HOMOGENEOUS STRUCTURES AND RIGIDITY OF ISOPARAMETRIC SUBMANIFOLDS IN HILBERT SPACE

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Dedicated to Richard Palais on the occasion of his 80th birthday

ABSTRACT. We study isoparametric submanifolds of rank at least two in a separable Hilbert space, which are known to be homogeneous by the main result in [HL99], and associate to such a submanifold M and a point x in M a canonical homogeneous structure Γ_x (a certain bilinear map defined on a subspace of $T_xM \times T_xM$). We prove that Γ_x together with the second fundamental form α_x encodes all the information about M, and deduce from this the rigidity result that M is completely determined by α_x and $(\nabla\alpha)_x$, thereby making such submanifolds accessible to classification. As an essential step, we show that the one-parameter groups of isometries constructed in [HL99] to prove their homogeneity induce smooth and hence everywhere defined Killing fields, implying the continuity of Γ (this result also seems to close a gap in [Chr02]). Here an important tool is the introduction of affine root systems of isoparametric submanifolds.

1. INTRODUCTION

In [HPTT95] Richard Palais and his co-authors discussed among other things relations between isoparametric submanifolds in Hilbert space and affine Kac-Moody algebras. The present paper may be seen as a continuation of that line of research. It contributes to the conjecture that all isoparametric submanifolds of rank at least two in an infinite dimensional Hilbert space arise as principal orbits of isotropy representations of symmetric spaces of affine Kac-Moody type (which are obtained from involutions of the second kind of affine Kac-Moody groups).

We begin with a simple example in finite dimensions that motivates our main construction in infinite dimensions. Let $M = G/K$ be a homogeneous space embedded in a Euclidean space V as an orbit of a compact connected group G of isometries of V, where K is the isotropy subgroup at $x \in M$. Then the Lie algebra of G admits a reductive decomposition as $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K. Each element of $\mathfrak{m}$ is a Killing field on V and this defines an isomorphism $\mathfrak{m} \to T_xM$ (by evaluating the Killing field at x) whose inverse we denote by $X \mapsto \check{X}$. Now the interesting point is that the bilinear mapping $\bar{\Gamma} := \bar{\Gamma}_x : T_xM \times T_xV \to T_xV$ defined by $\bar{\Gamma}_X Y := \frac{d}{dt}|_{t=0}(\exp t\check{X})_*(Y)$ determines M completely (as well as the reductive complement). In fact, the Killing fields $\check{X} \in \mathfrak{m}$ are completely determined by their value and derivative at x, which are $X := \check{X}(x)$ and $\bar{\Gamma}_X$. Thus $\bar{\Gamma}$ determines $\mathfrak{m}$ and,

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since the subgroup corresponding to the Lie algebra generated by $\mathfrak{m}$ acts transitively on the submanifold, M is determined as well.

In the special case in which M has flat normal bundle and G induces parallel translation along it (the only case in which we will be interested), we have $\bar{\Gamma}_X\xi = -A_\xi X$ and thus by skew-symmetry of $\bar{\Gamma}_X$, $\bar{\Gamma}_X Y = \Gamma_X Y + \alpha_x(X,Y)$ for all X, $Y \in T_xM$ and $\xi \in \nu_x M$, where α_x is the second fundamental form of M at x, A_ξ is the shape operator in the direction of ξ, and

$$\Gamma = \Gamma_x : T_xM \times T_xM \to T_xM$$

is the tangential component of $\bar{\Gamma}$, i.e.

$$\Gamma_X Y := \left(\frac{d}{dt}\Big|_{t=0} (\exp t\check{X})_*(Y) \right)^{\top}.$$

Thus the pair (α_x, Γ_x) contains the same information as $\bar{\Gamma}_x$ and we call Γ a *homogeneous structure* for M.

The main goal of this paper is to carry over these ideas to isoparametric submanifolds of rank at least 2 in Hilbert space with the ultimate goal of proving the above mentioned conjecture in a forthcoming paper by restricting the possibilities for α and Γ so much that only the examples coming from affine Kac-Moody algebras remain. The restrictions on α are essentially already known (as α is determined by the affine Weyl group and the multiplicities) and many restrictions for Γ will be derived in this paper.

Next we explain the main results of this paper in more detail (we refer to section 2 for the relevant terminology and notation). Let M be a connected, complete, full, irreducible isoparametric submanifold of rank at least 2 in an separable infinite dimensional Hilbert space V. By the main result in [HL99], M is extrinsically homogeneous, but it is unknown whether the group of isometries of V preserving M is a Banach-Lie subgroup of the group of isometries of V, not to speak of a reductive complement to the isotropy subalgebra. On the other hand, it is known from [HL99] that there exist canonically defined one-parameter groups $\{F_X^t\}_t$ of isometries of V leaving M invariant, for each $X \in E_{\mathbf{i}}(x)$ with $\mathbf{i} \neq \mathbf{0}$ and each $x \in M$. The restriction of F_X^t to any curvature sphere through x (including $S_{\mathbf{0}}(x) = x + E_{\mathbf{0}}(x)$) is differentiable and X is the initial direction of the curve $t \mapsto F_X^t(x)$. Thus we can define

$$\Gamma_X Y := (\Gamma_x)_X Y := \left(\frac{d}{dt}\Big|_{t=0} (F_X^t)_*(Y) \right)^{\top}$$

for all $X \in E_{\mathbf{i}}(x)$, $Y \in E_{\mathbf{j}}(x)$ and $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ with $\mathbf{i} \neq \mathbf{0}$, in analogy with the finite dimensional situation. We call Γ the *(canonical) homogeneous structure* for M at x. The drawback of this definition is that, for each $X \in E_{\mathbf{i}}$, $\Gamma_X : T_xM \to T_xM$, or equivalently $\frac{d}{dt}|_{t=0}(F_X^t)_* : V \to V$, is in principle only densely defined and might not be continuous and hence not extendable to the whole vector space. Geometrically speaking, it might happen that $(F_X^t)_*$ rotates in certain two-dimensional subspaces faster and faster, and analytically that the infinitesimal generator of $(F_X^t)_*$ is unbounded (cf. Remark 3.2). That this does not occur is one of the main results of this paper.

Theorem A. *Each Γ_X is continuous and thus extends to T_xM. Equivalently, the one-parameter groups $\{F_X^t\}_t$ are smooth curves in the Banach-Lie group of isometries of V. Moreover, Γ is continuous as a bilinear map.*

An immediate consequence is that the tangent vectors to the orbits of F_X^t yield Killing vector fields defined on the entire Hilbert space. As a side remark, we point out that this result fills apparently a gap in [Chr02] where the existence of globally defined Killing fields was taken for granted.

From Theorem A we conclude, as in finite dimensions, that M is completely determined by α_x and Γ_x for any $x \in M$. Moreover it is easily observed that Γ_x and $(\nabla\alpha)_x$ are closely related. Actually we can show that they contain equivalent information if α_x is given (Theorem 4.3) and therefore obtain the following rigidity result.

Theorem B. *For any $x \in M$, α_x and $(\nabla\alpha)_x$ determine M completely.*

Our proof of Theorem B also applies to finite dimensional homogeneous isoparametric submanifolds, and the result seems to be even new there. In more geometric terms the theorem states that M is completely determined by the curvature spheres at a single point x (the information α_x contains) and how for each finite dimensional curvature sphere $S_{\mathbf{i}}(x)$ any other curvature sphere $S_{\mathbf{j}}$ evolves along $S_{\mathbf{i}}(x)$ infinitesimally (the information $(\nabla\alpha)_x$ contains).

The proof of Theorem A requires several steps which cover almost the entire paper and can be outlined as follows. We fix $x \in M$, $\mathbf{i} \in \mathbf{I}$ with $\mathbf{i} \neq \mathbf{0}$ and $X_{\mathbf{i}} \in E_{\mathbf{i}} = E_{\mathbf{i}}(x)$. By construction and the smoothness properties of $F_{X_{\mathbf{i}}}^t$, $\Gamma_{X_{\mathbf{i}}}$ is continuous on $E_{\mathbf{0}}$. Thus it is enough to find a constant C that depends only on $X_{\mathbf{i}}$ such that

$$||\Gamma_{X_{\mathbf{i}}} Y|| \leqq C||Y||$$

for all Y belonging to the algebraic span of the $E_{\mathbf{j}}(x)$, $\mathbf{j} \neq \mathbf{0}$. This problem can be split into two parts, namely to find a constant C_1 that works for all Y such that $X_{\mathbf{i}}$ and Y are both tangent to a finite dimensional slice through x, and a constant C_2 that works for all Y tangent to the infinite dimensional rank one slice containing $E_{\mathbf{i}}$. The first part can be easily solved as in that case one can estimate Γ in the slice: since the slice is a homogeneous finite dimensional isoparametric submanifold and such submanifolds are classified, there is no problem to find C_1 which works uniformly for all slices.

The second part is more difficult. A special case, which we eventually prove to be sufficient, is to estimate $||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}||$ for all $Y_{\mathbf{j}} \in E_{\mathbf{j}}$ and all $\mathbf{j} \neq \mathbf{0}$ such that $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ are parallel. Essentially from the Gauss equation we obtain the formula

$$\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{Y_{\mathbf{j}}} X_{\mathbf{i}} \rangle = \frac{1}{2} \langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle \, ||X_{\mathbf{i}}||^2 \, ||Y_{\mathbf{j}}||^2$$

and the Codazzi equation allows us to interchange the two arguments of Γ if one restricts $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}$ to its components $(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}$ in $E_{\mathbf{k}}$. In fact, for all $\mathbf{i} \neq \mathbf{j}$

$$(\Gamma_{Y_{\mathbf{j}}} X_{\mathbf{i}})_{E_{\mathbf{k}}} = c_{ijk} \, (\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}},$$

where $c_{ijk} = \lambda$ if $v_{\mathbf{j}} - v_{\mathbf{k}} = \lambda(v_{\mathbf{i}} - v_{\mathbf{k}})$ for some λ and $c_{ijk} = 0$ otherwise (in particular, $(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}} = 0$ if $v_{\mathbf{i}}$, $v_{\mathbf{j}}$, $v_{\mathbf{k}}$ are not colinear). The two formulae combined together yield an explicit value for the sum over $\mathbf{k}$ of $||(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}||^2$ with certain coefficients which, however, in general cannot be immediately used to estimate the length of $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}$ as the coefficients may have different signs or tend to zero. However, in the particular case that $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}$ is contained in $E_{\mathbf{0}}$ for all $\mathbf{j} \neq \mathbf{0}$ with $v_{\mathbf{j}}$ parallel to $v_{\mathbf{i}}$, all terms with $\mathbf{k} \neq \mathbf{0}$ vanish yielding

$$||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}||^2 = \frac{1}{2} ||v_{\mathbf{i}}||^2 \, ||X_{\mathbf{i}}||^2 \, ||Y_{\mathbf{j}}||^2$$

if $\mathbf{i} \neq \mathbf{j}$ and thus the continuity of $\Gamma_{X_\mathbf{i}}$. Therefore Theorem A follows in many cases from the next result, which also gives interesting information on Γ itself.

Theorem C. *Assume the affine Weyl group of M is not of type $\tilde{B}_n$ or $\tilde{C}_n$. Then*

$$\Gamma_{E_\mathbf{i}} E_\mathbf{j} \subset E_\mathbf{0}$$

if $v_\mathbf{i}$ and $v_\mathbf{j}$ are parallel.

The proof of Theorem C in turn requires several steps. Crucial ingredients are a density theorem for the image of Γ (Theorem 4.1), a formula for $\Gamma_{X_\mathbf{i}}\Gamma_{Y_\mathbf{j}}Z_\mathbf{k} - \Gamma_{Y_\mathbf{j}}\Gamma_{X_\mathbf{i}}Z_\mathbf{k}$ which is obtained from a careful analysis of the Gauss equation (Corollary 5.10) and a simple lemma from plane geometry (Lemma 8.3).

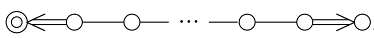
In the case of a general affine Weyl group, the statement of Theorem C does not hold as it is and the results about the image of Γ are more technical to describe. To do that, we have to refine the information on the affine Weyl group by associating an affine root system to the isoparametric submanifold M (cf. section 7). Like for finite root systems, this is described by a Dynkin diagram which is obtained from the Coxeter graph of the affine Weyl group by attaching arrows to the double and triple links and additional concentric circles around those vertices that correspond to roots for which also twice the root is a root. These concentric circles can only occur in the $\tilde{B}_n$ or $\tilde{C}_n$ cases and correspond to reducible eigenspaces $E_\mathbf{i}$, i. e. eigenspaces which split as $E_\mathbf{i} = E'_\mathbf{i} \oplus E''_\mathbf{i}$ under the isotropy representation of the isometry group of M, with $\dim E'_\mathbf{i}$ even and $\dim E''_\mathbf{i} = 1$ or 3 (cf. section 2).

At this point it is convenient to change the notation and identify $\mathbf{I} \setminus \{\mathbf{0}\}$ with $\mathcal{A} \times \mathbb{Z}$, where $\mathcal{A}$ parametrizes the infinite dimensional rank 1 slices (through x) and $\mathbb{Z}$ parametrizes, for each $\alpha \in \mathcal{A}$, the finite dimensional curvature distributions $E_{\alpha,i}$ in this rank 1 slice in such a way that the corresponding focal hyperplanes $H_{\alpha,i}$ (which are parallel to each other) occur in consecutive order. We then have the following two general results which include Theorem C as special case and are proved by similar but slightly more refined arguments. They suffice to finish the proof of Theorem A in all cases.

Theorem D. *Let $(\alpha, i) \in \mathcal{A} \times \mathbb{Z}$ with $E_{\alpha,i}$ irreducible and $j \in \mathbb{Z}$. Then*

$$\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_\mathbf{0}$$

unless the Dynkin diagram of M is

 *or* ,

$i - j$ is even, and the hyperplane $H_{\alpha,i}$ is conjugate under the affine Weyl group to the focal hyperplane corresponding the right extremal vertex while $H_{\alpha,\frac{i+j}{2}}$ is conjugate to the left one. In all cases we have

(i) $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_\mathbf{0} \oplus E_{\alpha,\frac{i+j}{2}}$ *if $i - j$ is even.*
(ii) $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_\mathbf{0}$ *if $i - j$ is divisible by 4.*
(iii) $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_\mathbf{0} \oplus E_{\alpha,2i-j} \oplus E_{\alpha,2j-i}$ *if $i - j$ is odd.*

Theorem E. *Let $(\alpha, i) \in \mathcal{A} \times \mathbb{Z}$ with $E_{\alpha,i} = E'_{\alpha,i} \oplus E''_{\alpha,i}$ reducible and $j \in \mathbb{Z}$. Then also $E_{\alpha,k}$ is reducible whenever $i - k$ is even and*

(i) $\Gamma_{E''_{\alpha,i}} E_{\alpha,j} \subset E_\mathbf{0} \oplus E'_{\alpha,2i-j}$ *if $i - j$ is even.*
(ii) $\Gamma_{E_{\alpha,i}} E''_{\alpha,j} \subset E_\mathbf{0} \oplus E'_{\alpha,2j-i}$ *if $i - j$ is even.*
(iii) $\Gamma_{E''_{\alpha,i}} E''_{\alpha,j} \subset E_\mathbf{0}$ *if $i - j$ is even.*

(iv) $\Gamma_{E'_{\alpha,i}} E'_{\alpha,j} \subset E_{\mathbf{0}} \oplus E''_{\alpha,\frac{i+j}{2}}$ *if* $i-j$ *is divisible by four.*

(v) $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_{\mathbf{0}} \oplus E_{\alpha,2i-j} \oplus E_{\alpha,2j-i}$ *if* $i-j$ *is odd.*

As an application of Theorems D and E, we show that $E_{\mathbf{0}}$ is always infinite dimensional.

It is a pleasure to thank Jost Eschenburg for several helpful discussions. Some of the questions treated here were also treated in the PhD thesis of K. Weinl [Wei06].

2. Preliminaries

We recall some basic facts about isoparametric submanifolds of Hilbert space (cf. [Ter89, PT88, HL99]) and introduce terminology and notation that will be used throughout the paper.

A submanifold M of an infinite dimensional separable Hilbert space V is called *proper Fredholm* if the normal exponential map $\nu M \to V$ restricted to any finite normal disk bundle is a proper Fredholm map. A proper Fredholm submanifold M is called *isoparametric* if its normal bundle is globally flat, and the shape operators along any parallel normal vector field are conjugate. Here globally flat means that every normal vector can be uniquely extended to a parallel normal vector field along the whole of M. The Fredholm condition implies that the codimension of M is finite and its shape operators are compact (self-adjoint) operators. Since the normal bundle νM is flat, the Ricci equation yields a splitting of the tangent bundle as $TM = \oplus_{\mathbf{i}\in\mathbf{I}} E_{\mathbf{i}}$ (closure of the algebraic direct sum) into the simultaneous eigendistributions $E_{\mathbf{i}}$ of the shape operators, where $\mathbf{I}$ is a countable index set containing $\mathbf{0}$; each $E_{\mathbf{i}}$ is called a *curvature distribution*. For each normal vector $\xi \in \nu M$, the corresponding shape operator satisfies $A_\xi|_{E_{\mathbf{i}}} = \langle \xi, v_{\mathbf{i}}\rangle \mathrm{id}_{E_{\mathbf{i}}}$, where $v_{\mathbf{i}}$ is a globally defined parallel normal vector field on M; each $v_{\mathbf{i}}$ is called a *curvature normal*. Unless explicitly stated, we will denote the zero curvature normal by $v_{\mathbf{0}}$ (if it occurs) and the corresponding curvature distribution by $E_{\mathbf{0}}$. For convenience, we also set $\mathbf{I}^* = \mathbf{I}\setminus\{\mathbf{0}\}$. Note that the substantial codimension of M equals the number of linearly independent curvature normals; this number is called the *rank* of M. We will always assume that M is full in V, that is, not contained in a proper affine subspace. It then follows that the curvature normals of M span the normal space.

As a consequence of the Codazzi equations, each curvature distribution $E_{\mathbf{i}}$ is integrable with totally geodesic leaves. The leaf of $E_{\mathbf{i}}$ passing through x is denoted by $S_{\mathbf{i}}(x)$. Due to the compactness of the shape operators, the dimension of $E_{\mathbf{i}}$ is finite if $\mathbf{i} \neq \mathbf{0}$, and then $m_{\mathbf{i}} = \dim E_{\mathbf{i}}$ is called a *multiplicity*, but $E_{\mathbf{0}}$ can have infinite dimension. Since the leaves of the $E_{\mathbf{i}}$ are umbilic in V (by definition), it follows that $S_{\mathbf{i}}(x)$ is a round sphere for $\mathbf{i} \neq \mathbf{0}$ (centered at $c_{\mathbf{i}}(x) := x + (v_{\mathbf{i}}(x)/||v_{\mathbf{i}}||^2)$, with radius $1/||v_{\mathbf{i}}||$) and $S_{\mathbf{0}}(x) = x + E_{\mathbf{0}}(x)$ is a closed affine subspace.

