DUALIDADE (§3.F) 25/01/22

$\mathcal{L}(V, \mathbb{F}) =: V'$ espaço dual de V

Elementos de V': funcionais lineares

Exs. • $\varphi : \mathbb{R}^3 \to \mathbb{R}$, $\varphi(x,y,z) = 4x - 5y + 2z$

$\varphi \in (\mathbb{R}^3)'$

• $\varphi(p) = 3p''(5) + 7p(4)$ $\quad \varphi \in (\mathcal{P}(\mathbb{R}))'$

$p \in \mathcal{P}(\mathbb{R})$

$$\begin{aligned}\varphi(p+q) &= 3(p+q)''(5) + 7(p+q)(4) \\ &= (3p''+3q'')(5) + 7(p+q)(4) \\ &= 3p''(5) + 3q''(5) + 7p(4) + 7q(4) \\ &= (3p''(5) + 7p(4)) + (3q''(5) + 7q(4)) \\ &= \varphi(p) + \varphi(q) \quad \text{etc.}\end{aligned}$$

• $\varphi(p) = \int_0^1 p(x)\,dx$ $\quad \varphi \in (\mathcal{P}(\mathbb{R}))'$

• $\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$ $\quad \varphi \in (\mathbb{F}^{2\times 2})'$

3.95 Se $\dim V < \infty$ então $\dim V' < \infty$ e

$\dim V' = \dim V$.

De fato $\dim V' = \dim \mathcal{L}(V, \mathbb{F}) = \dim V \underbrace{\dim \mathbb{F}}_{=1} = \dim V$ //

3.96 **Def.** [BASE DUAL]

Dada uma base B: $v_1, \ldots, v_n$ de V, a **base dual** de B é a base $\varphi_1, \ldots, \varphi_n$ de V' dada por

$$\varphi_i(v_j) = \begin{cases} 1, & \text{se } j = i \\ 0, & \text{se } j \neq i \end{cases} \qquad \varphi_i : V \to \mathbb{F}$$

Obs. Lembremos de (3.5): uma transf lin fica unicam/e determinada pela sua ação numa bar do domínio.

3.97 $\varphi_1, \ldots, \varphi_n$ é de fato uma base de V'.

Dem $\dim V' = \dim V = n$

$\varphi_1, \ldots, \varphi_n$ lista de compr n

Basta ver que $\varphi_1, \ldots, \varphi_n$ é L.I.

Tomemos uma relação linear

$$a_1 \varphi_1 + \ldots + a_n \varphi_n = \underset{\in V'}{0} \qquad (*)$$

onde $a_1, \ldots, a_n \in \mathbb{F}$.

Avaliando $(*)$ em v_j, $j = 1, \ldots, n$, vem que:

$$a_1\varphi_1(v_j) + \dots + a_n\varphi_n(v_j) = 0$$

$$a_1 . 0 + \dots + a_{j-1} . 0 + a_j . 1 + a_{j+1} 0 + \dots + a_n 0 = 0$$

$$\Rightarrow a_j = 0 \quad \forall j = 1, \dots, n$$

$\therefore$ $\varphi_1, \dots, \varphi_n$ é L.I. //

3.97 <u>Ex</u>. $e_1, \dots, e_n$ base canônica de $\mathbb{F}^n$

$e_1 = (1, 0, \dots, 0)$ Qual é a base dual?

$e_2 = (0, 1, 0, \dots, 0)$

$\vdots$

$e_n = (0, \dots, 0, 1)$

<u>Resp</u>. $\varphi_1, \dots, \varphi_n$ onde $\varphi_i(e_j) = \begin{cases} 1, \text{ se } i = j; \\ 0, \text{ se } i \neq j. \end{cases}$

