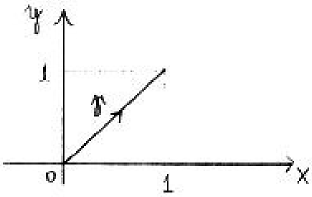
MAT 220 – Cálculo IV   Lista 4

1-) a) $f(z) = y - x - 3x^2 i$



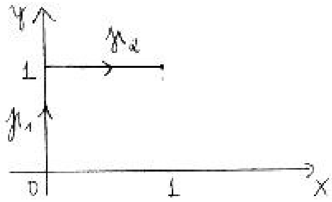
γ: $y = x$ $\qquad dz = dx + i\,dy$

$dy = dx \qquad dz = (1+i)\,dx$

$$\int_\gamma f(z)\,dz = \int_0^{1+i} (y - x - 3x^2 i)(dx + i\,dy) = \int_0^1 (x - x - 3x^2 i)(1+i)\,dx =$$

$$= \int_0^1 3x^2(1-i)\,dx = 3(1-i)\frac{x^3}{3}\Big|_0^1 = \underline{1-i}$$

b) $f(z) = y - x - 3x^2 i$



$\gamma = \gamma_1 + \gamma_2$

γ_1: $x = 0$, $y = y$ $\Rightarrow$ $dx = 0$, $dy = dy$

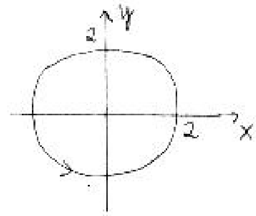
$dz = dx + i\,dy$

γ_2: $y = 1$, $x = x$ $\Rightarrow$ $dy = 0$, $dx = dx$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_0^1 y\,i\,dy + \int_0^1 (1 - x - 3x^2 i)\,dx =$$

$$= i\frac{y^2}{2}\Big|_0^1 + \left(x - \frac{x^2}{2} - x^3 i\right)\Big|_0^1 = \frac{i}{2} + 1 - \frac{1}{2} - i = \underline{\frac{1}{2}(1-i)}$$

c) $f(z) = \dfrac{z+2}{z}$



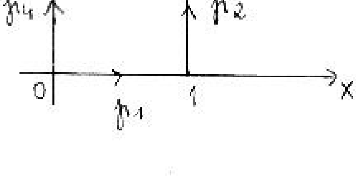
γ: $z = 2e^{i\theta}$; $\theta \in [-\pi, \pi]$

$dz = 2i\,e^{i\theta}\,d\theta$

$$\int_\gamma f(z)\,dz = \int_{-\pi}^{\pi} \frac{2e^{i\theta} + 2}{\cancel{2}e^{i\theta}} \cdot \cancel{2}\, i\, \cancel{e^{i\theta}}\,d\theta = \int_{-\pi}^{\pi} 2i\left(e^{i\theta} + 1\right)d\theta =$$

$$= 2i\left(\frac{e^{i\theta}}{i} + \theta\right)\Big|_{-\pi}^{\pi} = 2i\,2\pi = \underline{4\pi i}$$

d) $f(z) = 3z + 1 = 3x + 1 + 3yi$



$\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$

$\gamma_1: \begin{matrix} y=0 \\ x=t \end{matrix} \Rightarrow \begin{matrix} dy=0 \\ dx=dt \end{matrix}$; t de 0 a 1 $\quad \gamma_3: \begin{matrix} y=1 \\ x=t \end{matrix} \Rightarrow \begin{matrix} dy=0 \\ dx=dt \end{matrix}$; t de 0 a 1

$\gamma_2: \begin{matrix} x=1 \\ y=t \end{matrix} \Rightarrow \begin{matrix} dx=0 \\ dy=dt \end{matrix}$; t de 0 a 1 $\quad \gamma_4: \begin{matrix} x=0 \\ y=t \end{matrix} \Rightarrow \begin{matrix} dx=0 \\ dy=dt \end{matrix}$; t de 0 a 1