We will always assume that M is complete. A complete isoparametric submanifold M of V determines a singular foliation of V by parallel submanifolds, where the regular leaves are also isoparametric of the same codimension as M and the singular leaves are the focal manifolds of M. Indeed a parallel submanifold of M, denoted by M_ξ, is determined by a parallel normal vector field ξ along M so that the map $\pi_\xi : x \mapsto x + \xi(x)$ from M to M_ξ is a submersion. This map has differential $\mathrm{id} - A_\xi$ so its kernel is $\oplus\{E_{\mathbf{i}} \mid \mathbf{i} \in \mathbf{I}, \langle \xi, v_{\mathbf{i}}\rangle = 1\}$. We thus see that the focal set of M decomposes into focal manifolds, and M_ξ is a focal manifold precisely if $\ker d\pi_\xi$ is not zero in which case π_ξ is called a *focal map*. For $x \in M$, the affine normal space $x + \nu_x M$ meets the focal set of M along the union over $\mathbf{i} \in \mathbf{I}^*$ of the affine hyperplanes $H_{\mathbf{i}}(x) := x + \{\xi \in \nu_x M \mid \langle \xi, v_{\mathbf{i}}\rangle = 1\}$, called *focal hyperplanes* (with respect to x). An important result of Terng [Ter89] states that the group W which is generated by reflections in the $H_{\mathbf{i}}(x)$, $\mathbf{i} \in \mathbf{I}^*$, is an affine Weyl group acting on $x + \nu_x M$. We will always assume that M is irreducible, i.e. it cannot be split as an extrinsic product of lower dimensional isoparametric submanifolds. It then follows that W acts irreducibly on $x + \nu_x M$ (cf. [HL97]) and thus is isomorphic to one of $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_4$, $\tilde{G}_2$. It follows that the set of focal hyperplanes decomposes into finitely many families of parallel equidistant hyperplanes in $x + \nu_x M$. For each family, the corresponding curvature normals are thus of the form $v_k = (d_0 + kd)^{-1} v$, $k \in \mathbb{Z}$, where v is a unit vector, d is the distance between two consecutive hyperplanes and d_0 is the distance from x to the first hyperplane of this family in the direction of v. It also follows that there are only finitely many different multiplicities, since those are preserved by the action of W on the set of hyperplanes. Later in the paper it will be convenient to identify $\mathbf{I}^*$ with $\mathcal{A} \times \mathbb{Z}$, where $\mathcal{A}$ is a finite index set parametrizing the families of parallel curvature normals and, for each $\alpha \in \mathcal{A}$, $\mathbb{Z}$ parametrizes the curvature normals in that family so that the corresponding focal hyperplanes $\{H_{\alpha,i}(x)\}_{i \in \mathbb{Z}}$ are in consecutive order. Whereas typical indices in $\mathbf{I}$ will be denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$..., typical indices in $\mathcal{A} \times \mathbb{Z}$ will be denoted by (α, i), (β, j), (γ, k),...

Since the second fundamental form α of M is related to the shape operators by $\langle \alpha(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle$ for X, $Y \in T_x M$, $\xi \in \nu_x M$, it follows that

$$\alpha(X_{\mathbf{i}}, Y_{\mathbf{j}}) = \langle X_{\mathbf{i}}, Y_{\mathbf{j}} \rangle \, v_{\mathbf{i}} \tag{2.1}$$

for $X_{\mathbf{i}} \in E_{\mathbf{i}}(x)$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}(x)$. Hence the focal hyperplanes in $x + \nu_x M$ together with the multiplicities $m_{\mathbf{i}} = \dim E_{\mathbf{i}}$, $\mathbf{i} \in \mathbf{I}^*$, essentially determine the second fundamental form, up to passing to a parallel isoparametric submanifold. In turn the focal hyperplanes are already determined by the affine Weyl group, up to scaling of the ambient metric. Thus the affine Weyl group together with the multiplicities essentially determine the second fundamental form. Such data is usually encoded in a Coxeter graph with multiplicities (cf. section 7 for the Coxeter graph).

Another fundamental invariant of M is the covariant derivative of the second fundamental form. By taking derivatives, it follows from (2.1) that

$$\nabla_{X_{\mathbf{i}}} \alpha(Y_{\mathbf{j}}, Z_{\mathbf{k}}) = \langle \nabla_{X_{\mathbf{i}}} Y_{\mathbf{j}}, Z_{\mathbf{k}} \rangle (v_{\mathbf{j}} - v_{\mathbf{k}}) \tag{2.2}$$

for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}$ and $X_{\mathbf{i}} \in E_{\mathbf{i}}(x)$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}(x)$, $Z_{\mathbf{k}} \in E_{\mathbf{k}}(x)$. One uses the Codazzi equation, which is the symmetry of $\nabla \alpha$ in all three arguments, to derive strong restrictions on M. For instance if P is an affine subspace of $x + \nu_x M$ then $\mathcal{D}_P(x) := \oplus_{v_{\mathbf{i}}(x) \in P} E_{\mathbf{i}}(x)$ (closure of algebraic direct sum) is an integrable distribution with totally geodesic leaves. Moreover the leaf through $x \in M$, to be denoted by $L_P(x)$, is a complete full isoparametric submanifold of the affine subspace $W_P(x) := x + \mathcal{D}_P(x) + \operatorname{span}\{v_{\mathbf{i}}(x) \mid v_{\mathbf{i}}(x) \in P\}$ of rank $\dim P$ if $0 \in P$ and $\dim X + 1$ if $0 \notin P$. By taking in particular $P = \{v_{\mathbf{i}}\}$ for some $\mathbf{i} \in \mathbf{I}$ we see that $E_{\mathbf{i}}$ is integrable and $S_{\mathbf{i}}(x)$ is totally geodesic in M as mentioned above. In general $L_P(x)$ is called a *slice* through x. It is finite dimensional precisely if $0 \notin P$. In this case it can be focalized, that is, there exists a parallel normal vector field ξ_P with the property that $\langle \xi_P, v_{\mathbf{i}} \rangle = 1$ if and only if $v_{\mathbf{i}}(x) \in P$, so that the kernel of the differential $(\pi_{\xi_P})_*$ of the focal map is $\mathcal{D}_P$ and the connected components of the fibers of π_{ξ_P} are the leaves L_P. Note also in this case that $L_P(x)$ is contained in the round hypersphere in $W_P(x)$ of center $c_P(x) := x + \xi_P(x)$ and radius $||\xi_P||$. On the other hand, if we start with a point y in $x + \nu_x M$ lying in some focal hyperplanes, then there is a unique finite dimensional slice $L_P(x)$ focalizing at y coming from $P = \text{affine span}\{v_{\mathbf{i}}(x) \mid y \in H_{\mathbf{i}}(x)\}$. In this case $\mathcal{D}_P = \oplus_{y\in H_{\mathbf{i}}(x)} E_{\mathbf{i}}$, namely, P does not contain any other curvature normals besides those $v_{\mathbf{i}}(x)$ with $y \in H_{\mathbf{i}}(x)$.

In addition to the above assumptions (connectedness, completeness, fullness, irreducibility) we henceforth assume that M has rank at least 2. Then the main result of [HL99] asserts that M is homogeneous. In order to explain this result we first recall that for each curvature sphere $S_{\mathbf{i}}(x)$, $\mathbf{i} \in \mathbf{I}^*$, there exists a distinguished compact connected Lie group $\Phi^*_{\mathbf{i}} = \Phi^*_{\mathbf{i}}(x)$ which acts isometrically on the affine span $W_{\mathbf{i}}(x)$ of $S_{\mathbf{i}}(x)$ and has $S_{\mathbf{i}}(x)$ as an orbit. In fact $\Phi^*_{\mathbf{i}}$ is the identity component of the normal holonomy group of the focal manifold obtained from M by focalizing the distribution $E_{\mathbf{i}}$. The action of $\Phi^*_{\mathbf{i}}$ on $W_{\mathbf{i}}(x)$ is equivalent to the isotropy representation of a symmetric space of rank 1 different from the Cayley projective plane, where we view $W_{\mathbf{i}}(x)$ as a vector space with the origin at the center $c_{\mathbf{i}}(x)$ of $S_{\mathbf{i}}(x)$. Hence $\Phi^*_{\mathbf{i}}$ is isomorphic to one of $SO(m_{\mathbf{i}}+1)$, $U(\frac{m_{\mathbf{i}}+1}{2})$ ($m_{\mathbf{i}}$ odd) or $(Sp(\frac{m_{\mathbf{i}}+1}{4}) \times Sp(1))/\mathbb{Z}_2$ ($m_1 \equiv 3$ mod 4). Note that the isotropy group of $\Phi^*_{\mathbf{i}}$ at x acts irreducibly on $E_{\mathbf{i}}(x)$ in the first case only and that otherwise $E_{\mathbf{i}}(x)$ splits into two irreducible subspaces as $E_{\mathbf{i}}(x) = E'_{\mathbf{i}}(x) \oplus E''_{\mathbf{i}}(x)$ with $\dim E'_{\mathbf{i}}(x)$ even and $\dim E''_{\mathbf{i}}(x) = 1$ or 3. It is useful to note that the group $\Phi^*_{\mathbf{i}}$ is already determined by any irreducible finite dimensional slice of rank 2 containing $S_{\mathbf{i}}(x)$, and it follows that $\Phi^*_{\mathbf{i}} \cong SO(m_{\mathbf{i}}+1)$ unless W is isomorphic to $\tilde{B}_n$ or $\tilde{C}_n$. We also mention that the construction of $\Phi^*_{\mathbf{i}}$ can be generalized by replacing $S_{\mathbf{i}}(x)$ by any slice $L_P(x)$ to yield a compact connected Lie group $\Phi^*_P(x)$ acting isometrically on $W_P(x)$ and having $L_P(x)$ as an orbit. If $0 \notin P$, the group $\Phi^*_P(x)$ is the normal holonomy group of a focal manifold and acts on $W_P(x)$ as the isotropy representation of a symmetric space of rank equal to the rank of $L_P(x)$ as an isoparametric submanifold (viewing $W_P(x)$ as a vector space with origin at $c_P(x)$). The later statement is also known as the Homogeneous Slice Theorem [HL99].

The action of the group $\Phi^*_{\mathbf{i}}(x)$ on $S_{\mathbf{i}}(x)$ induces in a natural way an invariant connection on $S_{\mathbf{i}}(x)$ which coincides with the Levi-Cività connection if $\Phi^*_{\mathbf{i}} \cong SO(m_{\mathbf{i}}+1)$. It is defined via a reductive decomposition on the Lie algebra level $\mathbf{L}(\Phi^*_{\mathbf{i}}) = \mathbf{L}((\Phi^*_{\mathbf{i}})_x) + \mathfrak{m}_{\mathbf{i}}$ where $(\Phi^*_{\mathbf{i}})_x$ denotes the isotropy group of $\Phi^*_{\mathbf{i}}$ at x and $\mathfrak{m}_{\mathbf{i}}$ is the orthogonal complement of $\mathbf{L}((\Phi^*_{\mathbf{i}})_x)$ with respect the inner product $\langle X, Y\rangle = -B(X,Y) - \mathrm{trace}(\rho_* X)(\rho_* Y)$ for X, $Y \in \mathbf{L}(\Phi^*_{\mathbf{i}})$, where B denotes the Killing form of $\mathbf{L}(\Phi^*_{\mathbf{i}})$ and ρ denotes its representation on $W_{\mathbf{i}}(x)$. The invariant connection of course has the property that $\gamma_X(t) := \exp tX(x)$ is a geodesic for any $X \in \mathfrak{m}_{\mathbf{i}}$ and $(\exp tX)_*$ induces parallel translation along γ_X.

Finally the proof of the homogeneity of M is based on the explicit construction of certain one-parameter groups F^t_X of isometries of V leaving M invariant. In fact for each $x \in M$, $\mathbf{i} \in \mathbf{I}^*$ and $X \in E_{\mathbf{i}}(x)$, there exists such a one-parameter group having the following properties, by which they are also determined.

- (*a*) $F^t_X(x) := \gamma_X(t)$ is the geodesic in $S_{\mathbf{i}}(x)$ with initial speed X with respect to the above invariant connection and $(F^t_X)_* Y$ is a parallel vector field along γ_X for any $Y \in E_{\mathbf{i}}(x)$, i.e. $(F^t_X)_*|_{W_{\mathbf{i}}(x)} = (\exp t\check{X})_*$ where $\check{X}$ is the unique vector in $\mathfrak{m}_{\mathbf{i}}$ such that $\gamma_X(t) = \exp t\check{X}(x)$ has initial speed X.
- (*b*) For each $\mathbf{j} \in \mathbf{I} \setminus \{\mathbf{i}\}$ and $y \in S_{\mathbf{j}}(x)$, $F^t_X(y)$ is the unique smooth curve which is everywhere orthogonal to $E_{\mathbf{j}}$ and satisfies $F^t_X(y) \in S_{\mathbf{j}}(\gamma_X(t))$ for all t. Moreover $(t,y) \mapsto F^t_X(y)$ is smooth on $\mathbb{R} \times S_{\mathbf{j}}(x)$.
- (*c*) $(F^t_X)_*\xi$ is parallel in νM along γ_X for all $\xi \in \nu_x M$.

By means of these one-parameter groups one proves that the group of isometries of V preserving M is transitive on Q_x, for $x \in M$, where Q_x is defined as the set of points of M that can be reached from x by following piecewise differentiable curves along finite dimensional curvature spheres. One finally deduces the homogeneity of M from the results that this group is also transitive on the closure of Q_x in M [HL99, Prop. 4.4] and that Q_x is dense in M [HL99, Thm. B].

In this paper, V will always denote a finite or infinite dimensional separable Hilbert space, and M a connected, complete, full and irreducible isoparametric submanifold of V with rank at least 2. In case M is finite dimensional and of rank 2, we assume in addition that M is homogeneous. Thus M is always homogeneous, and more precisely, an orbit of its stabilizer in the isometry group of V. Moreover, for $x \in M$, $\mathbf{i} \in \mathbf{I}^*$ and $X \in E_{\mathbf{i}}(x)$ we have the one-parameter groups F_X^t leaving M invariant described above (in the finite dimensional case, this follows from [HL99, Lemma 4.2]). In the finite dimensional case we also recall that $E_{\mathbf{0}} = 0$ by irreducibility. Although in this paper we are ultimately interested in the infinite dimensional case, the reason to also treat finite dimensional isoparametric submanifolds is that they appear as (necessarily homogeneous) slices of the infinite dimensional ones.

3. The homogeneous structure Γ

3.1. **Definition of Γ and basic properties.** We now introduce the fundamental object of this paper.

Definition 3.1. For each $x \in M$, $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j} \in \mathbf{I}$ and $X_{\mathbf{i}} \in E_{\mathbf{i}}(x)$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}(x)$ let

$$(\Gamma_x)_{X_{\mathbf{i}}} Y_{\mathbf{j}} := \left(\frac{d}{dt}\Big|_{t=0} (F^t_{X_{\mathbf{i}}})_*(Y_{\mathbf{j}}) \right)^{\top}$$

where $(\cdot)^{\top}$ denotes the tangential part. We call Γ the *homogeneous structure* of M.

If x is fixed we write Γ instead of Γ_x and $E_{\mathbf{i}}$ instead of $E_{\mathbf{i}}(x)$.

Remark 3.2. By a theorem of Stone (see e. g. [Con90, p. 327]), for a given one-parameter group φ_t of unitary automorphisms of a complex Hilbert space, there exists a self-adjoint operator A such that $\varphi_t = \exp itA$ if and only if φ_t is strongly continuous, that is, in case $t \mapsto \varphi_t v$ is continuous for all v. The domain of definition of A is then the set of vectors v such that $\lim_{t\to 0} \frac{\varphi_t v - v}{t}$ exists and iAv is this limit. The operator A is called the infinitesimal generator of φ_t. In our setting, Stone's theorem can be applied to the complexification of the one-parameter group $(F^t_{X_{\mathbf{i}}})_*$. In fact, we know that $(F^t_{X_{\mathbf{i}}})_*$ is strongly continuous on a dense subset of V (namely, on the algebraic span of $\nu_x M$ and the $E_{\mathbf{i}}(x)$ for $\mathbf{i} \in \mathbf{I}$) and thus strongly continuous on V (since the one-parameter group $(F^t_{X_{\mathbf{i}}})_*$ consists of isometries). Thus $\Gamma_{X_{\mathbf{i}}}$ is essentially the infinitesimal generator of $(F^t_{X_{\mathbf{i}}})_*$. In particular the continuity of $\Gamma_{X_{\mathbf{i}}}$ is equivalent to the domain of this infinitesimal generator being the entire Hilbert space.

Lemma 3.3. *(i)* $\frac{d}{dt}\big|_{t=0} (F^t_{X_{\mathbf{i}}})_* Y_{\mathbf{j}} = \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}} + \alpha(X_{\mathbf{i}}, Y_{\mathbf{j}})$
(ii) $\frac{d}{dt}\big|_{t=0} (F^t_{X_{\mathbf{i}}})_* \xi = -A_\xi X_{\mathbf{i}}$
for $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j} \in \mathbf{I}$ *and* $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$, $\xi \in \nu_x M$.

Proof. Part (ii) follows from the definition of $F^t_{X_{\mathbf{i}}}$, and part (i) follows from (ii) by taking inner product with $Y_{\mathbf{j}}$. □

Lemma 3.4. *(i)* Γ_X *is skew-symmetric:* $\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, Z_{\mathbf{k}} \rangle + \langle Y_{\mathbf{j}}, \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}} \rangle = 0$ *for all* $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}$, *and* $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$, $Z_{\mathbf{k}} \in E_{\mathbf{k}}$.
(ii) Γ *is invariant under isometries:* $\Gamma_{g_* X_{\mathbf{i}}} g_* Y_{\mathbf{j}}|_{gx} = g_* \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}|_x$ *for any extrinsic isometry* g *of* M, $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$ *and* $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j} \in \mathbf{I}$.

Proof. Part (i) follows from $\langle (F^t_{X_{\mathbf{i}}})_* Y_{\mathbf{j}}, (F^t_{X_{\mathbf{i}}})_* Z_{\mathbf{k}} \rangle = \langle Y_{\mathbf{j}}, Z_{\mathbf{k}} \rangle$ by taking derivative at $t = 0$.

Part (ii) follows from $F^t_{g_* X_{\mathbf{i}}} = g F^t_{X_{\mathbf{i}}} g^{-1}$ which in turn is a direct consequence of the definition of $F^t_{X_{\mathbf{i}}}$. □

Lemma 3.5. *(i) Let L be a slice of M through x, choose $\mathbf{i} \in \mathbf{I}$ with $E_{\mathbf{i}}(x) \subset T_x L$ and $X \in E_{\mathbf{i}}(x)$. Then $F^t_X(L) = L$.*

(ii) If, in addition, L is irreducible and has rank at least 2, W denotes its affine span in V, and ${}^L F^t_X$ the one-parameter group of isometries of W associated to X, then

$$F^t_X|_W = {}^L F^t_X$$

for all t, and the homogeneous structure of L is the restriction of that of M.

Proof. (i) We have $L = L_P(x)$ for some affine subspace $P \subset \nu_x M$ and $F^t_X(L_P(x)) = L_P(F_X(x)) = L_P(x)$ since $F^t_X(x) \in L_P(x)$.

(ii)The result is a consequence of [HL99, Lemma 1.4], and the fact that the group $\Phi^*_{\mathbf{i}}$ which acts transitively on $S_{\mathbf{i}}(x)$ is already determined by L. The last assertion follows from [HL99, Remark, p. 162] in case L is finite dimensional, and otherwise by applying the same remark twice to the inclusions $S_{\mathbf{i}}(x) \subset L' \subset M$ and $S_{\mathbf{i}}(x) \subset L' \subset L$, where L' is a finite dimensional rank 2 slice. □

3.2. **The homogeneous structure of parallel isoparametric submanifolds.** Let ξ be a parallel normal vector field along M and denote by $\pi : M \to M_\xi$ the endpoint map $x \mapsto x + \xi(x)$. Assume that M_ξ is also isoparametric and denote its homogeneous structure by ${}^\xi\Gamma$.

Lemma 3.6. *For all $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j} \in \mathbf{I}$, $X_{\mathbf{i}} \in E_{\mathbf{i}}$ and $Y_{\mathbf{j}} \in E_{\mathbf{j}}$:*

(i) ${}^\xi\Gamma_{\pi_* X_{\mathbf{i}}} Y_{\mathbf{j}}|_{\pi(x)} = \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}|_x$;

(ii) $\pi_* X_{\mathbf{i}} = \pm \frac{||v_{\mathbf{i}}||}{||v^\xi_{\mathbf{i}}||} X_{\mathbf{i}}$, where $v^\xi_{\mathbf{i}}$ is the curvature normal of M_ξ with respect to $\pi_* E_{\mathbf{i}}$ (this subspace is a curvature distribution of M_ξ that coincides with $E_{\mathbf{i}}$ up to parallel translation in V, but may correspond to a different index in $\mathbf{I}^*$ if $M_\xi = M$).

