots, w_n) \in \mathbb{C}^n$ $\qquad \overline{\lambda w_j} = \bar{\lambda}\, \overline{w_j}$

$$z \cdot w = z_1 \overline{w_1} + \dots + z_n \overline{w_n} \in \mathbb{C}$$

$$z \cdot z = |z_1|^2 + \dots + |z_n|^2 = \|z\|^2 \in \mathbb{R}$$

PRODUTO INTERNO EM UM ESPAÇO VETORIAL REAL ($\mathbb{F} = \mathbb{R}$)

Seja V um esp vet / $\mathbb{R}$. Um produto interno em V é uma aplicação

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$$ ← forma bilinear

t.q.

$$\left.\begin{array}{l} \langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle \\ \langle \lambda u, v \rangle = \lambda \langle u, v \rangle \end{array}\right] \text{ linear no 1.º argumento}$$

$$\left.\begin{array}{l} \langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle \\ \langle u, \lambda v \rangle = \lambda \langle u, v \rangle \end{array}\right] \text{ linear no 2.º argumento}$$

$$\langle v, u \rangle = \langle u, v \rangle \quad \text{simetria}$$

$\langle u, u\rangle \geq 0$ e vale "=" $\Leftrightarrow$ $u = 0$

definido positivo

- $\mathbb{R}^n$ $\quad \langle x, y\rangle = x \cdot y$

$$= x_1 y_1 + \dots + x_n y_n$$

$$= \lambda(x_n y_n)$$

$$(\lambda x)\cdot y = (\lambda x_1) y_1 + \dots + (\lambda x_n) y_n$$

$$= \lambda(x_1 y_1 + \dots + x_n y_n)$$

$$= \lambda(x \cdot y) \quad \text{etc.}$$

$x \cdot x = \|x\|^2 \geq 0$ e $x \cdot x = \|x\|^2 = 0 \Leftrightarrow x = 0$.

- $V = C([-1,1]; \mathbb{R})$ fns. conts $[-1,1] \to \mathbb{R}$

$$\langle f, g\rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

- $V = P(\mathbb{R})$

$$\langle p, q\rangle = \int_0^\infty p(x) q(x) e^{-x}\, dx$$

- $V = P_2(\mathbb{R})$

$$\langle p, q\rangle = p(0)q(0) + 2\, p(1)q(1) + p(2)q(2)$$

$$\langle p, p\rangle = p(0)^2 + 2\, p(1)^2 + p(2)^2 \geq 0$$

Se $\langle p,p\rangle = 0$ então $p(0) = p(1) = p(2) = 0$,

mas $\deg p \leq 2 \Rightarrow p = 0$ (polinômio nulo).

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V com $\langle\, ,\, \rangle$: ESPAÇO COM PRODUTO INTERNO

ou

ESPAÇO EUCLIDEANO

Obs. $\langle 0, v\rangle = 0 \quad \forall v \in V$

linear no 1.º argumento : $V \to \mathbb{R}$

$0 \mapsto 0$

$\|v\| := \sqrt{\langle v,v\rangle}$ norma de v

$$\cos \measuredangle(u,v) := \frac{\langle u,v\rangle}{\|u\|\,\|v\|} \quad \text{se } u,v \neq 0$$

ângulo entre u e v

$\measuredangle(u,v) \in [0,\pi]$

Obs. $\|v\| = 0 \Leftrightarrow v = 0$

$$\begin{aligned} \|\lambda v\| &= \sqrt{\langle \lambda v, \lambda v\rangle} \\ &= \sqrt{\lambda^2 \langle v,v\rangle} \\ &= |\lambda| \sqrt{\langle v,v\rangle} \\ &= |\lambda|\, \|v\| \end{aligned}$$

<u>Def</u> $u \perp v \iff \langle u, v\rangle = 0$

<u>Obs</u> 0 é o único vetor $\perp V$.

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<u>PRODUTO INTERNO HERMITEANO EM UM ESP VET COMPLEXO</u>

Seja V um esp vet$/\mathbb{C}$. Um <u>produto interno Hermiteano</u> em V é uma aplicação

$V \times V \to \mathbb{C}$ t.q. $\leftarrow$ Forma <u>sesqui</u>linear

$$\left.\begin{aligned} \langle u_1+u_2, v\rangle &= \langle u_1, v\rangle + \langle u_2, v\rangle \\ \langle \lambda u, v\rangle &= \lambda\langle u, v\rangle \end{aligned}\right] \text{linear no 1.º arg.}$$

$$\left.\begin{aligned} \langle u, v_1+v_2\rangle &= \langle u, v_1\rangle + \langle u, v_2\rangle \\ \langle u, \lambda v\rangle &= \bar{\lambda}\langle u, v\rangle \end{aligned}\right] \text{conjugado-linear no 2º arg.}$$

$$\langle v, u\rangle = \overline{\langle u, v\rangle}$$

$\langle u, u\rangle \geq 0$ e a vale a "$=$" $\iff u = 0$

<u>Convenção.</u> Para $z \in \mathbb{C}$:

$z \geq 0$ ou $z > 0$ significa $z \in \mathbb{R}$ e $z \geq 0$ ou $z \in \mathbb{R}$ e $z > 0$.