$\varphi_i : \mathbb{F}^n \to \mathbb{F}$

$$\varphi_i(x_1, \dots, x_n) = x_1 \varphi_i(1, 0, \dots, 0) + \dots + x_n \varphi_i(0, \dots, 0, 1)$$

$$= x_1 \underbrace{\varphi_i(e_1)}_{=0} + \dots + x_i \underbrace{\varphi_i(e_i)}_{=1} + \dots + x_n \underbrace{\varphi_i(e_n)}_{=0}$$

$$= x_i$$

$\therefore$ $\varphi_i(x_1, \dots, x_n) = x_i \quad \forall i = 1, \dots, n$

3.99 Seja $T \in \mathcal{L}(V, W)$. A <u>transformação dual</u>

de T, denotada com T', é $T' \in \mathcal{L}(W', V')$,
definida por $T'(\varphi) = \varphi \circ T$. $(\in V')$

$$\varphi \in W'$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right] & W \arrow[r, "\varphi"] & \mathbb{F} \end{tikzcd}$$

$$\varphi \circ T =: T'(\varphi)$$

Verifiquemos que $\underline{T' \text{ é linear}}$:

- $\varphi, \psi \in W'$

$$T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T$$
$$= T'(\varphi) + T'(\psi)$$

- $\varphi \in W'$, $\lambda \in \mathbb{F}$.

$$T'(\lambda\varphi) = (\lambda\varphi) \circ T = \lambda(\varphi \circ T)$$
$$= \lambda T'(\varphi)$$

3.100 $\underline{\text{Ex}}$ $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$, $Dp = p'$

O que é $D' \in \mathcal{L}(\mathcal{P}(\mathbb{R})')$?

Seja $\varphi \in \mathcal{P}(\mathbb{R})'$. Então $D'(\varphi) \in \mathcal{P}(\mathbb{R})'$

$$D'(\varphi)(p) = (\varphi \circ D)p = \varphi(Dp) = \varphi(p')$$

$$p \in \mathcal{P}(\mathbb{R})$$

Por ex, se $\varphi(p) = p(3)$, então $D'(\varphi)(p) = p'(3)$.

Por ex, se $\psi(p) = \int_0^1 p$, então

$$D'(\psi)(p) = \psi(p') = \int_0^1 p' \underset{\substack{\text{T. Fund} \\ \text{Cálculo}}}{=} p(1) - p(0) \quad //$$

3.101 Propriedades algébricas da transf dual

(i) $(S+T)' = S' + T'$

(ii) $(\lambda T)' = \lambda T'$

$V \xrightarrow{S,T} W \qquad \lambda \in \mathbb{F}$

(iii) $(ST)' = T'S'$ $\leftarrow$

$$\varphi \circ S = S'(\varphi)$$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] \arrow[rr, bend left] & W \arrow[r, "\varphi"] & \mathbb{F} \end{tikzcd}$$

$$\begin{tikzcd} U' & V' \arrow[l, "T'"'] & W' \arrow[l, "S'"'] \arrow[ll, bend left, "T'S'"] \end{tikzcd}$$

Dem de (iii) $\quad ST : U \to W$

$$(ST)' : \underset{\substack{\cup \\ \varphi}}{W'} \to U' \quad ?$$

$$(ST)'\varphi = \varphi \circ (ST) = (\varphi \circ S) \circ T = \overbrace{(S'(\varphi))}^{\in V'} \circ T$$

$$= T'(S'\varphi) \qquad \therefore (ST)' = T'S' \quad //$$

$$\left[S'\varphi = \psi \in V' \qquad T'(\psi) = \psi \circ T \right]$$

3.102 Def. [ANULADOR]

Seja U um subconjunto de V. Então o anulador de U é

$$U^0 = \{ \varphi \in V' \mid \varphi(u) = 0, \ \forall u \in U \}.$$

3.103 Ex. $U = \{\text{múltiplos de } x^2\} \subset \underbrace{P(\mathbb{R})}_{= V}$

$\varphi, \psi, \eta \in P(\mathbb{R})'$

$\varphi(p) = p(0)$

$\psi(p) = p'(0)$

$\eta(p) = p''(0)$

$p \in U \Rightarrow 0$ é uma raiz dupla de p

$\Rightarrow \varphi(p) = 0$

$\psi(p) = 0$

$\eta(p) \neq 0$ em geral

$\therefore \varphi, \psi \in U^0, \ \eta \notin U^0.$

3.104 Seja $e_1, \dots, e_5$ a base canônica de $\mathbb{R}^5$, e seja $\varphi_1, \dots, \varphi_5$ a base dual de $(\mathbb{R}^5)'$.