Proof. (i) The diffeomorphism π maps curvature spheres of M through x to curvature spheres of M_ξ through $\pi(x)$. In fact $\pi_* = \mathrm{id} - A_\xi$ maps curvature distributions to curvature distributions and actually preserves $E_{\mathbf{k}}(x)$ as a subspace of V for all $\mathbf{k} \in \mathbf{I}$. Since $F^t_{X_{\mathbf{i}}}$ is an isometry of V preserving M and inducing parallel transport along νM, it immediately follows that $F^t_{X_{\mathbf{i}}}$ commutes with π and in particular preserves the parallel submanifold M_ξ. Note that the initial speed of $t \mapsto F^t_{X_{\mathbf{i}}}(\pi(x))$ is $\pi_* X_{\mathbf{i}}$. We claim that $F^t_{X_{\mathbf{i}}}$ is the canonical one-parameter group of isometries of V preserving M_ξ associated to $\pi_* X_{\mathbf{i}} \in T_{\pi(x)} M_\xi$. The result then follows by differentiating at $t = 0$ and taking tangential parts. In turn, the claim is proved by checking (a), (b) and (c) in section 2, and these conditions follow from the fact that M and M_ξ have parallel curvature distributions.

(ii) This follows from the fact that π maps curvature spheres of M through x of radii $1/||v_{\mathbf{i}}||$ to curvature spheres of M_ξ through $\pi(x)$ of radii $1/||v^\xi_{\mathbf{i}}||$. □

3.3. **Finite dimensional case.** The case of finite dimensional slices is particularly interesting, since they are congruent to principal orbits of isotropy representations of symmetric spaces, for which we have an effective way of computing the homogeneous structure.

Let M be a principal orbit of the isotropy representation of an irreducible symmetric space G/K of rank at least 2, say of noncompact type, with K connected. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G into the eigenspaces of the involution. Equip $\mathfrak{p}$ with an $\mathrm{Ad}_{\mathfrak{k}}$-invariant inner product. Choose a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, Then M is the adjoint orbit $\mathrm{Ad}_{\mathfrak{k}}(x)$ for some regular point $x \in \mathfrak{a}$ and $\nu_x M = \mathfrak{a}$. Consider the usual root space decompositions $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda\in\Lambda} \mathfrak{k}_\lambda$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\lambda\in\Lambda} \mathfrak{p}_\lambda$ where Λ denotes the root system with respect to $\mathfrak{a}$ and $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$, $\mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}$. Then $\mathfrak{p}_+ := \sum_{\lambda\in\Lambda} \mathfrak{p}_\lambda = T_x M$, the index set $\mathbf{I}$ can be identified with the set Λ^+_{red} of positive roots λ such that $\frac{1}{2}\lambda$ is not a root, and for such a root we have $E_\lambda = \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$, where we use the convention that $\mathfrak{p}_{2\lambda} = 0$ if $2\lambda \notin \Lambda$. Thus M carries the canonical homogeneous structure given by the reductive complement $\mathfrak{k}^+ = \sum_{\lambda\in\Lambda} \mathfrak{k}_\lambda$, and this coincides with the homogeneous structure in our definition. In fact, in the proof of [HL99, Lemma 4.2] it was shown that for any $X \in \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$, F^t_X equals $\exp t\check{X}$, where $\check{X} \in \mathfrak{k}_\lambda + \mathfrak{k}_{2\lambda}$ is the unique element in $\mathfrak{k}^+$ satisfying $[\check{X}, x] = X$. Hence

$$\Gamma_X Y = [\check{X}, Y]_{\mathfrak{p}^+} \tag{3.7}$$

(component in $\mathfrak{p}_+$) for X, $Y \in T_x M = \mathfrak{p}^+$.

As an application of the above discussion, we prove a result that will be used in the proof of Theorem 4.3.

Proposition 3.8. *If $v_{\mathbf{i}}$, $v_{\mathbf{j}}$, $v_{\mathbf{k}}$ are pairwise different and they span an affine subspace P in $\nu_x M$ that does not contain 0, then*

$$\Gamma_{X_{\mathbf{i}}}\Gamma_{Y_{\mathbf{j}}} Z_{\mathbf{k}} - \Gamma_{Y_{\mathbf{j}}}\Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}} = \Gamma_{\Gamma_{X_{\mathbf{i}}}Y_{\mathbf{j}} - \Gamma_{Y_{\mathbf{j}}}X_{\mathbf{i}}} Z_{\mathbf{k}} \tag{3.9}$$

where $X_{\mathbf{i}} \in E_{\mathbf{i}}(x)$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}(x)$, $Z_{\mathbf{k}} \in E_{\mathbf{k}}(x)$.

Proof. The assumption that $v_{\mathbf{i}}$, $v_{\mathbf{j}}$, $v_{\mathbf{k}}$ span P allows us to restrict to the corresponding finite dimensional slice, which can be assumed to be irreducible and is thus congruent to a principal orbit of the isotropy representation of a symmetric space, so formula (3.7) can be used. The Jacobi identity gives

$$[\check{X}_{\mathbf{i}}, [\check{Y}_{\mathbf{j}}, Z_{\mathbf{k}}]] + [\check{Y}_{\mathbf{j}}, [Z_{\mathbf{k}}, \check{X}_{\mathbf{i}}]] + [Z_{\mathbf{k}}, [\check{X}_{\mathbf{i}}, \check{Y}_{\mathbf{j}}]] = 0. \tag{3.10}$$

Note that $\Gamma_{Y_{\mathbf{j}}} Z_{\mathbf{k}} = [\check{Y}_{\mathbf{j}}, Z_{\mathbf{k}}]_{\mathfrak{p}_+} = [\check{Y}_{\mathbf{j}}, Z_{\mathbf{k}}]$ since $v_{\mathbf{j}}$, $v_{\mathbf{k}}$ are different. Taking $\mathfrak{p}_+$-components in (3.10), we now see that the first two terms in the formula thus obtained correspond to the left hand side of (3.9), and it remains to see that $-[Z_{\mathbf{k}}, [\check{X}_{\mathbf{i}}, \check{Y}_{\mathbf{j}}]]_{\mathfrak{p}_+}$ corresponds to the right hand side. In fact, this also follows from Jacobi, namely

$$\begin{aligned} [[\check{X}_{\mathbf{i}}, \check{Y}_{\mathbf{j}}], x] &= [X_{\mathbf{i}}, \check{Y}_{\mathbf{j}}] + [\check{X}_{\mathbf{i}}, Y_{\mathbf{j}}] \\ &= \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}} - \Gamma_{Y_{\mathbf{j}}} X_{\mathbf{i}} \end{aligned}$$

using that $v_{\mathbf{i}} \neq v_{\mathbf{j}}$, so $[[\check{X}_{\mathbf{i}}, \check{Y}_{\mathbf{j}}], Z_{\mathbf{k}}]_{\mathfrak{p}_+} = \Gamma_{\Gamma_{X_{\mathbf{i}}}Y_{\mathbf{j}} - \Gamma_{Y_{\mathbf{j}}}X_{\mathbf{i}}} Z_{\mathbf{k}}$. ☐

3.4. **First results on the image of Γ.** We start with some basic results about the components of $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}}$ in the eigenspaces $E_{\mathbf{k}}$. Later in section 8 they will be considerably refined.

Proposition 3.11. *Let $\mathbf{i} \in \mathbf{I}^*$.*

(i) If $E_{\mathbf{i}}$ is irreducible then $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{i}} = 0$.
(ii) If $E_{\mathbf{i}}$ is not irreducible then $\Gamma_{E''_{\mathbf{i}}} E''_{\mathbf{i}} = 0$, $\Gamma_{E'_{\mathbf{i}}} E''_{\mathbf{i}} \subset E'_{\mathbf{i}}$, $\Gamma_{E''_{\mathbf{i}}} E'_{\mathbf{i}} \subset E'_{\mathbf{i}}$ and $\Gamma_{E'_{\mathbf{i}}} E'_{\mathbf{i}} \subset E''_{\mathbf{i}}$.

Proof. (i) Note that $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{i}} \subset E_{\mathbf{i}}$ simply by the definition of $F^t_{X_{\mathbf{i}}}$ for $X_{\mathbf{i}} \in E_{\mathbf{i}}$. The irreducibility of $E_{\mathbf{i}}$ means $\Phi^*_{\mathbf{i}} \cong SO(m_{\mathbf{i}} + 1)$, where $m_{\mathbf{i}} = \dim S_{\mathbf{i}}(x)$. Thus the connection induced by $\Phi^*_{\mathbf{i}}$ on $S_{\mathbf{i}}(x)$ coincides with the standard Levi-Cività connection. Since $(F^t_X)_* Y$

is parallel along $t \mapsto F_X^t(x)$ in $S_{\mathbf{i}}(x)$ for all $X,\ Y \in E_{\mathbf{i}}(x)$, $\Gamma_X Y = \left(\frac{d}{dt}\big|_{t=0}(F_X^t)_* Y)\right)^\top = 0$. Alternatively, we could also prove this result by employing a line of reasoning like in part (ii).

(ii) We use Lemma 3.5. Let L be an irreducible finite dimensional rank two slice through x which contains $E_{\mathbf{i}}$. Then L is congruent to a principal orbit of the isotropy representation of a symmetric space of type BC_2, and the statements follow from the bracket relations in the corresponding Lie algebra and formula (3.7). □

Proposition 3.12. *We have* $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}} \perp E_{\mathbf{j}}$ *for* $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j} \in \mathbf{I}$ *and* $\mathbf{i} \neq \mathbf{j}$.

Proof. Let $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$ and let $c(s_1, s_2)$ be a differentiable parametrized surface in $S_{\mathbf{j}}(x)$ with $c(0,0) = x$ and $\frac{\partial c}{\partial s_1}(0,0) = Y_{\mathbf{j}}$. Put $\varphi(s_1, s_2, t) := F_{X_{\mathbf{i}}}^t(c(s_1, s_2))$. Then $\frac{\partial \varphi}{\partial s_1}(0,0,t) = (F_{X_{\mathbf{i}}}^t)_*(Y_{\mathbf{j}})$ and thus $\frac{D}{\partial t}\frac{\partial \varphi}{\partial s_1}(0,0,0) = \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}$. Hence

$$\left\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \frac{\partial c}{\partial s_2}(0,0) \right\rangle = \left\langle \frac{D}{\partial s_1}\frac{\partial \varphi}{\partial t}(0,0,0), \frac{\partial c}{\partial s_2}(0,0) \right\rangle = -\left\langle \frac{\partial \varphi}{\partial t}(0,0,0), \frac{D}{\partial s_1}\frac{\partial c}{\partial s_2}(0,0) \right\rangle = 0,$$

where we have used that $\frac{\partial \varphi}{\partial t}$ is orthogonal to $E_{\mathbf{j}}$ and $S_{\mathbf{j}}(x)$ is totally geodesic in M. As $\frac{\partial c}{\partial s_2}(0,0)$ can be chosen arbitrarily in $E_{\mathbf{j}}(x)$, the result follows. □

The next result shows that Γ determines $\nabla\alpha$.

Proposition 3.13. *(i)* $\nabla_{X_{\mathbf{i}}}\alpha(Y_{\mathbf{j}}, Z_{\mathbf{k}}) = \langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, Z_{\mathbf{k}}\rangle(v_{\mathbf{j}} - v_{\mathbf{k}})$;

(ii) $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}} = \nabla_{X_{\mathbf{i}}} \tilde{Y}_{\mathbf{j}}$ *mod* $E_{\mathbf{j}}$;

for all $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}$, $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$, $Z_{\mathbf{k}} \in E_{\mathbf{k}}$. *Here* $\tilde{Y}_{\mathbf{j}}$ *is any smooth local extension of* $Y_{\mathbf{j}}$ *to a section of* $E_{\mathbf{j}}$.

Proof. (i) follows from $\alpha((F_{X_{\mathbf{i}}}^t)_* Y_{\mathbf{j}}, (F_{X_{\mathbf{i}}}^t)_* Z_{\mathbf{k}}) = (F_{X_{\mathbf{i}}}^t)_*\alpha(Y_{\mathbf{j}}, Z_{\mathbf{k}})$ by taking derivative with respect to the normal connection at $t = 0$ and using e.g. $\alpha(X, Z_{\mathbf{k}}) = \langle X, Z_{\mathbf{k}}\rangle v_{\mathbf{k}}$ for all $X \in T_xM$ and the parallelism in the normal bundle of the right hand side.

Part (ii) follows by comparing (i) with the formula (2.2) for $\nabla\alpha$. □

Corollary 3.14. Γ *is also* $\mathbb{R}$*-linear in the lower argument, and thus can be extended as a bilinear map to* $\sum_{\mathbf{i}\in\mathbf{I}^*} E_{\mathbf{i}} \times \sum_{\mathbf{j}\in\mathbf{I}} E_{\mathbf{j}}$, *where* $\sum$ *means the algebraic sum.*

Proof. If $\mathbf{i} = \mathbf{j}$ this follows from the definition. Otherwise we use Propositions 3.12 and 3.13(ii). □

3.5. **Consequences of the Codazzi equation.** Important consequences of Proposition 3.13 are some formulae for permuting the arguments of Γ, which are obtained via the Codazzi equation. We introduce the following notation for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}$ with $\mathbf{i} \neq \mathbf{k}$:

$$\frac{v_{\mathbf{j}} - v_{\mathbf{k}}}{v_{\mathbf{i}} - v_{\mathbf{k}}} = \begin{cases} \lambda & \text{if } v_{\mathbf{j}} - v_{\mathbf{k}} = \lambda(v_{\mathbf{i}} - v_{\mathbf{k}}), \\ 0 & \text{if } v_{\mathbf{j}} - v_{\mathbf{k}} \text{ is not a multiple of } v_{\mathbf{i}} - v_{\mathbf{k}}. \end{cases}$$

We also denote by $(X)_{E_{\mathbf{k}}}$ the component of $X \in T_xM$ in $E_{\mathbf{k}}$.

Proposition 3.15.

$$(\Gamma_{Y_{\mathbf{j}}} X_{\mathbf{i}})_{E_{\mathbf{k}}} = \frac{v_{\mathbf{j}} - v_{\mathbf{k}}}{v_{\mathbf{i}} - v_{\mathbf{k}}}\, (\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}$$

for all $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$, $\mathbf{k} \in \mathbf{I}$ *with* $\mathbf{k} \neq \mathbf{i}$ *and* $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$.

Proof. Due to Proposition 3.13(i),

$$\begin{aligned}\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, Z_{\mathbf{k}}\rangle (v_{\mathbf{j}} - v_{\mathbf{k}}) &= \nabla_{X_{\mathbf{i}}} \alpha(Y_{\mathbf{j}}, Z_{\mathbf{k}}) \\ &= \nabla_{Y_{\mathbf{j}}} \alpha(X_{\mathbf{i}}, Z_{\mathbf{k}}) \\ &= \langle \Gamma_{Y_{\mathbf{j}}} X_{\mathbf{i}}, Z_{\mathbf{k}}\rangle (v_{\mathbf{i}} - v_{\mathbf{k}})\end{aligned}$$

which yields the desired result. □

Lemma 3.16. *(i) If $(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}} \neq 0$ for $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}$, then $v_{\mathbf{i}}$, $v_{\mathbf{j}}$, $v_{\mathbf{k}}$ are colinear.*
(ii) For $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}^*$, the condition $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{k}}} \neq 0$ is symmetric in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Proof. Part (i) is clear from Codazzi. For part (ii), note that the symmetry is obvious if $\mathbf{i} = \mathbf{j} = \mathbf{k}$ and a consequence of Codazzi if the indices are mutually different. In the remaining cases $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{k}}} = 0$ by Codazzi and Proposition 3.12. □

Remark 3.17. If $v_{\mathbf{i}}$, $v_{\mathbf{j}}$, $v_{\mathbf{k}}$ are colinear then the corresponding focal hyperplanes $H_{\mathbf{i}}$, $H_{\mathbf{j}}$, $H_{\mathbf{k}}$ share a point in common. The converse holds if M has rank 2. In fact for $x \in M$, $y = x + \xi \in H_{\mathbf{i}}(x) \cap H_{\mathbf{j}}(x) \cap H_{\mathbf{k}}(x)$ if and only if $\langle \xi, v_{\mathbf{i}}(x)\rangle = \langle \xi, v_{\mathbf{j}}(x)\rangle = \langle \xi, v_{\mathbf{k}}(x)\rangle = 1$ where $\xi \in \nu_x M$.

4. Density of the image of Γ and equivalence between Γ and $\nabla\alpha$

In the first theorem, we consider the algebraic span of the subsets $\Gamma_{E_{\mathbf{i}}(x)} E_{\mathbf{j}}$ for $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$, $\mathbf{i} \neq \mathbf{j}$.

Theorem 4.1. *$\sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I}^* \\ \mathbf{i} \neq \mathbf{j}}} \Gamma_{E_{\mathbf{i}}(x)} E_{\mathbf{j}}(x)$ is dense in $T_x M$ and moreover contains $\sum_{\mathbf{i} \in \mathbf{I}^*} E_{\mathbf{i}}(x)$.*

Proof. Set $\mathcal{D}(x) = \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I}^* \\ \mathbf{i} \neq \mathbf{j}}} \Gamma_{E_{\mathbf{i}}(x)} E_{\mathbf{j}}(x)$ for all $x \in M$. Then $\mathcal{D}$ is a possibly nonsmooth distribution of M which however is invariant under all isometries of V leaving M invariant (Lemma 3.4(ii)).

We first show that $E_{\mathbf{k}} \subset \mathcal{D}(x)$ for each $\mathbf{k} \in \mathbf{I}^*$. Here we may assume that M is finite dimensional, by passing to an irreducible finite dimensional slice of rank at least 2 containing $E_{\mathbf{k}}$. Fix $x \in M$ and suppose $Y \in T_x M$ is orthogonal to $\mathcal{D}(x)$. We are going to show that for each $\mathbf{l} \in \mathbf{I}$, the $E_{\mathbf{l}}$-component $Y_{\mathbf{l}}$ lies in $T_y M$ for all $y \in M$ and thus has to vanish, as otherwise the line in the direction of $Y_{\mathbf{l}}$ could be split off, contradicting the irreducibility of M.

So we fix $\mathbf{l}$ and, by Theorem D in [HOT91], select indices $\mathbf{i}_1, \dots, \mathbf{i}_r \in \mathbf{I}$ different from $\mathbf{l}$ such that any point $y \in M$ can be reached from x by following piecewise smooth curves whose smooth arcs are contained in curvature spheres $S_{\mathbf{i}}$ with $\mathbf{i} \in \{\mathbf{i}_1, \dots, \mathbf{i}_r\}$. For such an $\mathbf{i}$, consider $X \in E_{\mathbf{i}}(x)$. Then $Y(t) := (F^t_{X_{\mathbf{i}}})_* Y$ is a smooth extension of Y along the curve $\gamma(t) := F^t_{X_{\mathbf{i}}}(x)$ in $S_{\mathbf{i}}(x)$ (here we use the finite dimensionality of M), which is everywhere orthogonal to $\mathcal{D}$. Next we split $Y(t)$ into its $E_{\mathbf{i}}$- and $E_{\mathbf{i}}^{\perp}$-components and note that the latter, which is given by $(F^t_X)_*(Y - Y_{\mathbf{i}})$, is constant. In fact

$$\frac{d}{dt}\Big|_{t=s} (F^t_X)_*(Y - Y_{\mathbf{i}}) = (F^s_X)_*(\Gamma_X(Y - Y_{\mathbf{i}}) + \alpha(X, Y - Y_{\mathbf{i}})) = 0$$

using $\alpha(E_{\mathbf{i}}, E_{\mathbf{i}}^{\perp}) = 0$, $\langle \Gamma_{E_{\mathbf{i}}} E_{\mathbf{i}}^{\perp}, E_{\mathbf{i}}\rangle = \langle \Gamma_{E_{\mathbf{i}}} E_{\mathbf{i}}, E_{\mathbf{i}}^{\perp}\rangle = 0$, and $\langle \Gamma_{E_{\mathbf{i}}} Y, E_{\mathbf{i}}^{\perp}\rangle = \langle Y, \Gamma_{E_{\mathbf{i}}} E_{\mathbf{i}}^{\perp}\rangle = 0$ by the choice of Y. In particular the $E_{\mathbf{l}}$-component of $Y(t)$ is the constant vector $Y_{\mathbf{l}}$. Changing $X \in E_{\mathbf{i}}(x)$ we obtain in this way that, for any $y \in S_{\mathbf{i}}(x)$, the $E_{\mathbf{l}}$-component $Y_{\mathbf{l}}$ of Y lies in $T_y M$ for all $y \in S_{\mathbf{i}}(x)$ and that moreover $Y_{\mathbf{l}}$ is the $E_{\mathbf{l}}$-component of a vector $\tilde{Y} \in T_y M$ which is orthogonal to $\mathcal{D}(y)$. Repeating this argument with $\tilde{Y}$ in place of Y and any $\mathbf{i}' \in \{\mathbf{i}_1, \ldots, \mathbf{i}_r\}$ in place of $\mathbf{i}$ eventually proves by induction the existence for any $y \in M$ of a vector in T_yM whose $E_{\mathbf{l}}$-component is $Y_{\mathbf{l}}$. This finishes the proof of the second statement.