Ex. $V = \mathbb{C}^n$

$$\langle z, w\rangle = c_1 z_1 \overline{w_1} + \dots + c_n z_n \overline{w_n}$$

$$c_1, \dots, c_n > 0$$

$\mathbb{F} = \mathbb{C}$ $\quad V$ com prod interno Herm. $\langle \cdot, \cdot \rangle$:

V : espaço Hermiteano.

- $\langle 0, v\rangle = \langle v, 0\rangle = 0 \quad \forall v \in V$
- $\|v\| = \sqrt{\langle v, v\rangle}$ norma
- $u \perp v \iff \langle u, v\rangle = 0$
- $u \perp V \iff u = 0$

6.13 TEOREMA DE PITÁGORAS

$$u \perp v \implies \|u+v\|^2 = \|u\|^2 + \|v\|^2 \quad (\mathbb{F} = \mathbb{R} \text{ ou } \mathbb{C})$$

Dem. $\|u+v\|^2 = \langle u+v, u+v\rangle$

$$= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle$$

$$= \|u\|^2 + \langle u, v\rangle + \overline{\langle u, v\rangle} + \|v\|^2$$

$$= \|u\|^2 + 2\,\mathrm{Re}\,\langle u, v\rangle + \|v\|^2$$

$= 0$ pois $u \perp v$

$= \|u\|^2 + \|v\|^2$ //

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No caso $\mathbb{F} = \mathbb{R}$ vale a recíproca do T. Pitágoras:

$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$$

Portanto:

$$u \perp v \Leftrightarrow \langle u, v\rangle = 0 \Leftrightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$$

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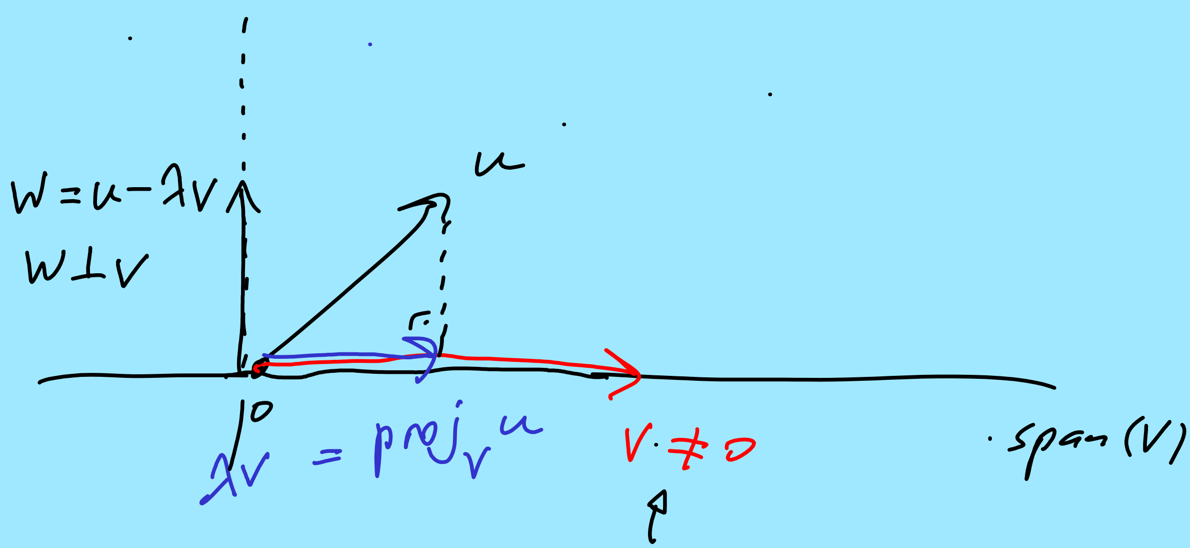
PROJEÇÃO ORTOGONAL SOBRE UM SUBESPAÇO UNIDIMENSIONAL



$$W \perp v \Leftrightarrow \langle w, v\rangle = 0 \Leftrightarrow \langle u - \lambda v, v\rangle = 0$$

$$\Leftrightarrow \langle u, v\rangle - \lambda\langle v, v\rangle = 0 \Leftrightarrow \lambda = \frac{\langle u, v\rangle}{\|v\|^2}$$

$(\|v\| \neq 0)$

$$\therefore \boxed{\mathrm{proj}_v u = \frac{\langle u, v\rangle}{\|v\|^2} v}$$

$$= \left\langle u, \frac{v}{\|v\|} \right\rangle \frac{v}{\|v\|} \leftarrow$$

$$u = \underbrace{\mathrm{proj}_v u}_{/\!/ v} + \underbrace{(u - \mathrm{proj}_v u)}_{\perp v}$$

DESIGUALDADE DE CAUCHY-SCHWARTZ

Sejam $u, v \in V$, onde V é um esp vet c/ prod interno (resp. Hermiteano). ($\mathbb{F} = \mathbb{R}$ ou $\mathbb{C}$) Então

$\iff \langle u,v\rangle^2 \leq \|u\|^2 \|v\|^2$

$$|\langle u,v\rangle| \leq \|u\| \, \|v\|$$

Além disso, a "=" vale $\iff$ u, v LD.