$dz = dx + i\,dy$

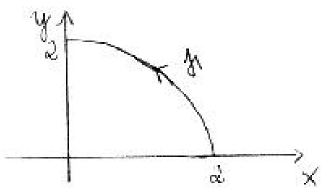
$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz - \int_{\gamma_3} f(z)\,dz - \int_{\gamma_4} f(z)\,dz =$$

$$= \int_0^1 (3t+1)\,dt + \int_0^1 (3+1+3ti)\,i\,dt - \int_0^1 (3t+1+3i)\,dt - \int_0^1 (1-3ti)\,i\,dt =$$

$$= \left(3\frac{t^2}{2}+t\right)\Big|_0^1 + \left(4ti - 3\frac{t^2}{2}\right)\Big|_0^1 - \left[3\frac{t^2}{2} + (1+3i)t\right]\Big|_0^1 - \left(it - 3\frac{t^2}{2}\right)\Big|_0^1 =$$

$$= \frac{3}{2} + \cancel{1} + 4i - \cancel{\frac{3}{2}} - \frac{3}{2} - \cancel{1} - 3i - i + \frac{3}{2} = \underline{0}$$

2-)



$|z| = 2 \Rightarrow z = 2e^{i\theta}$

$$\left|\int_\gamma \frac{dz}{z^2+1}\right| \leqslant \int_\gamma \frac{|dz|}{|z^2+1|} = \int_\gamma \frac{|dz|}{|4e^{i2\theta}+1|} = \int_\gamma \frac{|dz|}{|(4\cos 2\theta + 1) + i\,4\,\mathrm{sen}\,2\theta|} =$$

$$= \int_\gamma \frac{|dz|}{\sqrt{16\cos^2 2\theta + 8\cos 2\theta + 1 + 16\,\mathrm{sen}^2 2\theta}} = \int_\gamma \frac{|dz|}{\sqrt{17+8\cos 2\theta}} \leqslant \int_\gamma \frac{|dz|}{\sqrt{17-8}} =$$

é máximo p/ $\cos 2\theta = -1$

$$= \frac{1}{3} \underbrace{\int_\gamma |dz|}_{\text{comprimento de } \gamma} = \frac{1}{3} \cdot \frac{1}{4} \cdot 2\pi \cdot 2 = \frac{\pi}{3}$$

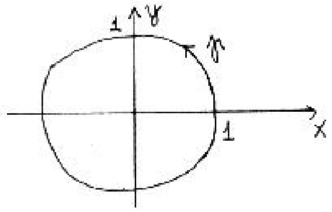
$$\Rightarrow \boxed{\left|\int_\gamma \frac{dz}{z^2+1}\right| \leqslant \frac{\pi}{3}}$$

3-) Teorema Integral de Cauchy:

Se uma função $f(z)$ é bem definida e analítica dentro e sobre uma curva γ, então

$$\int_\gamma f(z)\,dz = 0$$

3-) a)



pontos singulares:

$\cos z = 0$

$z = (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$

não há pontos singulares internos a γ

$\int_\gamma \tan z \, dz = \int_\gamma \frac{\operatorname{sen} z}{\cos z} dz$

$\tan z$ é analítica dentro e sobre γ

$\therefore \int_\gamma \tan z \, dz = 0$

b) $\int_\gamma \frac{z^2}{z-3} dz$

pontos singulares:

$z - 3 = 0$

$z = 3$ é externo a γ

$\frac{z^2}{z-3}$ é analítica dentro e sobre γ

$\therefore \int_\gamma \frac{z^2}{z-3} dz = 0$

c) $f(z) = \log(z-2)$

pontos singulares:

$z - 2 = 0$

$z = 2$ é externo a γ

$\int_\gamma \log(z-2) \, dz$

$\log(z-2)$ é analítica dentro e sobre γ

$\therefore \int_\gamma \log(z-2) \, dz = 0$