The proof of the first statement is similar but now simpler. M is infinite dimensional and we suppose $Y \in T_xM$ is orthogonal to $\mathcal{D}(x)$. Then $Y \in E_{\mathbf{0}}(x)$ by the first part. A calculation similar to the above shows that $Y(t) := (F_X^t)_* Y$ is constant for all $X \in E_{\mathbf{i}}(x)$, $\mathbf{i} \in \mathbf{I}^*$. As the set of points that can be reached from x by following piecewise smooth curves whose smooth arcs lie in finite dimensional curvature spheres is dense in M, we deduce that the constant vector Y is everywhere tangent to M and thus has to vanish so as not to contradict the irreducibility of M. $\square$

Corollary 4.2. *(i) For each* $\mathbf{k} \in \mathbf{I}^*$, $E_{\mathbf{k}} = \sum\{(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{k}}} \mid v_{\mathbf{i}}, v_{\mathbf{j}} \notin \mathbb{R}v_{\mathbf{k}}\}$.
(ii) $\sum\{(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{0}}} \mid \mathbf{i}, \mathbf{j} \in \mathbf{I}^*$ *with* $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ *lin. dep.*$\}$ *is dense in* $E_{\mathbf{0}}$.

Proof. (i) Since $\mathbf{k} \neq \mathbf{0}$ we may assume M is finite dimensional by passing to an appropriate slice. By Theorem 4.1, $E_{\mathbf{k}} = \sum_{\mathbf{i} \neq \mathbf{j}} (\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{k}}}$. Now for $\mathbf{i} \neq \mathbf{j}$ and $\mathbf{k} \in \{\mathbf{i}, \mathbf{j}\}$ we have $\langle \Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}}, E_{\mathbf{k}} \rangle = 0$. Hence $E_{\mathbf{k}} = \sum_{\substack{\mathbf{i} \neq \mathbf{j} \\ \mathbf{i}, \mathbf{j} \neq \mathbf{k}}} (\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{k}}}$ proving (i).

(ii) The statement follows directly from Theorem 4.1 as $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{0}}} = 0$ if $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ are linearly independent by Codazzi. $\square$

Theorem 4.3. *If* α_x *is given, then* Γ_x *and* $(\nabla\alpha)_x$ *determine one the other.*

Proof. Suppose α_x and Γ_x are given. Proposition 3.13(i) can be rewritten as saying

$$\nabla_{X_{\mathbf{i}}} \alpha(Y_{\mathbf{j}}, Z_{\mathbf{k}}) = -\alpha(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, Z_{\mathbf{k}}) - \alpha(Y_{\mathbf{j}}, \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}}).$$

Since $\nabla\alpha$ is a tensor (and thus continuous in all three arguments), $\nabla_X \alpha(Y, Z)$ is thus determined for X, Y, $Z \in T_xM$ with $X \perp E_{\mathbf{0}}$, and hence by Codazzi for all X, Y, $Z \in T_xM$ as $\nabla_X \alpha(Y, Z) = 0$ if two arguments lie in $E_{\mathbf{0}}$.

Conversely, we will show that Γ_x is determined by α_x and $\nabla\alpha_x$. So let $\tilde{M}$ be a second isoparametric submanifold in the same Hilbert space V. We assume that $x \in M \cap \tilde{M}$, $T_xM = T_x\tilde{M}$, $\alpha_x = \tilde{\alpha}_x$ and $\nabla\alpha_x = \nabla\tilde{\alpha}_x$, and want to show that $\Gamma_x = \tilde{\Gamma}_x$. The assumption that the second fundamental forms coincide implies that M and $\tilde{M}$ have the same curvature distributions and the same curvature normals at x, so more precisely we need to show that Γ_x and $\tilde{\Gamma}_x$ coincide as mappings $E_{\mathbf{i}}(x) \times E_{\mathbf{j}}(x) \to T_xM$ for all $\mathbf{i} \in \mathbf{I}^*$, $\mathbf{j} \in \mathbf{I}$.

We first note that $\Gamma_x = \tilde{\Gamma}_x$ on $E_{\mathbf{i}} \times E_{\mathbf{j}}$ if $\mathbf{i} \neq \mathbf{j}$. Indeed $(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{j}}} = (\tilde{\Gamma}_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{j}}} = 0$ by Proposition 3.12, and $(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}$ (resp. $(\tilde{\Gamma}_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}$) for $\mathbf{k} \neq \mathbf{j}$ is determined by $\nabla\alpha_x = \nabla\tilde{\alpha}_x$ in view of Proposition 3.13(i), where $X_{\mathbf{i}} \in E_{\mathbf{i}} = E_{\mathbf{i}}(x)$, $Y_{\mathbf{j}} \in E_{\mathbf{j}} = E_{\mathbf{j}}(x)$.

Moreover for $\mathbf{i} \neq \mathbf{0}$ we have $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{i}} = \tilde{\Gamma}_{E_{\mathbf{i}}} E_{\mathbf{i}} = 0$ if $E_{\mathbf{i}}$ is irreducible due to Proposition 3.11. So we only need to understand Γ_x, $\tilde{\Gamma}_x : E_{\mathbf{i}}(x) \times E_{\mathbf{i}}(x) \to E_{\mathbf{i}}(x)$ in case $E_{\mathbf{i}}$ is reducible. Fix such an $\mathbf{i} \in \mathbf{I}^*$. Decompose $E_{\mathbf{i}} = E'_{\mathbf{i}} \oplus E''_{\mathbf{i}}$. Again by Proposition 3.11 and using Lemma 3.4(i), there are two cases that need to be discussed:

$$\Gamma,\ \tilde{\Gamma} : E'_{\mathbf{i}} \times E''_{\mathbf{i}} \to E'_{\mathbf{i}} \quad \text{and} \quad \Gamma,\ \tilde{\Gamma} : E''_{\mathbf{i}} \times E'_{\mathbf{i}} \to E'_{\mathbf{i}}. \tag{4.4}$$

The discussion is similar in both cases so we restrict it to the first one.

There exists an irreducible finite dimensional rank 2 slice L of M (resp. $\tilde{L}$ of $\tilde{M}$) through x containing $E_{\mathbf{i}}$, which of course is of nonreduced type and hence of type BC_2. Recall that L, $\tilde{L}$ are homogeneous. Since α_x, $(\nabla\alpha)_x$ and Γ_x restrict to the corresponding objects for

L, and similarly for $\tilde{L}$, it is enough to assume that $M = L$, $\tilde{M} = \tilde{L}$ are finite dimensional, homogeneous of type BC_2, which we henceforth do.

Let $X_{\mathbf{i}} \in E'_{\mathbf{i}}$, $W_{\mathbf{i}} \in E''_{\mathbf{i}}$. Owing to Theorem 4.1, we can write $W_{\mathbf{i}}$ as a finite sum

$$W_{\mathbf{i}} = \sum_{\ell} \Gamma_{Y_{\mathbf{j}_\ell}} Z_{\mathbf{k}_\ell} \tag{4.5}$$

where for each ℓ, $v_{\mathbf{i}}$, $v_{\mathbf{j}_\ell}$, $v_{\mathbf{k}_\ell}$ are pairwise different curvature normals of L, $\tilde{L}$ (Note that the pair $(v_{\mathbf{j}_\ell}, v_{\mathbf{k}_\ell})$ can repeat for different ℓ.) Using formula (3.9), we get

$$\begin{aligned} \Gamma_{X_{\mathbf{i}}} W_{\mathbf{i}} &= \sum_{\ell} \Gamma_{X_{\mathbf{i}}} \Gamma_{Y_{\mathbf{j}_\ell}} Z_{\mathbf{k}_\ell} \\ &= \sum_{\ell} \left(\Gamma_{Y_{\mathbf{j}_\ell}} \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}_\ell} + \Gamma_{\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}_\ell} - \Gamma_{Y_{\mathbf{j}_\ell}} X_{\mathbf{i}}} Z_{\mathbf{k}_\ell} \right), \end{aligned}$$

and a similar formula for $\tilde{\Gamma}$. In order to finish the proof, we need only to check the claim that on the right hand side of this formula Γ is being computed always on a pair of vectors lying in different curvature distributions, as we already know that Γ and $\tilde{\Gamma}$ coincide for such pairs of vectors.

Recall that the positive root system of type BC_2 can be described as $\Lambda^+ = \{\theta_1, \theta_2, 2\theta_1, 2\theta_2, \theta_1 + \theta_2, \theta_1 - \theta_2\}$ so that we can write $\mathbf{I} = \{\theta_1, \theta_2, \theta_1 + \theta_2, \theta_1 - \theta_2\}$ and assume that the index $\mathbf{i}$ corresponds to θ_1. Now in the formula (4.5) we may assume that $(\mathbf{j}_\ell, \mathbf{k}_\ell) = (\theta_1 + \theta_2, \theta_1 - \theta_2)$ or $(\theta_1 - \theta_2, \theta_1 + \theta_2)$ for all ℓ, and the claim follows from remarking that $\Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}_\ell}$, $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}_\ell}$, $\Gamma_{Y_{\mathbf{j}_\ell}} X_{\mathbf{i}}$ all lie in E'_{θ_2}. □

5. Implications of the Gauss equation for Γ

The Gauss equation yields another sort of formulae for permuting arguments of Γ. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}^*$, $\mathbf{l} \in \mathbf{I}$ and let $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$, $Z_{\mathbf{k}} \in E_{\mathbf{k}}$, $W_{\mathbf{l}} \in E_{\mathbf{l}}$. On one hand the Gauss equation yields

$$\begin{aligned} \langle R(X_{\mathbf{i}}, Y_{\mathbf{j}}) Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle &= \langle \alpha(X_{\mathbf{i}}, W_{\mathbf{l}}), \alpha(Y_{\mathbf{j}}, Z_{\mathbf{k}}) \rangle - \langle \alpha(X_{\mathbf{i}}, Z_{\mathbf{k}}), \alpha(Y_{\mathbf{j}}, W_{\mathbf{l}}) \rangle \\ &= -\langle X_{\mathbf{i}} \wedge Y_{\mathbf{j}}, Z_{\mathbf{k}} \wedge W_{\mathbf{l}} \rangle \langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle. \end{aligned}$$

On the other hand, extending these vectors to smooth local sections of the corresponding eigenbundles and using the same letters for the extensions, we get

$$\begin{aligned} \langle R(X_{\mathbf{i}}, Y_{\mathbf{j}}) Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle &= \langle \nabla_{X_{\mathbf{i}}} \nabla_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle - \langle \nabla_{Y_{\mathbf{j}}} \nabla_{X_{\mathbf{i}}} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle - \langle \nabla_{[X_{\mathbf{i}}, Y_{\mathbf{j}}]} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle \\ &= X_{\mathbf{i}} \langle \nabla_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle - \langle \nabla_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, \nabla_{X_{\mathbf{i}}} W_{\mathbf{l}} \rangle \\ &\qquad - Y_{\mathbf{j}} \langle \nabla_{X_{\mathbf{i}}} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle + \langle \nabla_{X_{\mathbf{i}}} Z_{\mathbf{k}}, \nabla_{Y_{\mathbf{j}}} W_{\mathbf{l}} \rangle - \langle \nabla_{[X_{\mathbf{i}}, Y_{\mathbf{j}}]} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle \\ &= \langle \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}}, \Gamma_{Y_{\mathbf{j}}} W_{\mathbf{l}} \rangle - \langle \Gamma_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, \Gamma_{X_{\mathbf{i}}} W_{\mathbf{l}} \rangle - \langle \nabla_{[X_{\mathbf{i}}, Y_{\mathbf{j}}]} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle \\ &\qquad + X_{\mathbf{i}} \langle \nabla_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle - \langle \nabla'_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, \nabla_{X_{\mathbf{i}}} W_{\mathbf{l}} \rangle - \langle \Gamma_{Y_{\mathbf{j}}} Z_{\mathbf{k}}, \nabla'_{X_{\mathbf{i}}} W_{\mathbf{l}} \rangle \\ &\qquad - Y_{\mathbf{j}} \langle \nabla_{X_{\mathbf{i}}} Z_{\mathbf{k}}, W_{\mathbf{l}} \rangle + \langle \nabla'_{X_{\mathbf{i}}} Z_{\mathbf{k}}, \nabla_{Y_{\mathbf{j}}} W_{\mathbf{l}} \rangle + \langle \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}}, \nabla'_{Y_{\mathbf{j}}} W_{\mathbf{l}} \rangle, \end{aligned} \tag{5.1}$$

where e.g. $\nabla'_{Y_{\mathbf{j}}} Z_{\mathbf{k}} = \nabla_{Y_{\mathbf{j}}} Z_{\mathbf{k}} - \Gamma_{Y_{\mathbf{j}}} Z_{\mathbf{k}}$. Note that ∇' satisfies the properties of a metric connection (Leibniz rule and compatibility with the metric) when both arguments are sections of curvature distributions, and that $\nabla'_{Y_{\mathbf{j}}} Z_{\mathbf{k}} \in E_{\mathbf{k}}$ and $\nabla'_{Y_{\mathbf{j}}} Z_{\mathbf{k}} = (\nabla_{Y_{\mathbf{j}}} Z_{\mathbf{k}})_{E_{\mathbf{k}}}$ if $\mathbf{j} \neq \mathbf{k}$. Moreover, Γ is parallel with respect to ∇' in the following sense.

Lemma 5.2. $\nabla'_{X_\mathbf{i}}\left(\Gamma_{Y_\mathbf{j}}Z_\mathbf{k}\right)_{E_\mathbf{l}} = \left(\Gamma_{\nabla'_{X_\mathbf{i}}Y_\mathbf{j}}Z_\mathbf{k}\right)_{E_\mathbf{l}} + \left(\Gamma_{Y_\mathbf{j}}\nabla'_{X_\mathbf{i}}Z_\mathbf{k}\right)_{E_\mathbf{l}}.$

Proof. To check this formula at $p \in M$, it suffices to differentiate at $t = 0$ the relevant vector fields along $\gamma(t) = F^t_{X_\mathbf{i}}(p)$. From the definition of Γ, we see that $(F^t_{X_\mathbf{i}})_* Y_\mathbf{j}$ is ∇'-parallel along γ, as well as $\left(\Gamma_{(F^t_{X_\mathbf{i}})_*Y_\mathbf{j}}(F^t_{X_\mathbf{i}})_*Z_\mathbf{k}\right)_{E_\mathbf{l}}$, since this is equal to $\left((F_{X^t_\mathbf{i}})_*\Gamma_{\mathbf{Y}_j}Z_\mathbf{k}\right)_{E_\mathbf{l}} = (F^t_{X_\mathbf{i}})_*\left(\Gamma_{\mathbf{Y}_j}Z_\mathbf{k}\right)_{E_\mathbf{l}}$ by Lemma 3.4(ii). This already shows that the result holds for ∇'-parallel vector fields $Y_\mathbf{j}$, $Z_\mathbf{k}$. Hence the result follows in general, since $E_\mathbf{j}$ and $E_\mathbf{k}$ are finite dimensional, and so an arbitrary smooth vector field along γ with values in one of those curvature distributions can be written as a finite linear combination of parallel vector fields with smooth functions as coefficients. □

To take care of the term with the bracket we introduce the following notation. For any $A \in T_xM$ and $Z_\mathbf{k}$, $W_\mathbf{l}$ as above (with in particular $\mathbf{k} \neq \mathbf{l}$), we put

$$\langle \Gamma_A Z_\mathbf{k}, W_\mathbf{l}\rangle := \sum_{\mathbf{m}\in\mathbf{I}} \langle \Gamma_{Z_\mathbf{k}} W_\mathbf{l}, A_\mathbf{m}\rangle \frac{v_\mathbf{l} - v_\mathbf{m}}{v_\mathbf{k} - v_\mathbf{l}}, \tag{5.3}$$

where $A_\mathbf{m}$ is the component of A in $E_\mathbf{m}$. Note that the sum converges as the coefficients $\frac{v_\mathbf{l}-v_\mathbf{m}}{v_\mathbf{k}-v_\mathbf{l}}$ are bounded, and that this definition indeed extends the domain of Γ relative to the first argument in view of Proposition 3.15. The proof of that proposition also shows that

$$\langle \Gamma_A Z_\mathbf{k}, W_\mathbf{l}\rangle = \langle \nabla_A Z_\mathbf{k}, W_\mathbf{l}\rangle. \tag{5.4}$$

Lemma 5.5.

$$\langle \nabla_{[X_\mathbf{i},Y_\mathbf{j}]} Z_\mathbf{k}, W_\mathbf{l}\rangle = \langle \Gamma_{\Gamma_{X_\mathbf{i}}Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}}X_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle + \langle \Gamma_{\nabla'_{X_\mathbf{i}}Y_\mathbf{j}} Z_\mathbf{k}, W_\mathbf{l}\rangle - \langle \Gamma_{\nabla'_{X_\mathbf{j}}Y_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle.$$

Proof. We have $\langle \nabla_{[X_\mathbf{i},Y_\mathbf{j}]} Z_\mathbf{k}, W_\mathbf{l}\rangle = \langle \nabla_{\Gamma_{X_\mathbf{i}}Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}}X_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle + \langle \nabla_{\nabla'_{X_\mathbf{i}}Y_\mathbf{j} - \nabla'_{Y_\mathbf{j}}X_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle$, and the result follows from equation (5.4). □

Next we extend the domain of Γ relative to its second argument by putting

$$\langle \Gamma_{X_\mathbf{i}} Y, A_\mathbf{m}\rangle = -\langle Y, \Gamma_{X_\mathbf{i}} A_\mathbf{m}\rangle \tag{5.6}$$

for all $Y \in T_xM$, $\mathbf{m} \in \mathbf{I}$ and $A_\mathbf{m} \in E_\mathbf{m}$ to formulate the main result of this section.