Dem. Podemos assumir $v \neq 0$

$$u = \underbrace{\mathrm{proj}_v u}_{/\!/ v} + w = \frac{\langle u,v\rangle}{\|v\|^2} v + w \qquad (1)$$

$w \perp v$

T. Pitágoras:

$$\|u\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^{\cancel{4}\,2}} \cancel{\|v\|^2} + \underbrace{\|w\|^2}_{\geq 0} \leftarrow$$

$$\geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$\therefore \quad \|u\|^2\,\|v\|^2 \geq |\langle u,v\rangle|^2 .$$

Pelo cálculo acima, vale a "=" $\iff$ $w=0$

por (1) $\iff$ u é múltiplo de v $\iff$ u,v LD //

Exs (a) $V = \mathbb{R}^n$ (Cauchy 1821)

$$(x_1y_1+\dots+x_ny_n)^2 \leq (x_1^2+\dots+x_n^2)(y_1^2+\dots+y_n^2)$$

(b) $V = C([-1,1];\mathbb{R})$ $\quad \mathbb{F}=\mathbb{R}$ (Schwarz 1886)

$$\left(\int_{-1}^{1} f(x)g(x)\,dx\right)^2 \leq \left(\int_{-1}^{1} f(x)^2\,dx\right)\left(\int_{-1}^{1} g(x)^2\,dx\right)$$

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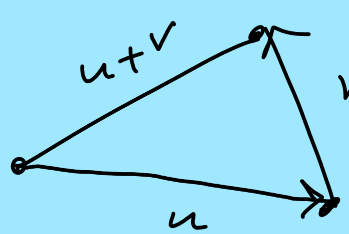
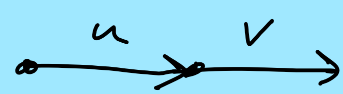
DESIGUALDADE TRIANGULAR

$$\|u+v\| \leq \|u\|+\|v\|$$

e vale a "=" $\iff$ Um dentre u e v ($\mathbb{F}=\mathbb{R}$ ou $\mathbb{C}$) é múltiplo não-negativo do outro





Dem. $\|u+v\|^2 = \|u\|^2 + \underbrace{2\operatorname{Re}\langle u,v\rangle}_{\leq 2|\langle u,v\rangle|} + \|v\|^2$

$z = x+iy$

$\operatorname{Re} z = x$

$|z| = \sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$

$$\overset{*}{\leq} \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|$$

$$\overset{C-S}{\underset{**}{\leq}} \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\|$$

$$= (\|u\| + \|v\|)^2$$

$$\therefore \|u+v\| \leq \|u\| + \|v\|$$

A "=" vale $\iff$ Vale a igualdade em $*$ e $**$

$\iff \operatorname{Re}\langle u,v\rangle = |\langle u,v\rangle|$ e $|\langle u,v\rangle| = \|u\|\,\|v\|$

$\iff \langle u,v\rangle \geq 0$ e u,v LD

$\iff u = \lambda v$ com $\lambda > 0$

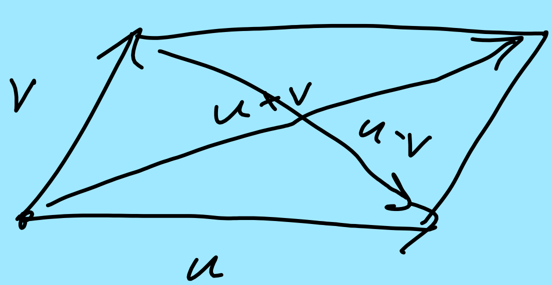
ou

$v = \lambda u$ com $\lambda > 0$. //

$z = x+iy$

$x = |z| \implies y = 0$ e $x \geq 0$

LEI DOS PARALELOGRAMOS



Vale: $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

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Q. É sempre possível dotar um esp vet V de um prod interno?

SIM

Resp Suponhamos $\dim V < \infty$ e $\mathbb{F} = \mathbb{R}$.

Seja $v_1, \dots, v_n$ uma base de V.

Dados $u, v \in V$, escrevemos

$$u = \sum_i a_i v_i \qquad v = \sum_i b_i v_i$$

e pomos $\langle u, v \rangle := \sum_i a_i b_i$

$\mathbb{F} = \mathbb{C}$ $\quad \langle u, v \rangle := \sum_i a_i \overline{b_i}$