Tomemos $U = \operatorname{span}(e_1, e_2)$

$$= \{ (x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 \mid x_1, x_2 \in \mathbb{R} \}$$

Determinar U^0.

Resolução. Já vimos que

$$\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j \qquad j = 1, \dots, 5$$

É claro que $\varphi_3, \varphi_4, \varphi_5 \in U^0$.

$(a_3\varphi_3 + a_4\varphi_4 + a_5\varphi_5)(u) \qquad u \in U$

$$= a_3 \underbrace{\varphi_3(u)}_{=0} + a_4 \underbrace{\varphi_4(u)}_{=0} + a_5 \underbrace{\varphi_5(u)}_{=0} = 0$$

$$\Rightarrow \boxed{\operatorname{Span}(\varphi_3, \varphi_4, \varphi_5) < U^0}$$

Seja $\varphi \in U^0 < V'$. Usando a base $\varphi_1, \dots, \varphi_5$ de V':

$$\varphi = b_1\varphi_1 + \dots + b_5\varphi_5 \qquad (\star)$$

onde $b_1, \dots, b_5 \in \mathbb{R}$.

Como $e_1 \in U$ e $\varphi \in U^0$, temos

$$\varphi(e_1) = 0$$

$$\overset{(\star)}{\Rightarrow} b_1 \underbrace{\varphi_1(e_1)}_{=1} + b_2 \underbrace{\varphi_2(e_1)}_{=0} + \dots + b_5 \underbrace{\varphi_5(e_1)}_{=0} = 0$$

$$\Rightarrow b_1 = 0 \qquad (\star\star)$$

Analogamente, $e_2 \in U$ e $\varphi \in U^0 \Rightarrow \varphi(e_2) = 0$

$$\overset{(\star)}{\Rightarrow} b_2 = 0 \qquad (\star\star\star)$$

Substituindo (★★) e (★★★) em (★):

$$\varphi = b_3\varphi_3 + b_4\varphi_4 + b_5\varphi_5$$

$$\in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \quad \therefore \; U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$$

$$\therefore \; U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \; //.$$

3.105 Seja U um subconjunto de V. Então U^0 sempre é um subespaço de V!

<u>Dem.</u> • $0 \in U^0$
↑ funcional linear nulo

• $\varphi, \psi \in U^0 \Rightarrow (\varphi+\psi)u = \underbrace{\varphi(u)}_{=0} + \underbrace{\psi(u)}_{=0}$
$u \in U$

$= 0 + 0 = 0 \Rightarrow \varphi + \psi \in U^0$

• $\varphi \in U^0, \; \lambda \in \mathbb{F} \Rightarrow (\lambda\varphi)u = \lambda(\varphi u) = \lambda 0 = 0$
$u \in U$

$\Rightarrow \lambda\varphi \in U^0 \; //.$

3.106 Se $\dim V < \infty$ e U é um subespaço de V, então $\boxed{\dim U + \dim U^0 = \dim V}$

Dem. Seja $i : U \to V$ a inclusão.
$u \mapsto u$

(i é linear). O que é $i' : V' \to U'$?
$\varphi \mapsto \varphi \circ i$

$i'(\varphi) = \varphi \circ i$

$\varphi \in V'$

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[rr, bend right, "\varphi \circ i"'] & V \arrow[r, "\varphi"] & \mathbb{F} \end{tikzcd}$$

$$\boxed{\varphi \circ i = \varphi|_U}$$

i' é a restrição de V a U!

Vamos aplicar o T.F.A.L. a i':

$$\dim V' = \dim \ker i' + \dim \operatorname{im} i' \quad (1)$$

$$\ker i' = \{\varphi \in V' \mid \varphi|_U = 0\} = U^0$$

$$\dim V' = \dim V$$

De (1), vem:

$$\dim V = \dim U^0 + \dim \operatorname{im} i' \quad (2)$$

Falta apenas identificar $\operatorname{im} i' = U'$!