Theorem 5.7. *For any* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}^*$ *and* $\mathbf{l} \in \mathbf{I}$ *with* $\mathbf{k} \neq \mathbf{l}$, *and* $X_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$, $Z_\mathbf{k} \in E_\mathbf{k}$, $W_\mathbf{l} \in E_\mathbf{l}$,

$$\left\langle \left([\Gamma_{X_\mathbf{i}}, \Gamma_{Y_\mathbf{j}}] - \Gamma_{\Gamma_{X_\mathbf{i}}Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}}X_\mathbf{i}}\right) Z_\mathbf{k}, W_\mathbf{l}\right\rangle = -\langle X_\mathbf{i} \wedge Y_\mathbf{j}, Z_\mathbf{k} \wedge W_\mathbf{l}\rangle \langle v_\mathbf{i}, v_\mathbf{j}\rangle.$$

Proof. We combine equation (5.1) with Lemma 5.5 to write

$$\begin{aligned}
-\langle X_\mathbf{i} \wedge Y_\mathbf{j}, Z_\mathbf{k} \wedge W_\mathbf{l}\rangle \langle v_\mathbf{i}, v_\mathbf{j}\rangle &= \left\langle \left([\Gamma_{X_\mathbf{i}}, \Gamma_{Y_\mathbf{j}}] - \Gamma_{\Gamma_{X_\mathbf{i}}Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}}X_\mathbf{i}}\right) Z_\mathbf{k}, W_\mathbf{l}\right\rangle \\
&\quad + X_\mathbf{i}\langle \nabla_{Y_\mathbf{j}} Z_\mathbf{k}, W_\mathbf{l}\rangle + \langle \nabla'_{X_\mathbf{i}} Z_\mathbf{k}, \nabla_{Y_\mathbf{j}} W_\mathbf{l}\rangle - \langle \Gamma_{\nabla'_{X_\mathbf{i}Y_\mathbf{j}}} Z_\mathbf{k}, W_\mathbf{l}\rangle - \langle \Gamma_{Y_\mathbf{j}} Z_\mathbf{k}, \nabla'_{X_\mathbf{i}} W_\mathbf{l}\rangle \\
&\quad - Y_\mathbf{j}\langle \nabla_{X_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle - \langle \nabla'_{Y_\mathbf{j}} Z_\mathbf{k}, \nabla_{X_\mathbf{i}} W_\mathbf{l}\rangle + \langle \Gamma_{\nabla'_{Y_\mathbf{j}X_\mathbf{i}}} Z_\mathbf{k}, W_\mathbf{l}\rangle + \langle \Gamma_{X_\mathbf{i}} Z_\mathbf{k}, \nabla'_{Y_\mathbf{j}} W_\mathbf{l}\rangle.
\end{aligned}$$

The four last terms on the right hand side vanish (and similarly the four terms preceeding those) by Lemma 5.2 as

$$\langle \nabla_{X_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle = \langle \Gamma_{X_\mathbf{i}} Z_\mathbf{k}, W_\mathbf{l}\rangle$$

(due to $\mathbf{k} \neq \mathbf{l}$) and

$$\langle \nabla'_{Y_\mathbf{j}} Z_\mathbf{k}, \nabla_{X_\mathbf{i}} W_\mathbf{l} \rangle = \langle \nabla'_{Y_\mathbf{j}} Z_\mathbf{k}, \nabla'_{X_\mathbf{i}} W_\mathbf{l} \rangle + \langle \nabla'_{Y_\mathbf{j}} Z_\mathbf{k}, \Gamma_{X_\mathbf{i}} W_\mathbf{l} \rangle = -\langle \Gamma_{X_\mathbf{i}} \nabla'_{Y_\mathbf{j}} Z_\mathbf{k}, W_\mathbf{l} \rangle$$

(due to $\nabla'_{Y_\mathbf{j}} Z_\mathbf{k} \in E_\mathbf{k}$, $\nabla'_{X_\mathbf{i}} W_\mathbf{l} \in E_\mathbf{l}$ and $\mathbf{k} \neq \mathbf{l}$). □

We will use only special cases of this theorem. By using definitions (5.3), (5.6) and Proposition 3.15,

$$(\Gamma_{X_\mathbf{i}} Y_\mathbf{j} - \Gamma_{Y_\mathbf{j}} X_\mathbf{i})_{E_\mathbf{m}} = (\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}} \frac{v_\mathbf{i} - v_\mathbf{j}}{v_\mathbf{i} - v_\mathbf{m}},$$

for all $\mathbf{m} \neq \mathbf{i}$, and under the assumption that $(\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{i}} \perp \Gamma_{Z_\mathbf{k}} W_\mathbf{l}$ (which holds for instance if $\mathbf{i} \neq \mathbf{j}$ or $v_\mathbf{i}$, $v_\mathbf{k}$, $v_\mathbf{l}$ are not colinear), we can reformulate Theorem 5.7 more explicitly as

$$\begin{aligned} \langle \Gamma_{X_\mathbf{i}} Z_\mathbf{k}, \Gamma_{Y_\mathbf{j}} W_\mathbf{l} \rangle - \langle \Gamma_{Y_\mathbf{j}} Z_\mathbf{k}, \Gamma_{X_\mathbf{i}} W_\mathbf{l} \rangle &= \sum_{\mathbf{m} \in \mathbf{I} \setminus \{\mathbf{i}\}} \langle \Gamma_{Z_\mathbf{k}} W_\mathbf{l}, (\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}} \rangle \frac{v_\mathbf{l} - v_\mathbf{m}}{v_\mathbf{k} - v_\mathbf{l}} \frac{v_\mathbf{i} - v_\mathbf{j}}{v_\mathbf{i} - v_\mathbf{m}} \\ &\quad - \langle X_\mathbf{i} \wedge Y_\mathbf{j}, Z_\mathbf{k} \wedge W_\mathbf{l} \rangle \langle v_\mathbf{i}, v_\mathbf{j} \rangle. \end{aligned} \tag{5.8}$$

The following two corollaries of Theorem 5.7 are obtained from equation (5.8): the first one, by taking $\mathbf{i} = \mathbf{l} \neq \mathbf{k} = \mathbf{j}$; the second one, in case $\mathbf{i} \neq \mathbf{j}$, by using that $(\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}} \neq 0$ only if $v_\mathbf{i}$, $v_\mathbf{j}$, $v_\mathbf{m}$ are colinear.

Corollary 5.9. *Let* $\mathbf{i}, \mathbf{j} \in \mathbf{I}^*$, $\mathbf{i} \neq \mathbf{j}$. *Then*

$$\langle \Gamma_{X_\mathbf{i}} Y_\mathbf{j}, \Gamma_{Z_\mathbf{j}} W_\mathbf{i} \rangle + \langle \Gamma_{X_\mathbf{i}} Z_\mathbf{j}, \Gamma_{Y_\mathbf{j}} W_\mathbf{i} \rangle = \langle X_\mathbf{i}, W_\mathbf{i} \rangle \langle Y_\mathbf{j}, Z_\mathbf{j} \rangle \langle v_\mathbf{i}, v_\mathbf{j} \rangle$$

for all $X_\mathbf{i}$, $W_\mathbf{i} \in E_\mathbf{i}$ *and* $Y_\mathbf{j}$, $Z_\mathbf{j} \in E_\mathbf{j}$. *In particular*

$$\langle \Gamma_{X_\mathbf{i}} Y_\mathbf{j}, \Gamma_{Y_\mathbf{j}} X_\mathbf{i} \rangle = \frac{1}{2} \langle v_\mathbf{i}, v_\mathbf{j} \rangle ||X_\mathbf{i}||^2 ||Y_\mathbf{j}||^2.$$

Corollary 5.10. *Let* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}^*$, $\mathbf{l} \in \mathbf{I}$, *and assume that* $v_\mathbf{i}$, $v_\mathbf{k}$, $v_\mathbf{l}$ *are not colinear. Then there exists a constant* $c \in \mathbb{R}$ *such that*

$$\langle \Gamma_{X_\mathbf{i}} Z_\mathbf{k}, \Gamma_{Y_\mathbf{j}} W_\mathbf{l} \rangle = \langle \Gamma_{Y_\mathbf{j}} Z_\mathbf{k}, \Gamma_{X_\mathbf{i}} W_\mathbf{l} \rangle + c \cdot \langle \Gamma_{X_\mathbf{i}} Y_\mathbf{j}, \Gamma_{Z_\mathbf{k}} W_\mathbf{l} \rangle$$

for all $X_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$, $Z_\mathbf{k} \in E_\mathbf{k}$, $W_\mathbf{l} \in E_\mathbf{l}$. *Moreover if* $\mathbf{i} = \mathbf{j}$ *or the lines spanned by* $v_\mathbf{i}$, $v_\mathbf{j}$ *and* $v_\mathbf{k}$, $v_\mathbf{l}$ *do not meet at a curvature normal, then* $c = 0$. *On the other hand, if those lines meet at* $v_\mathbf{m}$ *we have* $c = \dfrac{v_\mathbf{l} - v_\mathbf{m}}{v_\mathbf{k} - v_\mathbf{l}} \cdot \dfrac{v_\mathbf{i} - v_\mathbf{j}}{v_\mathbf{i} - v_\mathbf{m}}$.

Corollary 5.11. *Let* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbf{I}^*$ *with* $\mathbf{k} \neq \mathbf{i}$, $\mathbf{j}$ *and* $\frac{v_\mathbf{j}}{v_\mathbf{k}} \neq \frac{v_\mathbf{i} - v_\mathbf{j}}{v_\mathbf{k} - v_\mathbf{i}}$. *Assume further that* $(\Gamma_{E_\mathbf{i}} E_\mathbf{j})_{E_\mathbf{m}} \perp \Gamma_{E_\mathbf{i}} E_\mathbf{k}$ *for all* $\mathbf{m} \in \mathbf{I}^*$ *and* $(\Gamma_{E_\mathbf{i}} E_\mathbf{j})_{E_\mathbf{k}} = 0$ *(both conditions hold if e.g.* $\Gamma_{E_\mathbf{i}} E_\mathbf{j} \subset E_\mathbf{0}$*). Then*

$$\Gamma_{E_\mathbf{i}} E_\mathbf{j} \perp \Gamma_{E_\mathbf{i}} E_\mathbf{k}.$$

Proof. We may assume $\mathbf{i} \neq \mathbf{j}$ since $\Gamma_{E_\mathbf{i}} E_\mathbf{i} = (\Gamma_{E_\mathbf{i}} E_\mathbf{i})_{E_\mathbf{i}} \perp \Gamma_{E_\mathbf{i}} E_\mathbf{k}$ by assumption. If $\mathbf{0}$ does not belong to one of the lines through $v_\mathbf{i}$, $v_\mathbf{j}$ and $v_\mathbf{i}$, $v_\mathbf{k}$, then $\Gamma_{E_\mathbf{i}} E_\mathbf{j} \perp E_\mathbf{0}$ or $\Gamma_{E_\mathbf{i}} E_\mathbf{k} \perp E_\mathbf{0}$ and the assertion follows again from the assumptions. Thus we are left with the case $v_\mathbf{i}$, $v_\mathbf{j}$, $v_\mathbf{k}$ are multiples of each other.

Let $X_\mathbf{i}$, $W_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$ and $Z_\mathbf{k} \in E_\mathbf{k}$. Then equation (5.8) with $\mathbf{l} = \mathbf{i}$ yields

$$\langle \Gamma_{X_\mathbf{i}} Z_\mathbf{k}, \Gamma_{Y_\mathbf{j}} W_\mathbf{i} \rangle = \langle \Gamma_{Z_\mathbf{k}} W_\mathbf{i}, \Gamma_{X_\mathbf{i}} Y_\mathbf{j} \rangle \frac{v_\mathbf{i} - v_\mathbf{j}}{v_\mathbf{k} - v_\mathbf{i}},$$

and hence

$$\langle \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}}, \Gamma_{W_{\mathbf{i}}} Y_{\mathbf{j}} \rangle \frac{v_{\mathbf{j}}}{v_{\mathbf{i}}} = \langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{W_{\mathbf{i}}} Z_{\mathbf{k}} \rangle \frac{v_{\mathbf{k}}}{v_{\mathbf{i}}} \frac{v_{\mathbf{i}} - v_{\mathbf{j}}}{v_{\mathbf{k}} - v_{\mathbf{i}}}. \tag{5.12}$$

Here we have used that $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{k}}} = 0$ implies $(\Gamma_{E_{\mathbf{j}}} E_{\mathbf{k}})_{E_{\mathbf{i}}} = 0$, and $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{m}}} \perp \Gamma_{E_{\mathbf{i}}} E_{\mathbf{k}}$ implies $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_{\mathbf{m}}} \perp \Gamma_{E_{\mathbf{k}}} E_{\mathbf{i}}$ and $(\Gamma_{E_{\mathbf{j}}} E_{\mathbf{i}})_{E_{\mathbf{m}}} \perp \Gamma_{E_{\mathbf{i}}} E_{\mathbf{k}}$. Choosing $W_{\mathbf{i}} = X_{\mathbf{i}}$ in (5.12) gives $\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}} \rangle = 0$, and then by polarization, $\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{X'_{\mathbf{i}}} Z_{\mathbf{k}} \rangle + \langle \Gamma_{X'_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}} \rangle = 0$ for all $X'_{\mathbf{i}} \in E_{\mathbf{i}}$. Again by (5.12), we have $\langle \Gamma_{X'_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{X_{\mathbf{i}}} Z_{\mathbf{k}} \rangle = \langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{X'_{\mathbf{i}}} Z_{\mathbf{k}} \rangle \frac{v_{\mathbf{k}}}{v_{\mathbf{j}}} \frac{v_{\mathbf{i}} - v_{\mathbf{j}}}{v_{\mathbf{k}} - v_{\mathbf{i}}}$, and owing to $\mathbf{k} \neq \mathbf{j}$, we get $\frac{v_{\mathbf{k}}}{v_{\mathbf{j}}} \frac{v_{\mathbf{i}} - v_{\mathbf{j}}}{v_{\mathbf{k}} - v_{\mathbf{i}}} \neq -1$, from which $\langle \Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}, \Gamma_{X'_{\mathbf{i}}} Z_{\mathbf{k}} \rangle = 0$, as desired. □

From Proposition 3.15 and Corollary 5.9 we get

Corollary 5.13.

$$\sum_{\mathbf{k} \in \mathbf{I} \setminus \{\mathbf{i}\}} ||(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}||^2 \frac{v_{\mathbf{k}} - v_{\mathbf{j}}}{v_{\mathbf{k}} - v_{\mathbf{i}}} = \frac{1}{2} \langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle ||X_{\mathbf{i}}||^2 ||Y_{\mathbf{j}}||^2$$

for all $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$, $\mathbf{i} \neq \mathbf{j}$ *and* $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$.

6. Reduction to rank one slices

In this section we prove Proposition 6.3, which reduces the proof of the continuity of $\Gamma_{X_{\mathbf{i}}}$ to that on the infinite dimensional rank one slice containing $E_{\mathbf{i}}$. We also explain our strategy to reduce the proof of continuity of $\Gamma_{X_{\mathbf{i}}}$ along that slice to finding a constant C such that $||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}|| \leq C\, ||v_{\mathbf{i}}||\, ||X_{\mathbf{i}}||\, ||Y_{\mathbf{j}}||$ for all $\mathbf{j} \in \mathbf{I} \setminus \{\mathbf{i}\}$ with $v_{\mathbf{j}} \in \mathbb{R} v_{\mathbf{i}}$ and $Y_{\mathbf{j}} \in E_{\mathbf{j}}$ (Lemma 6.4).

We first consider a special case which is simpler.

Proposition 6.1. *Assume the affine Weyl group of* M *to be of type* $\tilde{A} - \tilde{D} - \tilde{E}$. *Let* $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$ *where we assume that*

$v_{\mathbf{i}}$, $v_{\mathbf{j}}$ *are linearly independent,*

and choose $X_{\mathbf{i}} \in E_{\mathbf{i}}$, $Y_{\mathbf{j}} \in E_{\mathbf{j}}$. *If* $v_{\mathbf{i}} \perp v_{\mathbf{j}}$ *then* $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}} = 0$; *otherwise*

$$||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}|| = \frac{1}{2} ||v_{\mathbf{i}}||\, ||X_{\mathbf{i}}||\, ||Y_{\mathbf{j}}||.$$

Proof. Let P be the affine span of $\{v_{\mathbf{i}}(x), v_{\mathbf{j}}(x)\}$ for some $x \in M$. Then $L_P(x)$ is a slice of rank two and, by the assumption on M, of type $A_1 \times A_1$ or A_2. In the first case $v_{\mathbf{i}}$ and $v_{\mathbf{j}}$ are orthogonal, P does not contain any other curvature normal and $\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}} = 0$. In the second case there is precisely one more curvature normal $v_{\mathbf{k}} \in P$ for $\mathbf{k} \in \mathbf{I}^* \setminus \{\mathbf{i}, \mathbf{j}\}$. For the sake of simplicity, by applying a translation of V to M we may assume that the origin of V lies at the intersection of $H_{\mathbf{i}}$, $H_{\mathbf{j}}$, $H_{\mathbf{k}}$. Let $u_{\mathbf{i}}$, $u_{\mathbf{j}}$, $u_{\mathbf{k}}$ unit vectors orthogonal to $H_{\mathbf{i}}$, $H_{\mathbf{j}}$, $H_{\mathbf{k}}$, respectively. We may assume $u_{\mathbf{k}} = u_{\mathbf{i}} + u_{\mathbf{j}}$ by eventually multiplying some of these vectors by -1. As $c_{\mathbf{i}}(x) = x + \frac{v_{\mathbf{i}}}{||v_{\mathbf{i}}||^2} \in H_{\mathbf{i}}$, we have $v_{\mathbf{r}} = -\frac{u_{\mathbf{r}}}{\langle x, u_{\mathbf{r}} \rangle}$ for $\mathbf{r} = \mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Thus Corollary 5.13 yields

$$||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}||^2 = ||(\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}})_{E_{\mathbf{k}}}||^2 = \frac{1}{2} \langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle \frac{v_{\mathbf{k}} - v_{\mathbf{i}}}{v_{\mathbf{k}} - v_{\mathbf{j}}} ||X_{\mathbf{i}}||^2 ||Y_{\mathbf{j}}||^2 = \frac{1}{4} ||v_{\mathbf{i}}||^2 ||X_{\mathbf{i}}||^2 ||Y_{\mathbf{j}}||^2$$

by a straightforward computation. □

If the affine Weyl group of M is not of type $\tilde{A} - \tilde{D} - \tilde{E}$, Corollary 5.13 does not suffice to estimate $||\Gamma_{X_{\mathbf{i}}} Y_{\mathbf{j}}||$ for linearly independent $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ as the sum on the left hand side in its statement contains more than one term and in general with different signs. We circumvent this difficulty by using the classification of homogeneous compact rank two isoparametric submanifolds.

Proposition 6.2. *There exists a positive constant C such that*

$$||\Gamma_{X_{\mathbf{i}}}Y|| \leq C\,||v_{\mathbf{i}}||\,||X_{\mathbf{i}}||\,||Y||$$

whenever $X_{\mathbf{i}}$, Y are tangent to a finite dimensional slice through x, $\mathbf{i} \in \mathbf{I}^*$ and $X_{\mathbf{i}} \in E_{\mathbf{i}}$.

Proof. Assume first that such a slice L is irreducible. Any curvature sphere of M is contained in a finite dimensional irreducible slice of rank at least two, so we may assume that the rank of L is at least two and apply Lemma 3.5 to compute Γ along L. Since the rank of L is bounded by the rank of M and multiplicities of L are also multiplicities of M, there are only finitely many possibilities for L up to parallel translation (notice that scaling of the ambient metric can be viewed as parallel translation in the radial direction). Moreover, for given L, there exists the desired constant simply by finite dimensionality, so we have only to check that the same constant works for the isoparametric submanifolds parallel to L. However, this is a direct consequence of Lemma 3.6.

If L is a product of irreducible slices, then the result follows from the case above together with the remark that $\Gamma_X Y = 0$ whenever X, Y are tangent to different factors. To prove it, we may assume that $X \in E_{\mathbf{i}}$, $Y \in E_{\mathbf{j}}$ where $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$, $\mathbf{i} \neq \mathbf{j}$. Since the line through $v_{\mathbf{i}}$, $v_{\mathbf{j}}$ contains no other curvature normals, we finish by noting that $\Gamma_{E_{\mathbf{i}}}E_{\mathbf{j}} \perp E_{\mathbf{i}}$, $E_{\mathbf{j}}$. □

Proposition 6.3. *Let $\mathbf{i} \in \mathbf{I}^*$ and $X_{\mathbf{i}} \in E_{\mathbf{i}} = E_{\mathbf{i}}(x)$. Then $\Gamma_{X_{\mathbf{i}}}$ is continuous (that is, can be extended continuously to T_xM) if and only if $\Gamma_{X_{\mathbf{i}}}$ is continuous on $\sum_{\mathbf{j}}\{E_{\mathbf{j}} \mid \mathbf{j} \in \mathbf{I}^*,\, v_{\mathbf{j}} \in \mathbb{R}v_{\mathbf{i}}\}$.*

Proof. Write $T_xM = V_0 \oplus V_1 \oplus V_2$ where $V_0 = E_{\mathbf{0}}(x)$ and V_1, V_2 are the closures of

$$\sum_{\mathbf{j}}\{E_{\mathbf{j}} \mid \mathbf{j} \in \mathbf{I}^*,\, v_{\mathbf{j}} \notin \mathbb{R}v_{\mathbf{i}}\} \quad \text{and} \quad \sum_{\mathbf{j}}\{E_{\mathbf{j}} \mid \mathbf{j} \in \mathbf{I}^*,\, v_{\mathbf{j}} \in \mathbb{R}v_{\mathbf{i}}\},$$

respectively. By the closed graph theorem $\Gamma_{X_{\mathbf{i}}}$ is continuous on V_0 (as $E_{\mathbf{0}}(x)$ is a closed subspace lying in the domain of definition of $\Gamma_{X_{\mathbf{i}}}$ and $\Gamma_{X_{\mathbf{i}}}$ is skew-symmetric). We orthogonally decompose V_1 into $\Gamma_{X_{\mathbf{i}}}$-invariant subspaces

$$\sum\{E_{\mathbf{j}} \mid \mathbf{j} \in \mathbf{I}^* \setminus \{\mathbf{i}\},\, v_{\mathbf{j}} \in \ell\},$$

where ℓ runs over the lines in $\nu_xM \setminus \{0\}$ passing through $v_{\mathbf{i}}$. Of course for only countably many lines these subspaces are nonzero. Each such ℓ determines a finite dimensional rank two slice through x to which E_i is tangent so, by Proposition 6.2, $\Gamma_{X_{\mathbf{i}}}$ is continuous on V_1 with $||\Gamma_{X_{\mathbf{i}}}||$ bounded by a constant independent of ℓ. The result follows. □

Identify $\mathbf{I}^*$ with $\mathcal{A} \times \mathbb{Z}$ as explained in section 2 and set $\mathbf{i} = (\alpha, i)$. Since $L_P(x)$, $P = \mathbb{R}v_{\alpha,i}(x)$, is totally geodesic, we have $\Gamma_{X_{\alpha,i}}Y_{\alpha,j} \in E_{\mathbf{0}} \oplus \bigoplus_{k\in\mathbb{Z}} E_{\alpha,k}$. In these terms, Proposition 6.3 reduces the proof of continuity of $\Gamma_{X_{\alpha,i}}$ to finding a constant C such that $||\Gamma_{X_{\alpha,i}}Y|| \leq C||Y||$ for all $Y \in \sum_{j\in\mathbb{Z}} E_{\alpha,j}$; since $E_{\alpha,i}$ is finite dimensional, we may even take $Y \in \sum_{j\in\mathbb{Z}\setminus\{i\}} E_{\alpha,j}$. The following lemma gives a sufficient condition to further simplify the proof in that the estimate only needs to be checked for all $Y \in E_{\alpha,j}$ (and all $j \in \mathbb{Z} \setminus \{i\}$). Grosso modo it is required that for all $j \neq i$, the subspace $\Gamma_{E_{\alpha,i}}E_{\alpha,j}$ be orthogonal to $\Gamma_{E_{\alpha,i}}E_{\alpha,k}$ for all but finitely many $k \in \mathbb{Z}$.

Lemma 6.4. *Let W be a Hilbert space with an orthogonal decomposition $W = \oplus_{i\in\mathbb{Z}} W_i$, and let $f : \sum_{i\in\mathbb{Z}} W_i \to W$ be a linear map. Assume there exists a constant $C > 0$ such that $||fw_i|| \le C||w_i||$ for all $i \in \mathbb{Z}$ and $w_i \in W_i$, and that there exist injective maps $m_1, \dots, m_r : \mathbb{Z} \to \mathbb{Z}$ such that $f(W_i) \perp f(W_j)$ unless $j \in \{m_1(i), \dots, m_r(i)\}$. Then $||f|| \le \sqrt{r}C$ and thus f can be continuously extended to W.*

Proof. Let $w = \sum_{i\in\mathbb{Z}} w_i$ where $w_i \in W_i$ and w_i is nonzero for only finitely many indices. Then

$$\begin{aligned}
||f(w)||^2 &= \sum_{i,j\in\mathbb{Z}} \langle f(w_i), f(w_j)\rangle \\
&= \sum_{i\in\mathbb{Z}} \sum_{k=1}^{r} \langle f(w_i), f(w_{m_k(i)})\rangle \\
&\le \sum_{i\in\mathbb{Z}} \sum_{k=1}^{r} ||f(w_i)||||f(w_{m_k(i)})|| \\
&\le C^2 \sum_{k=1}^{r} \sum_{i\in\mathbb{Z}} ||w_i||||w_{m_k(i)}|| \\
&\le C^2 \sum_{k=1}^{r} \left(\sum_{i\in\mathbb{Z}} ||w_i||^2 \sum_{i\in\mathbb{Z}} ||w_{m_k(i)}||^2 \right)^{1/2} \\
&\le C^2 \sum_{k=1}^{r} (||w||^2||w||^2)^{1/2} \\
&= rC^2||w||^2,
\end{aligned}$$

as we wished. □

Let $v \in P$ be a unit vector and define $\lambda_{\alpha,k} \in \mathbb{R}$ by $v_{\alpha,k} = \frac{1}{\lambda_{\alpha,k}} v$. Since $x + \lambda_{\alpha,k} v = x + \frac{v_{\alpha,k}}{||v_{\alpha,k}||^2} \in H_{\alpha,k}$, we see that $\lambda_{\alpha,k}$ is the directed distance from x to $H_{\alpha,k}$. Moreover we have

$$\frac{v_{\alpha,k} - v_{\alpha,j}}{v_{\alpha,k} - v_{\alpha,i}} = \frac{\lambda_{\alpha,k} - \lambda_{\alpha,j}}{\lambda_{\alpha,k} - \lambda_{\alpha,i}} \frac{\lambda_{\alpha,i}}{\lambda_{\alpha,j}} = \frac{k-j}{k-i} \frac{\langle v_{\alpha,i}, v_{\alpha,j}\rangle}{||v_{\alpha,i}||^2}$$

as the hyperplanes $\{H_{\alpha,k}\}$ form an equidistant family for each fixed α. Therefore Corollary 5.13 yields:

Proposition 6.5. *For each $\alpha \in \mathcal{A}$ and i, $j \in \mathbb{Z}$ with $i \ne j$ and for $X_{\alpha,i} \in E_{\alpha,i}$, $Y_{\alpha,j} \in E_{\alpha,j}$ we have*

$$\sum_{k\in\mathbb{Z}\setminus\{i\}} \frac{k-j}{k-i} ||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,k}}||^2 + ||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\mathbf{0}}}||^2 = \frac{1}{2} ||v_{\alpha,i}||^2 \, ||X_{\alpha,i}||^2 \, ||Y_{\alpha,j}||^2.$$

Unless the $E_{\alpha,k}$-components all vanish, a bound on $||\Gamma_{X_{\alpha,i}} Y_{\alpha,j}||$ is not immediately clear from this formula because the sign of the factor $\frac{k-j}{k-i}$ changes with k, and its size could be arbitrarily small for certain values of k, j. On the other hand, if $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_{\mathbf{0}}$ for all $j \in \mathbb{Z} \setminus \{i\}$, then Corollary 5.11 says that $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \perp \Gamma_{E_{\alpha,i}} E_{\alpha,k}$ unless $k = 2i - j$. Therefore Lemma 6.4 can be applied and continuity of $\Gamma_{X_{\alpha,i}}$ already follows from Proposition 6.5. In general, the applicability of both of these results relies on the vanishing of sufficiently many components of $\Gamma_{E_{\alpha,i}} E_{\alpha,j}$. As it turns out that, we shall see in section 8 that $(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,k}}$ vanishes for all $j,\ k \in \mathbb{Z}$, $j \neq i$, in most cases, and for sufficiently many in the remaining ones.

7. Root systems of isoparametric submanifolds

As mentioned in section 2, for a finite or infinite dimensional isoparametric submanifold, Terng proved that the group generated by reflections in the focal hyperplanes in a fixed affine normal space is a finite or affine Weyl group, respectively [Ter85, Ter89]. Under our assumptions, such isoparametric submanifolds are always irreducible and homogeneous of rank at least two. It turns out then to be possible to refine the data given by the Weyl group into a root system associated to the focal hyperplanes. The refinement amounts to specializing the Coxeter graph of the Weyl group to a Dynkin diagram by adjoining arrows to the double and triple links, and adding concentric circles around certain vertices in the nonreduced case. This root system is unique up to scaling, and is a root system in the ordinary sense if M is finite dimensional and an affine root system otherwise. To distinguish one from the other, we also call the former a finite root system. The aim of this section is to describe this construction. We start by considering root systems attached to a set of hyperplanes, discuss affine root systems, including an outline of their classification and then associate a root system to the set of focal hyperplanes of an isoparametric submanifold.

Weyl groups and their Coxeter graphs. Let E be an affine Euclidean space and denote by T its group of translations (a finite dimensional real vector space). Let $\mathcal{H}$ be a given set of affine hyperplanes in E which is invariant under the group W generated by all the orthogonal reflections in the elements of $\mathcal{H}$. It is assumed that the normal vectors to the $H \in \mathcal{H}$ span T and that W is a finite or an affine Weyl group. In the first case W has a fixed point that necessarily is contained in all $H \in \mathcal{H}$; taking this point as the origin in E, we can identify E with T and view the hyperplanes as linear subspaces. The second case may be characterized as $\mathcal{H}$ consisting of finitely many families of equidistant hyperplanes. It is well known [Bou68, ch. VI, § 2, no. 5, Prop. 8] that these actually are described by a unique reduced root system Δ in T as the set of hyperplanes

$$L_{\alpha,k} = \{x \in T \mid \alpha(x) = k\}, \quad \alpha \in \Delta, \quad k \in \mathbb{Z} \tag{7.1}$$

after choosing an appropriate point in E (any *special point*, namely, one through which there passes one hyperplane from each family of parallel hyperplanes) to identify E with T. We also recall that W acts simply transitively on the set of Weyl chambers, which are the connected components of the complement of the union of the hyperplanes in $\mathcal{H}$. Weyl chambers are also called alcoves in case W is an affine Weyl group. The Coxeter graph of W (or $\mathcal{H}$) is obtained from a Weyl chamber $\mathcal{C}$ by taking as vertices the walls of $\mathcal{C}$ (hyperplanes bounding $\mathcal{C}$) and linking two vertices by 0, 1, 2, 3 or infinitely many edges according to whether the corresponding walls make an angle $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$ or are parallel (other cases cannot occur). W is called irreducible if its Coxeter graph is connected. In this case $\mathcal{C}$ is a simplicial cone (resp. simplex) if W is finite (resp. affine) and hence the Coxeter graph has n (resp. $n+1$) vertices, where $n = \dim E$ is called the rank of W. The isomorphism type of the Coxeter graph is independent of the chosen Weyl chamber and determines $\mathcal{H}$ and W up to isomorphism. Here an isomorphism between two sets of hyperplanes $\mathcal{H} \subset E$, $\mathcal{H}' \subset E'$ as above with irreducible Weyl groups is a map $f : E \to E'$ that is the composition of an isometry with a homothety and takes $\mathcal{H}$ onto $\mathcal{H}'$. It turns out that the isomorphism classes of irreducible finite (resp. affine) Weyl groups correspond bijectively to the irreducible reduced root systems Δ in T and are correspondingly denoted by A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_n ($n = 6, 7, 8$), F_4 and G_2 in case W is finite, and $\tilde{A}_n$ ($n \geq 1$), $\tilde{B}_n$ ($n \geq 3$), $\tilde{C}_n$ ($n \geq 2$), $\tilde{D}_n$ ($n \geq 4$), $\tilde{E}_n$ ($n = 6, 7, 8$), $\tilde{F}_4$, $\tilde{G}_2$ in case W is affine.

It will be important to understand the orbits of the action of W on $\mathcal{H}$ in case W is affine. For a fixed Weyl chamber $\mathcal{C}$, it follows from the transitiveness of W on the set of Weyl chambers that each $H \in \mathcal{H}$ is conjugate under W to some wall of $\mathcal{C}$ and thus to (at least) one vertex of the Coxeter graph defined by $\mathcal{C}$. The W-orbits in $\mathcal{H}$ are thus described by the next result, which is a simple consequence of [Bou68, ch. IV, § 1, no. 3, Prop. 3].

Proposition 7.2. *Two vertices of the Coxeter graph lie in the same W-orbit if and only if they belong to a connected subgraph containing only simple links. In particular, any two hyperplanes in $\mathcal{H}$ are conjugate if W is of type $\tilde{A}_n$ ($n \geq 2$), $\tilde{D}_n$ ($n \geq 4$), $\tilde{E}_n$ ($n = 6, 7, 8$), and W acts with two (resp. three) orbits in $\mathcal{H}$ if W is of type $\tilde{A}_1$, $\tilde{B}_n$ ($n \geq 3$), $\tilde{F}_4$, $\tilde{G}_2$ (resp. $\tilde{C}_n$ ($n \geq 2$)).*

Root systems associated to a set of hyperplanes. We now come to the main definition in this section. Let E, T, $\mathcal{H}$ and W be as above.

Definition 7.3. A *root system* associated to $\mathcal{H}$ is a subset R of $T \times \mathcal{H}$ such that

(i) $v \neq 0$ and $v \perp H$ for all $(v, H) \in R$.
(ii) $2\langle v, v'\rangle/||v||^2 \in \mathbb{Z}$ for all (v, H), $(v', H') \in R$.
(iii) The projection $T \times \mathcal{H} \to \mathcal{H}$ maps R onto $\mathcal{H}$.
(iv) R is invariant under W, that is, $(w_* v, wH) \in R$ for all $(v, H) \in R$ and $w \in W$.

The *rank* of R is defined to be the rank of W. Each $(v, H) \in R$ may be identified with the nonconstant affine mapping $E \to \mathbb{R}$ whose gradient is v and whose zero set is H. In case W is finite and E is identified with T by taking the point $x \in \cap_{H\in\mathcal{H}} H$ as origin, these affine mappings are linear functionals and one gets a root system in the ordinary sense. In case W is affine, one gets an affine root system in the sense of Macdonald [Mac72]. In this case we thus call R an *affine root system* associated to $\mathcal{H}$. An equivalent definition, under a different name, has been given by Bruhat and Tits [BT72].

It follows immediately from (ii) that if (v, H), $(v', H') \in R$ and $v' = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\lambda \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. In particular, R associates to each $H \in \mathcal{H}$ either a pair $\{\pm v\}$ or a quadruple $\{\pm v, \pm 2v\}$ of nonzero normal vectors. Moreover it is clear that together with R also $R_{\text{red}} := \{(v, H) \in S \mid (\frac{1}{2}v, H) \notin S\}$, $R_{\text{red}'} := \{(v, H) \in S \mid (2v, H) \notin S\}$, and $\check{R} = \{(\check{v}, H) \mid (v, H) \in S\}$ are root systems associated to $\mathcal{H}$, where $\check{v} := 2v/||v||^2$. R is called *reduced* if it coincides with R_{red}, i. e. if R associates a pair $\{\pm v\}$ of normal vectors to each $H \in \mathcal{H}$, and *nonreduced* otherwise. R is called *irreducible* if W is irreducible.

Examples of affine root systems. Assume W is an irreducible affine Weyl group associated to a family of hyperplanes $\mathcal{H}$ and let Δ denote the unique reduced root system satisfying (7.1). It follows from [Bou68, ch. VI, § 2, Prop. 2] that

$$R = \{(h_\alpha, L_{\alpha,k}) \mid \alpha \in \Delta,\ k \in \mathbb{Z}\}$$

is a reduced affine root system, where $h_\alpha \in T$ is defined by $\langle h_\alpha, x\rangle = \alpha(x)$ for all $x \in T$. Since the distance between $L_{\alpha,k}$ and $L_{\alpha,k+1}$ is $1/||h_\alpha||$, R can be equivalently described as

$$R = \{(v, H) \mid H \in \mathcal{H},\ v \perp H,\ ||v|| = 1/d_H\},$$

where d_H denotes the minimal distance from H to a parallel $H' \in \mathcal{H} \setminus \{H\}$. Thus to each $\mathcal{H}$ is associated a canonical reduced affine root system.

Restrictions of root systems. Let E, T, $\mathcal{H}$ and W as above and suppose R is a root system associated to $\mathcal{H}$. Let E' be an affine subspace of E, T' its group of translations and $\mathcal{H}' = \{H \cap E' \mid H^\perp \subset T'\}$. Assume that the set of $v \in T'$ such that $(v, H) \in R$ for some $H \in \mathcal{H}$ spans T'. Then we have the following result whose proof is simple.

Lemma 7.4. *The set R' of pairs $(v', H') \in T' \times \mathcal{H}'$ such that there exists $(v, H) \in R$ with $v = v'$ and $H \cap E' = H'$ is a root system in E' associated to $\mathcal{H}'$.*

R' is called the root system obtained from R by restriction to E'.

The classification of irreducible affine root systems. Let R be an irreducible affine root system associated to $\mathcal{H}$. For simplicity, we assume that the rank of R is at least two (the rank one case can also be done easily, but is not relevant to us). Suppose first that R is *reduced*. Then it is completely determined by its *length function* $\ell : \mathcal{H} \to \mathbb{R}$, which is given by $\ell(H) := ||v||$ for $(v, H) \in R$. Namely, $R = \{(v, H) \in T \times \mathcal{H} \mid v \perp H,\ ||v|| = \ell(H)\}$. Since ℓ is invariant under W and W acts transitively on the set of alcoves, ℓ is determined by its values on the walls of one fixed alcove, and thus on the vertices of the Coxeter graph. Moreover ℓ has to take the same value on any two vertices that are linked by a single edge. More generally, it follows from (ii) in Definiton 7.3 that $\ell(H) = c\,\ell(H')$ with $c \in \{2^{\pm\frac{1}{2}}\}$ or $\{3^{\pm\frac{1}{2}}\}$ if the vertices associated to H and H' are linked by 2 or 3 edges, respectively. In each case, we encode the actual choices of $c \neq 1$ in the Coxeter graph by adding to the corresponding link an arrow pointing to the vertex of shorter length. The so obtained diagram is called the Dynkin diagram of R. It determines the root system uniquely up to scaling once the correspondence between vertices of the diagram and walls of the fixed alcove is given, and up to similarity without this extra piece of information. Here two irreducible root systems R, R' associated to families $\mathcal{H}$, $\mathcal{H}'$ in E, E', respectively, are called similar if there exists $\lambda > 0$ and an affine mapping $\varphi : E \to E'$ which is the composition of an isometry with a homothety such that $R' = \{(\lambda \varphi_* v, \varphi H) \mid (v, H) \in R\}$. From the above restrictions, one obtains precisely the list in Table 1 for the possible Dynkin diagrams of irreducible reduced affine root systems of rank at least 2 .

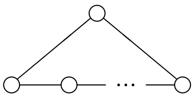
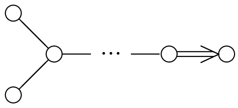
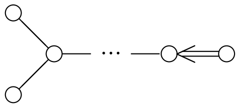
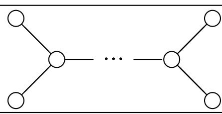
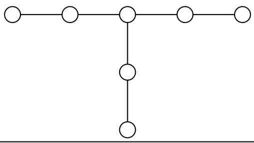
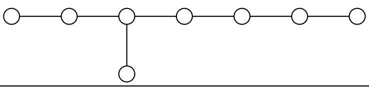
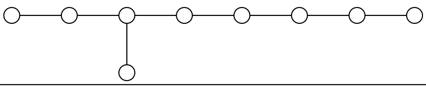
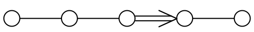
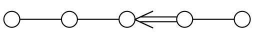
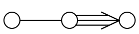
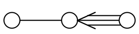
Type	*Diagram*
$\tilde{A}_n,\ n \geq 2$	
$\tilde{B}_n,\ n \geq 3$	
$\tilde{B}_n^\vee,\ n \geq 3$	
$\tilde{C}_n,\ n \geq 2$	
$\tilde{C}_n^\vee,\ n \geq 2$	
$\tilde{C}_n',\ n \geq 2$	
$\tilde{D}_n,\ n \geq 4$	
$\tilde{E}_6$	
$\tilde{E}_7$	
$\tilde{E}_8$	
$\tilde{F}_4$	
$\tilde{F}_4^\vee$	
$\tilde{G}_2$	
$\tilde{G}_2^\vee$	

Table 1: Dynkin diagrams of reduced irreducible affine root systems of rank at least 2.