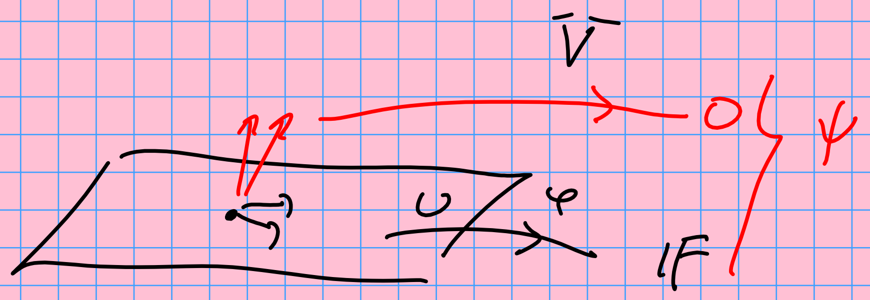
Queremos ver que $i' : V' \to U'$ é sobrejetora.

Ou seja, que dada $\varphi \in U'$, existe $\psi \in V'$

t.q. $i'(\psi) = \varphi$ ou $\psi \circ i = \varphi$ ou $\psi|_U = \varphi$.

Então queremos saber se toda $\varphi : U \to \mathbb{F}$

pode ser estendida a uma $\psi : V \to \mathbb{F}$



(Ex. 11, §3.A)

Sim, existe uma extensão.

Agora $\operatorname{im} i' = U' \Rightarrow \dim \operatorname{im} i' = \dim U' = \dim U$

e substituímos em (2) //.

3.107 $\boxed{\ker T'}$

Seja $T \in \mathcal{L}(V, W)$. Então

(a) $\ker T' = (\operatorname{im} T)^0$

(b) Se $\dim V, \dim W < \infty$, então

$\dim \ker T' = \dim \ker T + \dim W - \dim V$.

<u>Dem.</u> $V \xrightarrow{T} W \supset \operatorname{im} T \Rightarrow (\operatorname{im} T)^0 \subset W'$

$$V' \xleftarrow{T'} \begin{matrix} W' \\ \cup \\ \ker T' \end{matrix}$$

(a) Seja $\varphi \in \ker T'$. Então

$$0 = T'(\varphi) = \varphi \circ T$$

$$\Rightarrow \varphi(Tv) = 0 \quad \forall v \in V \Rightarrow \varphi|_{\operatorname{im} T} = 0$$

$$\Rightarrow \varphi \in (\operatorname{im} T)^0$$

$$\therefore \underline{\ker T' \subset (\operatorname{im} T)^0}$$

Seja agora $\varphi \in (\operatorname{im} T)^0$. Então

$$\varphi|_{\operatorname{im} T} = 0$$

$$\Rightarrow \underbrace{\varphi(Tv)}_{= T'(\varphi)v} = 0 \quad \forall v \in V$$

$$\Rightarrow T'(\varphi) = 0 \Rightarrow \varphi \in \ker T' \quad \therefore \underline{(\operatorname{im} T)^0 \subset \ker T'}$$

$$\therefore \ker T' = (\operatorname{im} T)^0$$

(b)
$$\begin{aligned}
\dim \ker T' &= \dim (\operatorname{im} T)^0 \quad \text{(por (a))} \\
&= \dim W - \dim \operatorname{im} T \quad \text{(por (3.106))} \\
&= \dim W - (\dim V - \dim \ker T) \quad \text{(T.F.A.L.)} \\
&= \dim \ker T + \dim W - \dim V \;//
\end{aligned}$$

3.108 Suponhamos que $\dim V, \dim W < \infty$, e $T \in \mathcal{L}(V, W)$. Então

T é sobrejetora $\Leftrightarrow$ T' é injetora.

<u>Dem</u>. T sobrej $\Leftrightarrow$ $\operatorname{im} T = W$ $\overset{3.106}{\Leftrightarrow}$ $(\operatorname{im} T)^0 = \{0\}$

$\overset{3.107(a)}{\Leftrightarrow}$ $\ker T' = \{0\}$ $\Leftrightarrow$ T' inj //.

3.106: $\dim \underline{\operatorname{im} T} + \dim (\underline{\operatorname{im} T})^0 = \dim W$