That all diagrams in Table 1 occur indeed as diagrams of affine root systems follows from the examples discussed above. The canonical affine root system associated to $\mathcal{H}$ (more precisely, its similarity type) is denoted by the same symbol as the corresponding affine Weyl group. The geometric meaning of the arrows in this case can be explained as follows. Each vertex is associated to a family of equally spaced parallel hyperplanes, and an arrow always points to a family that is wider spaced. The difference between the diagrams of R and $\check{R}$ is that the directions of the arrows are all reversed as the product of their length functions is

2. In this way, all diagrams listed above are obtained, except $\tilde{C}'_n$ which can be obtained by a modification of the construction of the canonical root system [Mac72].

Suppose now R is *nonreduced*. In this case R is completely determined by R_{red} and the information for which $H \in \mathcal{H}$ there exists $v \in T$ with (v, H), $(2v, H) \in \mathcal{H}$. This property is invariant under the Weyl group and thus can be encoded in the Dynkin diagram of R_{red} by adding a second, larger concentric circle around the corresponding vertex, following the notation of [Loo69]. The diagram so obtained is called the Dynkin diagram of R and determines R as before up to scaling (similarity). Again (ii) in Definition 7.3 restricts which vertices can admit a second concentric circle. Namely, if (v, H), $(2v, H)$, $(v', H') \in R$ then $m = \frac{\langle v,v'\rangle}{||v||^2}$, $n = 2\frac{\langle v,v'\rangle}{||v'||^2} \in \mathbb{Z}$ implying $mn \leq 2$ and hence $||v'||^2 = 2||v||^2$ unless v, v' are parallel or orthogonal. As we are assuming the rank of R to be at least two, a concentric circle can be added only to a vertex v in the Dynkin diagram of R_{red} which is only doubly linked to other vertices and for which the arrows point to v . The possible diagrams are listed in Table 2, where the type refers to $(R_{\text{red}}, R_{\text{red}'})$.

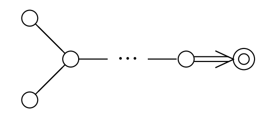
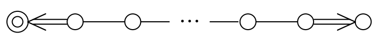
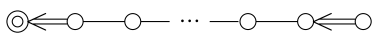
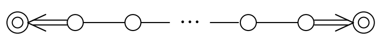
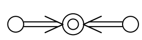
Type	*Diagram*
$(\tilde{B}_n, \tilde{B}^\vee_n)$	
$(\tilde{C}'^\vee_n, \tilde{C}'_n)$	
$(\tilde{C}'_n, \tilde{C}_n)$	
$(\tilde{C}^\vee_n, \tilde{C}_n)$	
$(\tilde{C}_2, \tilde{C}^\vee_2)$	

Table 2: Dynkin diagrams of nonreduced irreducible affine root systems of rank at least 2.

That all diagrams in Table 2 actually occur can be seen by the following construction. Enlarge R_{red} with the elements $(2v, H)$ for all $(v, H) \in R_{\text{red}}$ such that H is conjugate under W to a vertex with a double circle. Then we only need to check condition (ii) in Definition 7.3 to show that this enlarged set is a root system with the required properties. If there is only one vertex with an additional concentric circle this follows by the remark that for the two diagrams one obtains by deleting the additional circle and keeping or reversing the direction of the arrows to that vertex there exists always a root system. If there are two additional circles one applies the same argument to either of them using the first step.

Root systems of isoparametric submanifolds. Consider first the case of a finite dimensional homogeneous compact isoparametric submanifold M of rank at least two in Euclidean space, which we may assume to be contained in a sphere around the origin. By Dadok's theorem [Dad85], M can be identified with a principal orbit of the isotropy representation of an irreducible symmetric space G/K of compact type, where (G, K) is an effective symmetric pair and K is connected. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition and fix $x \in M$. The affine and linear normal spaces of M at x coincide, and $\nu_x M$ is a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Recall that the root system Δ of G/K with respect to $\mathfrak{a}$ is given by $\Delta = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{p}_\alpha \neq 0\}$, where $\mathfrak{p}_\alpha = \{X \in \mathfrak{p} \mid ad_H^2 X = -\alpha(H)^2 X\}$ for all $H \in \mathfrak{a}\}$. One sees that the focal hyperplanes of M in $\nu_x M = \mathfrak{a}$ coincide with the kernels of the roots in Δ. We identify each $\alpha \in \Delta$ with the pair $(h_\alpha, \ker \alpha)$, where $h_\alpha \in \mathfrak{a}$ is defined by $\langle h_\alpha, v\rangle = \alpha(v)$ for all $v \in \mathfrak{a}$, in order to associate a root system to the set $\mathcal{H}$ of focal hyperplanes of M in $\nu_x M$. We call this the root system of M. It follows from the next result that it is independent of the identification of M with a principal orbit of the isotropy representation of a symmetric space. Hence it is well defined, up to scaling.

Proposition 7.5. *Let G/K, G'/K' be irreducible symmetric spaces of compact type of rank at least two, where (G,K), (G',K') are effective symmetric pairs and K, K' are connected. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ be the corresponding decompositions of the Lie algebras of G, G' into the ±1-eigenspaces of the involutions, respectively. Assume $\varphi : \mathfrak{p} \to \mathfrak{p}'$ is an isometry that maps a principal K-orbit onto a principal K'-orbit. Then, after multiplication by a suitable constant, φ maps the root system of G/K with respect to $\mathfrak{a}$ onto the root system of G'/K' with respect to $\varphi(\mathfrak{a})$, where $\mathfrak{a}$ is any normal space to the principal K-orbit.*

Proof. By irreducibility, after multiplying φ by a suitable constant, we may assume that $\langle \varphi x, \varphi y\rangle' = \langle x, y\rangle$ for all x, $y \in \mathfrak{p}$, where $\langle,\rangle$, $\langle,\rangle'$ denote the negatives of the Killing forms of $\mathfrak{g}$, $\mathfrak{g}'$, respectively. Now it suffices to show that φ extends to a Lie algebra isomorphism from $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ to $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ preserving the decompositions.

By effectiveness, we can identify K, K' with subgroups of $O(\mathfrak{p})$, $O(\mathfrak{p}')$, respectively. Then, using the rank assumption, K (and similarly K') is the maximal connected subgroup of $O(\mathfrak{p})$ with its orbits (this follows from [Sim62], cf. [BCO03, Prop. 4.3.9], or [EH99]). Since an isoparametric foliation is determined by any regular leaf, φ has to map the K-orbit foliation to the K'-orbit foliation and thus conjugates K to K'. Now φ can be extended to a linear bijective map $\mathfrak{g} \to \mathfrak{g}'$ preserving the decompositions by putting $\varphi(A) = \varphi A \varphi^{-1}$ for all $A \in \mathfrak{k}$. This is clearly a Lie algebra isomorphism from $\mathfrak{k}$ to $\mathfrak{k}'$, but also from $\mathfrak{g}$ to $\mathfrak{g}'$ as

$$[\varphi(A), \varphi x] = \varphi A \varphi^{-1}(\varphi x) = \varphi[A, x],$$

and

$$\begin{aligned} \langle \varphi(A), [\varphi x, \varphi y]\rangle' &= \langle [\varphi(A), \varphi x], \varphi y\rangle' \\ &= \langle \varphi[A, x], \varphi y\rangle' \\ &= \langle [A, x], y\rangle \\ &= \langle A, [x, y]\rangle \\ &= \langle \varphi(A), \varphi[x, y]\rangle' \end{aligned}$$

for all $A \in \mathfrak{k}$ and x, $y \in \mathfrak{p}$. Here we have used that φ preserves the inner products as $\mathfrak{k} \perp \mathfrak{p}$, $\mathfrak{k}' \perp \mathfrak{p}'$ and

$$\begin{aligned} \langle \varphi(A), \varphi(B)\rangle' &= \langle \varphi(A), \varphi(B)\rangle_{\mathfrak{k}'} - \mathrm{tr}_{\mathfrak{p}'} \varphi(A)\varphi(B) \\ &= \langle A, B\rangle_{\mathfrak{k}} - \mathrm{tr}_{\mathfrak{p}} AB \\ &= \langle A, B\rangle \end{aligned}$$

for all A, $B \in \mathfrak{k}$ where $\langle,\rangle_{\mathfrak{k}}$, $\langle,\rangle_{\mathfrak{k}'}$ denote the negatives of the Killing forms of $\mathfrak{k}$, $\mathfrak{k}'$ and $\mathrm{tr}_{\mathfrak{p}}$, $\mathrm{tr}_{\mathfrak{p}'}$ denote the traces of operators on $\mathfrak{p}$, $\mathfrak{p}'$, respectively. □

By construction it is clear that the root system is invariant under isometries. More precisely we have

Lemma 7.6. *Assume M is finite dimensional. Let φ be an isometry from V to another Euclidean space V' and let $M' = \varphi M$. Then, for any $x \in M$, φ maps the root system of M in $x + \nu_x M$ to the root system of M' in $\varphi(x) + \nu_{\varphi(x)} M'$, up to a scaling factor.*

Since a finite dimensional isoparametric submanifold M is congruent to an orbit of the isotropy representation of a symmetric space and the root system of M coincides with that of the symmetric space, we get from the standard theory of symmetric spaces

Proposition 7.7. *Assume M is a finite dimensional homogeneous compact isoparametric submanifold of rank at least two in an Euclidean space V and let L be an irreducible slice of M of rank at least two through $x \in M$ with affine span W. Then the root system of L associated to the focal hyperplanes in the affine normal space of L at x in W is obtained by restriction from the root system of M associated to the focal hyperplanes in $x + \nu_x M$, up to scaling.*

After this preparation, we come the main result of this section.

Theorem 7.8. *For each infinite dimensional connected complete full irreducible isoparametric submanifold M of rank at least two in a separable Hilbert space V and $x \in M$, there exists a naturally defined affine root system associated to the family of focal hyperplanes in $x + \nu_x M$ which is unique up to scaling.*

Proof. Let $E = x + \nu_x M$ and let $\mathcal{H}$ be the family of focal hyperplanes in E. Then the group W generated by the reflections in the elements of $\mathcal{H}$ is an affine Weyl group. Fix an alcove $\mathcal{A}$ and denote its walls by $H_{\mathbf{i}_1}, \dots H_{\mathbf{i}_{n+1}}$, which also parametrize the vertices of the Coxeter graph of W. In order to have an affine root system associated to $\mathcal{H}$, up to scaling, we need to specify in the Coxeter graph the arrows attached to the double and triple links, and the possible additional circles around vertices that are only doubly linked to other vertices (with arrows pointing to the given one). We proceed as follows. For any two vertices $H_{\mathbf{i}_a}$, $H_{\mathbf{i}_b}$ that are doubly or triply linked, consider the finite dimensional rank two slice $L_P(x)$ where P is the affine span of the curvature normals $v_{\mathbf{i}_a}$, $v_{\mathbf{i}_b}$ and transfer the information about arrows and additional circles from the Dynkin diagram of $L_P(x)$ to the subdiagram of M with vertices $H_{\mathbf{i}_a}$, $H_{\mathbf{i}_b}$. In this way we get well defined arrows. That also additional circles are well defined — the only problem arises for the middle vertex in the $\tilde{C}_2$ graph which lies in two such subgraphs — follows from the fact that the information about additional circles can also be read off from the multiplicities (cf. section 2).

The construction of the affine root system of M is also independent of the chosen alcove. If $\mathcal{A}'$ is another alcove then there exists an element $w \in W$ with $\mathcal{A}' = wA$. The walls of $\mathcal{A}'$ are $wH_{\mathbf{i}_1}, \dots, wH_{\mathbf{i}_{n+1}}$. Let P be the affine span of $v_{\mathbf{i}_a}$, $v_{\mathbf{i}_b}$ for $a \neq b$. There exists an isometry φ of V that preserves M and maps x to wx. Then $\varphi(L_P(x)) = L_P(wx)$ and $\varphi H_{\mathbf{i}_a} = wH_{\mathbf{i}_a}$, $\varphi H_{\mathbf{i}_b} = wH_{\mathbf{i}_b}$ as φ coincides with w on the affine normal space $x + \nu_x M$ due to $\varphi v_{\mathbf{i}}(x) = v_{\mathbf{i}}(wx) = wv_{\mathbf{i}}(x)$ for all $\mathbf{i} \in \mathbf{I}^*$. Therefore φ, which maps the root system of $L_P(x)$ isometrically onto that of $L_P(wx)$, maps the Dynkin diagram of $L_P(x)$ with vertices $H_{\mathbf{i}_a}$, $H_{\mathbf{i}_b}$ isomorphically to the Dynkin diagram of $L_P(wx)$ with vertices $wH_{\mathbf{i}_a}$, $wH_{\mathbf{i}_b}$. ☐

It is now clear that Lemma 7.6 and Proposition 7.7 carry over to to the infinite dimensional setting.

Corollary 7.9. *(i) The (finite or affine) root system of an irreducible slice of rank at least two of M is obtained from that of M by restriction.*

(ii) *Let φ be an isometry from V to another Hilbert space V' and let $M' = \varphi M$. Then, for any $x \in M$, φ maps the root system of M in $x + \nu_x M$ to the root system of M' in $\varphi(x) + \nu_{\varphi(x)} M'$, up to a scaling factor.*

8. Γ along rank one slices

In order to use Proposition 6.5 effectively, we have to understand which components $(\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}})_{E_k}$ of $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}}$ might be non-zero if $v_{\mathbf{i}}$ and $v_{\mathbf{j}}$ are linearly dependent.

The first result is a reduction to the rank two case.

Lemma 8.1. (i) *If the affine Weyl group W of M is isomorphic to $\tilde{A}_n$ $(n \geq 2)$, $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_n$ $(n = 6$, 7 or $8)$ or $\tilde{F}_4$ then any rank one slice is contained in a slice L of type $\tilde{A}_2$.*

(ii) *If W is isomorphic to $\tilde{B}_n$ or $\tilde{C}_n$ $(n \geq 2)$ then any rank one slice is contained in a slice of type $\tilde{A}_2$ or in a slice whose Dynkin diagram has a symbol which is obtained from that of the Dynkin diagram of M by replacing n by 2 and (if W is isomorphic to $\tilde{B}_n$) B by C (e.g. if M is of type $(\tilde{B}_n, \tilde{B}_n^\vee)$ then any rank one slice is contained in a slice of type $\tilde{A}_2$ or $(\tilde{C}_2, \tilde{C}_2^\vee)$).*

Proof. We may assume the rank one slice to be infinite dimensional (since otherwise it is simply a curvature sphere which is contained in an infinite dimensional rank one slice) and thus of the form L_P with $P = \mathbb{R}v_{\mathbf{i}}$ for some $\mathbf{i} \in \mathbf{I}^*$. The focal hyperplane $H_{\mathbf{i}}$ bounds an alcove and thus corresponds to a vertex in the Dynkin diagram. If there exists a vertex which is joined by a single link to the vertex corresponding to $H_{\mathbf{i}}$, then there exists a finite dimensional slice L_Q of type A_2 containing $S_{\mathbf{i}}(x)$ such that Q is the affine span of $\{v_{\mathbf{i}}, v_{\mathbf{j}}\}$ for some $\mathbf{j} \in \mathbf{I}^*$. If now $\tilde{Q}$ is the linear span of $\{v_{\mathbf{i}}, v_{\mathbf{j}}\}$ then we see that $L_{\tilde{Q}}$ is of type $\tilde{A}_2$ and contains L_P. This proves (i) and shows in case (ii) that any rank one slice $L_P, P = \mathbb{R}v_{\mathbf{i}}$, is contained in a slice of type $\tilde{A}_2$ if the hyperplane $H_{\mathbf{i}}$ is not conjugate under the Weyl group to an extremal vertex of the Dynkin diagram that is connected by a double link to another vertex.

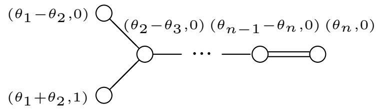
To study the remaining cases it is convenient to assume, after possibly translating M in V, that the origin of V lies in $\nu_x M$ and in fact in one focal hyperplane from each family of parallel focal hyperplanes. In particular, the affine normal space is identified with the normal space and the origin is a special point in the sense of Bourbaki [Bou68, ch. V, § 3, nr. 10].

Consider first the case W isomorphic to $\tilde{B}_n$. Up to rescaling the metric in $\nu_x M$, we can choose an orthonormal basis $\theta_1, \ldots, \theta_n$ of the dual space $(\nu_x M)^*$ such that $H_{\alpha,i} = \{\xi \in \nu_x M \mid \alpha(\xi) = i\}$ are the focal hyperplanes, where $\alpha \in \Delta^+ := \{\theta_a,\ \theta_a \pm \theta_b \mid 1 \leq a < b \leq n\}$ and $i \in \mathbb{Z}$. In fact, $\Delta := \Delta^+ \cup (-\Delta^+)$ is a a root system of type B_n with Δ^+ the positive roots (cf. [Bou68, ch. V, planche II]). Since $\theta_1 - \theta_2, \ldots, \theta_{n-1} - \theta_n$, θ_n is a basis of Δ and $\theta_1 + \theta_2$ a highest root, the hyperplanes

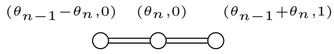
$$\theta_1 - \theta_2 = 0, \ldots, \theta_{n-1} - \theta_n = 0,$$

$$\theta_n = 0, \quad \theta_1 + \theta_2 = 1$$

are the walls of an alcove whose associated Coxeter graph has the form



By the above discussion we may assume that $v_{\mathbf{i}} = v_{(\theta_n,0)}$ and thus can take $P \subset \nu_x M$ to be the linear span of $\{v_{(\theta_{n-1}-\theta_n,0)}, v_{(\theta_n,0)}\}$. Then the Coxeter graph of $L_P(x)$ is



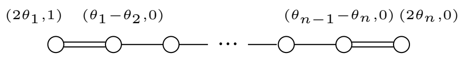
Denote the reflection in $H_{\alpha,i}$ by $s_{\alpha,i}$. Since $n \geq 3$, there is an element in W that maps the pair $(H_{\theta_{n-1}+\theta_n,1}, H_{\theta_n,0})$ to $(H_{\theta_{n-1}-\theta_n,0}, H_{\theta_n,0})$, namely the composition $s_{\theta_{n-2}-\theta_{n-1},1} s_{\theta_{n-2}-\theta_{n-1},0} s_{\theta_n,0}$. Hence both arrows of the Dynkin diagram of the slice point inward or outward in accordance with the direction of the arrow in the Dynkin diagram of M. The desired result follows.

Next, consider the case W isomorphic to $\tilde{C}_n$. Up to rescaling the metric in $\nu_x M$, we can choose an orthonormal basis $\theta_1, \ldots, \theta_n$ of $(\nu_x M)^*$ such that the hyperplanes

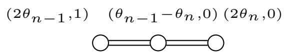
$$\theta_1 - \theta_2 = 0, \ldots, \theta_{n-1} - \theta_n = 0,$$

$$2\theta_n = 0, \quad 2\theta_1 = 1$$

are the walls of an alcove (cf. [Bou68, ch. V, planche III]). Now the associated Coxeter graph has the form



We may assume that $v_{\mathbf{i}} = v_{(2\theta_n,0)}$ and can take $P \subset \nu_x M$ to be the linear span of $\{v_{(\theta_{n-1}-\theta_n,0)}, v_{(2\theta_n,0)}\}$. Then the Coxeter graph of $L_P(x)$ is



Since the finite Weyl group of type C_n contains the full permutation group on the $\theta_{\mathbf{i}}$ (and their flips of signs), the pair $(H_{2\theta_{n-1},1}, H_{\theta_{n-1}-\theta_n,0})$ is W-conjugate to $(H_{2\theta_1,1}, H_{\theta_1-\theta_2,0})$. This implies the stated result. □

To deal with the case of $\tilde{G}_2$, we need the following result.

Lemma 8.2. *If the affine Weyl group of M is isomorphic to $\tilde{G}_2$ then $\Gamma_{E_{\mathbf{i}}} E_{\mathbf{j}} = 0$ for all $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$ with $v_{\mathbf{i}} \perp v_{\mathbf{j}}$.*

Proof. Let P be the affine span of $\{v_{\mathbf{i}}, v_{\mathbf{j}}\}$ for some $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$ with $v_{\mathbf{i}} \perp v_{\mathbf{j}}$ (fixing some point $x \in M$ as usually). Then L_P is finite dimensional (since $0 \notin P$) and focalizes at the intersection point of $H_{\mathbf{i}}$ and $H_{\mathbf{j}}$ in the affine normal space. If there is no further focal line passing through this point then L_P is of type $A_1 \times A_1$ and the statement is clear. Otherwise there must be exactly six focal lines passing through this point and L_P is of type G_2. Therefore it suffices to check the statement for finite dimensional homogeneous isoparametric submanifolds of type G_2, in which case it follows from simple properties of the associated root system. In fact, such a submanifold is congruent to a principal orbit of the isotropy representation of a symmetric space G/K and we can apply the discussion in subsection 3.3. We employ the notation from there and recall that we are dealing with a reduced root system. For $X_\lambda \in E_\lambda$, $Y_\mu \in E_\mu$ we have

$$\Gamma_{X_\lambda} Y_\mu = [\check{X}_\lambda, Y_\mu] \in [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu},$$

where $\check{X}_\lambda$ is the unique element in $\mathfrak{k}_\lambda$ satisfying $[\check{X}_\lambda, x] = X_\lambda$. In the specific case of G_2, it is a standard fact that orthogonal roots are always strongly orthogonal, i.e. $\lambda \perp \mu$ implies that $\lambda \pm \mu$ are not roots, which can also be immediately seen from the explicit description of the root system: $\Lambda = \{\pm\lambda_1, \pm\lambda_2, \pm(\lambda_1+\lambda_2), \pm(2\lambda_1+\lambda_2), \pm(3\lambda_1+\lambda_2), \pm(3\lambda_1+2\lambda_2)\}$. The desired result follows. □

One of the main ingredients to describe the image of Γ is surprisingly the following elementary lemma in Euclidean plane geometry.

Lemma 8.3. *Let ℓ_1, ℓ_2, ℓ_3 be three different parallel lines in the plane and let x_i be a point in ℓ_i for each i. Let ℓ_{ij} for $1 \le i < j \le 3$ be the line through x_i and x_j.*

(i) If the angles between any two of the six lines above are multiples of $\pi/6$ but never $\pi/2$ then x_1, x_2, x_3 are colinear.

(ii) If the angles between any two of the six lines above are multiples of $\pi/4$ then either x_1, x_2, x_3 are colinear or one of the lines ℓ_1, ℓ_2, ℓ_3, say ℓ_2, lies exactly in half way distance in between the other two. In the later case ℓ_{13} is orthogonal to ℓ_2.

Proof. Without loss of generality we may assume that ℓ_2 lies between ℓ_1 and ℓ_3. Suppose that x_1, x_2, x_3 are not colinear, which is equivalent to saying that ℓ_{12}, ℓ_{13}, ℓ_{23} are pairwise different. Plainly from the formula for the angle sum, it is readily seen that the angles of the triangle with vertices x_1, x_2, x_3 can only be $\pi/6$, $\pi/6$, $2\pi/3$, or all $\pi/3$ in case (i) and are necessarily $\pi/4$, $\pi/4$, $\pi/2$ in case (ii). Therefore neither ℓ_{12} nor ℓ_{23} is orthogonal to ℓ_2 and ℓ_2 bisects the angle between $x_1 - x_2$ and $x_3 - x_2$. This implies that the angles of the triangle at x_1 and x_3 are equal and that ℓ_{13} is orthogonal to ℓ_2. This is a contradiction in case (i) and proves the result in case (ii). □

Next we prove Theorem C, as stated in the introduction. It will be further sharpened by Theorem 8.12.

Theorem 8.4. *If the affine Weyl group W of M is isomorphic to $\tilde{A}_n$ $(n \ge 2)$, $\tilde{D}_n$ $(n \ge 4)$, $\tilde{E}_n$ $(n = 6, 7$ or $8)$, $\tilde{F}_4$ or $\tilde{G}_2$ then*

$$\Gamma_{E_\mathbf{i}} E_\mathbf{j} \subset E_\mathbf{0}$$

for all $\mathbf{i}$, $\mathbf{j} \in \mathbf{I}^*$ with $v_\mathbf{j} \in \mathbb{R} v_\mathbf{i}$.

Proof. By Lemma 8.1 we may assume W is isomorphic to $\tilde{A}_2$ or $\tilde{G}_2$ and by Proposition 3.11 we may assume $\mathbf{i} \neq \mathbf{j}$. Assume that there exist $X_\mathbf{i} \in E_\mathbf{i}$, $Y_\mathbf{j} \in E_\mathbf{j}$ and $\mathbf{m} \in \mathbf{I}^*$ with $(\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}} \neq 0$. Since L_P for $P = \mathbb{R} v_\mathbf{i}(x)$ is totally geodesic, we must then have $v_\mathbf{m} \in \mathbb{R} v_\mathbf{i}$, and by Propositions 3.11 and 3.12, $\mathbf{m} \neq \mathbf{i}, \mathbf{j}$. According to Theorem 4.1 there exist $\mathbf{k}, \mathbf{l} \in \mathbf{I}^*$ with $v_\mathbf{k}$, $v_\mathbf{l}$ linearly independent and $Z_\mathbf{k} \in E_\mathbf{k}$, $W_\mathbf{l} \in E_\mathbf{l}$ such that $\langle (\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}}, \Gamma_{Z_\mathbf{k}} W_\mathbf{l} \rangle \neq 0$. Since the line through $v_\mathbf{k}$ and $v_\mathbf{l}$ meets P in at most one point, we have $\langle \Gamma_{X_\mathbf{i}} Y_\mathbf{j}, \Gamma_{Z_\mathbf{k}} W_\mathbf{l} \rangle = \langle (\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}}, \Gamma_{Z_\mathbf{k}} W_\mathbf{l} \rangle \neq 0$ and no three among $v_\mathbf{i}$, $v_\mathbf{j}$, $v_\mathbf{k}$, $v_\mathbf{l}$ are colinear. Thus Corollary 5.10 yields

$$0 \neq \langle (\Gamma_{X_\mathbf{i}} Y_\mathbf{j})_{E_\mathbf{m}}, \Gamma_{Z_\mathbf{k}} W_\mathbf{l} \rangle = \langle \Gamma_{Z_\mathbf{k}} Y_\mathbf{j}, \Gamma_{X_\mathbf{i}} W_\mathbf{l} \rangle + c \cdot \langle \Gamma_{X_\mathbf{i}} Z_\mathbf{k}, \Gamma_{Y_\mathbf{j}} W_\mathbf{l} \rangle$$

for some $c \in \mathbb{R}$. At least one of the terms on the right hand side is nonzero, so at least one of the following two cases must be true: (i) the lines through $v_\mathbf{k}$, $v_\mathbf{j}$ and $v_\mathbf{i}$, $v_\mathbf{l}$ meet at some curvature normal, say $v_{\mathbf{n}}$; (ii) the lines through $v_{\mathbf{i}}$, $v_{\mathbf{k}}$ and $v_{\mathbf{j}}$, $v_{\mathbf{l}}$ meet at some curvature normal. The analysis is completely similar in both cases, so we assume (i) is true. Now we have the following picture for the focal lines: $H_{\mathbf{i}}$, $H_{\mathbf{j}}$, $H_{\mathbf{m}}$ are parallel and pairwise different, and $H_{\mathbf{k}}$, $H_{\mathbf{l}}$, $H_{\mathbf{n}}$ are three other lines such that $H_{\mathbf{k}}$, $H_{\mathbf{l}}$ meet at $H_{\mathbf{m}}$, $H_{\mathbf{k}}$, $H_{\mathbf{n}}$ meet at $H_{\mathbf{j}}$, and $H_{\mathbf{l}}$, $H_{\mathbf{n}}$ meet at $H_{\mathbf{i}}$. The intersection points are not colinear as $\mathbf{k} \neq \mathbf{l}$. However this is in contradiction with Lemma 8.3 since the angle between any two of these lines is a multiple of $\pi/6$ and no two among them are orthogonal. The non-orthogonality is of course automatic in the $\tilde{A}_2$ case and follows from Lemmata 3.16(ii) and 8.2 in the $\tilde{G}_2$ case. □

In view of the discussion at the end of section 6, Theorem 8.4 already yields continuity of Γ in all cases but $\tilde{B}_n$ and $\tilde{C}_n$.

Corollary 8.5. *If the affine Weyl group of M is of type $\tilde{A}$, $\tilde{D}$, $\tilde{E}$, $\tilde{F}$ or $\tilde{G}$ then $\Gamma_{X_{\mathbf{i}}}$ is continuous for all $X_{\mathbf{i}} \in E_{\mathbf{i}}$ and $\mathbf{i} \in \mathbf{I}^*$.*

The next theorem extends Theorem 8.4 to the remaining cases of $\tilde{B}$ and $\tilde{C}$. However, to get continuity of $\Gamma_{X_{\mathbf{i}}}$ in those cases, it will be necessary to refine its information. The required refinements are given by Propositions 8.10 (case $E_{\mathbf{i}}$ is irreducible) and 8.13 (case $E_{\mathbf{i}}$ is reducible).

As already in the last section, we identify the index set $\mathbf{I}^*$ with $\mathcal{A} \times \mathbb{Z}$.

Theorem 8.6. *Let $\alpha \in \mathcal{A}$ and i, $j \in \mathbb{Z}$ with $i \neq j$. Then*

$$\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_{\mathbf{0}} \oplus E_{\alpha,2i-j} \oplus E_{\alpha,2j-i} \oplus E_{\alpha,\frac{i+j}{2}},$$

where the last term is to be omitted if $i+j$ is odd.

Proof. We may assume $W \cong \tilde{C}_2$ due to Theorem 8.4 and Lemma 8.1(ii). Since the rank one slice L_P for $P = \mathbb{R}v_{\alpha,i}(x)$ is totally geodesic, $\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_{\mathbf{0}} \oplus \bigoplus_{k\in\mathbb{Z}} E_{\alpha,k}$. Assume $(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,m}} \neq 0$ for some $X_{\alpha,i} \in E_{\alpha,i}$, $Y_{\alpha,j} \in E_{\alpha,j}$ and some $m \in \mathbb{Z}$, where $m \neq i$, j necessarily. Then we find by Theorem 4.1 (β, k), $(\gamma, \ell) \in \mathcal{A} \times \mathbb{Z}$ with $\beta \neq \gamma$ and $Z_{\beta,k} \in E_{\beta,k}$, $W_{\gamma,\ell} \in E_{\gamma,\ell}$ such that $\langle (\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,m}}, \Gamma_{Z_{\beta,k}} W_{\gamma,\ell} \rangle \neq 0$ and deduce as in the proof of Theorem 8.4 that $(\Gamma_{E_{\alpha,i}} E_{\gamma,\ell})_{E_{\delta,n}}$ and $(\Gamma_{E_{\alpha,j}} E_{\beta,k})_{E_{\delta,n}}$ are not zero for some $(\delta, n) \in \mathcal{A} \times I$. Therefore we have again three parallel lines $H_{\alpha,i}$, $H_{\alpha,j}$, $H_{\alpha,m}$ in the affine normal space, and three further lines $H_{\beta,k}$, $H_{\gamma,\ell}$, $H_{\delta,n}$ which are pairwise different and such that two of which intersect on $H_{\alpha,r}$ for each $r \in \{i, j, m\}$. Thus Lemma 8.3 implies that one of the parallel lines has to lie exactly in the middle between the other two lines or, equivalently, that one of the indices i, j, m is the arithmetic mean of the other two, that is $m = \frac{i+j}{2}$, $2i-j$ or $2j-i$. □

Combining Theorem 8.6 with Corollary 5.11 yields the following result, which makes Lemma 6.4 applicable in general.

Corollary 8.7. *Let $\alpha \in \mathcal{A}$ and i, j, $k \in \mathbb{Z}$ with $i \neq j$. Then*

$$\Gamma_{E_{\alpha,i}} E_{\alpha,j} \perp \Gamma_{E_{\alpha,i}} E_{\alpha,k}$$

if k is not one of

$$4j-3i,\ 2j-i,\ j,\ \frac{i+j}{2},\ \frac{3i+j}{4},\ \frac{3i-j}{2},\ 2i-j,\ 3i-2j.$$

Proof. We may assume that $k \neq i$. Let $\mathbf{i} = (\alpha, i)$, $\mathbf{j} = (\alpha, j)$, $\mathbf{k} = (\alpha, k)$. Now the condition $\frac{v_\mathbf{j}}{v_\mathbf{k}} \neq \frac{v_\mathbf{i} - v_\mathbf{j}}{v_\mathbf{k} - v_\mathbf{i}}$ in Corollary 5.11 is equivalent to $k \neq 2i - j$. Moreover $(\Gamma_{E_\mathbf{i}} E_\mathbf{j})_{E_\mathbf{m}} \perp \Gamma_{E_\mathbf{i}} E_\mathbf{k}$ for all $\mathbf{m} \in \mathbf{I}^*$ follows from Theorem 8.6 together with $\{2i-j, 2j-i, \frac{i+j}{2}\} \cap \{2i-k, 2k-i, \frac{i+k}{2}\} = \varnothing$. Finally $(\Gamma_{E_\mathbf{i}} E_\mathbf{j})_{E_\mathbf{k}} = 0$ follows from Theorem 8.6 and $k \neq 2i - j,\ 2j - i,\ \frac{i+j}{2}$. □

Let $P \subset x + \nu_x M$ be an affine line containing precisely four curvature normals, say $v_\mathbf{i}$, $v_\mathbf{j}$, $v_\mathbf{k}$, $v_\mathbf{l}$. Then the corresponding slice is necessarily finite dimensional and either of type B_2 or $(BC)_2$. We may assume $v_\mathbf{i} \perp v_\mathbf{j}$ and $v_\mathbf{k} \perp v_\mathbf{l}$ after an eventual permutation of the indices. The slice is of type B_2 if $E_\mathbf{i}$, $E_\mathbf{j}$, $E_\mathbf{k}$, $E_\mathbf{l}$ are all irreducible, and of type $(BC)_2$ if one pair among $E_\mathbf{i}$, $E_\mathbf{j}$ and $E_\mathbf{k}$, $E_\mathbf{l}$ is irreducible and the other is reducible.

Lemma 8.8. *Let $P \subset x + \nu_x M$ be an affine line containing precisely four curvature normals, say $v_\mathbf{i}$, $v_\mathbf{j}$, $v_\mathbf{k}$, $v_\mathbf{l}$ with $v_\mathbf{i} \perp v_\mathbf{j}$ and $v_\mathbf{k} \perp v_\mathbf{l}$.*

- *(i)* $\Gamma_{E_\mathbf{i}} E_\mathbf{j} = 0$ *if the corresponding slice is of type B_2 and $v_\mathbf{i}$, $v_\mathbf{j}$ correspond to the long roots in this slice.*
- *(ii)* $(\Gamma_{E_\mathbf{i}} E_\mathbf{j})_{E'_\mathbf{k}} = 0$ *if $E_\mathbf{k}$ is reducible (and $E_\mathbf{i}$, $E_\mathbf{j}$ are irreducible).*
- *(iii)* $\Gamma_{E''_\mathbf{i}} E_\mathbf{j} = \Gamma_{E_\mathbf{i}} E''_\mathbf{j} = 0$ *if $E_\mathbf{i}$ and $E_\mathbf{j}$ are reducible.*

Proof. The corresponding slice is homogeneous and thus congruent to a principal orbit of the isotropy representation of a symmetric space. The roots of the symmetric space are, up to sign, in a natural bijection with the focal lines and therefore with $v_\mathbf{i}$, $v_\mathbf{j}$, $v_\mathbf{k}$, $v_\mathbf{l}$. We may assume that the root system is of the form $\{\pm\theta_1, \pm\theta_2, \pm(\theta_1 \pm \theta_2)\}$ in the B_2 case and $\{\pm\theta_1, \pm\theta_2, \pm 2\theta_1, \pm 2\theta_2, \pm(\theta_1 \pm \theta_2)\}$ in the $(BC)_2$ case. Now (i)-(iii) follow from the discussion in subsection 3.3 by using the bracket relations

- (i') $[\mathfrak{k}_{\theta_1+\theta_2}, \mathfrak{p}_{\theta_1-\theta_2}] = [\mathfrak{k}_{\theta_1-\theta_2}, \mathfrak{p}_{\theta_1+\theta_2}] = 0$ (case B_2),
- (ii') $[\mathfrak{k}_{\theta_1+\theta_2}, \mathfrak{p}_{\theta_1-\theta_2}] = [\mathfrak{k}_{\theta_1-\theta_2}, \mathfrak{p}_{\theta_1+\theta_2}] \subset \mathfrak{p}_{2\theta_1} + \mathfrak{p}_{2\theta_2}$ (case BC_2),
- (iii') $[\mathfrak{k}_{2\theta_1}, \mathfrak{p}_{\theta_2} + \mathfrak{p}_{2\theta_2}] = [\mathfrak{k}_{2\theta_2}, \mathfrak{p}_{\theta_1} + \mathfrak{p}_{2\theta_1}] = 0$ (case BC_2),

respectively. □

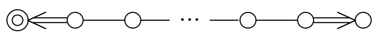
Remark 8.9. Let W be an affine Weyl group isomorphic to $\tilde{C}_2$ acting on an Euclidean plane, which is generated by reflections on a family of lines $\mathcal{H}$. We call a point an *intersection point* if it lies on at least two different lines in $\mathcal{H}$. It is clear that along any reflection line the intersection points are equally spaced, and there are exactly two possibilities for their spacing, one being wider than the other. Let H and H' be two nonparallel lines in $\mathcal{H}$.

- (i) If the spacing of intersection points along H is wide (or $H \perp H'$) then there passes through each intersection point on H a line in $\mathcal{H}$ which is parallel to H'.
- (ii) In general, there passes at least through each second intersection point on H a line in $\mathcal{H}$ which is parallel to H'.

Proposition 8.10. *Let i, $j \in \mathbb{Z}$ with $i - j$ even and $m = \frac{i+j}{2}$. Let $\alpha \in \mathcal{A}$ with $E_{\alpha,i}$ irreducible. Then*

$$(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,m}} = 0 \tag{8.11}$$

unless the diagram of M is

 *or* 

and (α, i) corresponds to the right extremal vertex while (α, m) corresponds to the left one. In particular equation (8.11) holds if $i - j$ is divisible by 4 or the affine Weyl group of M is not of type $\tilde{C}_n$ $(n \geq 2)$.

Proof. If (α, i) and (α, m) correspond to the two extremal vertices of a $\tilde{C}_n$ diagram then $i - m$ is odd necessarily, that is, $i - j$ is not divisible by 4. Hence it suffices to prove the first assertion, i.e. that $(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,m}} \neq 0$ can only occur in the two special cases described above.

By Theorem 8.4 and Lemma 8.1 we may assume M to be of type $\tilde{C}_2$. Note that according to Lemma 8.1 the arrows in the diagram of a slice of type $\tilde{C}_2$ both point inward or both point outward if M is of type $\tilde{B}_n$ and do not change direction if M is of type $\tilde{C}_n$.

So let M be of type $\tilde{C}_2$ and assume that $(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,m}} \neq 0$. Exactly as in the proof of Theorem 8.6 we find β, γ, $\delta \in \mathcal{A}$ and k, ℓ, $n \in \mathbb{Z}$ with $\beta \neq \gamma$ and such that $(\Gamma_{E_{\beta,k}} E_{\gamma,\ell})_{E_{\alpha,m}}$, $(\Gamma_{E_{\beta,k}} E_{\alpha,i})_{E_{\delta,n}}$ and $(\Gamma_{E_{\gamma,\ell}} E_{\alpha,j})_{E_{\delta,n}}$ are all nonzero. Lemma 8.3 yields that the focal lines $H_{\beta,k}$ and $H_{\gamma,\ell}$ are orthogonal to each other and make an angle $\pi/4$ with $H_{\alpha,m}$, and that $H_{\delta,n}$ is orthogonal to $H_{\alpha,i}$. Let $x_i \in H_{\alpha,i}$ and $x_m \in H_{\alpha,m}$ be the intersection points of $H_{\delta,n}$, $H_{\beta,k}$ and of $H_{\beta,k}$, $H_{\gamma,\ell}$, respectively.

We first observe that $E_{\beta,k}$ and $E_{\gamma,\ell}$ are necessarily irreducible. In fact, if one of them were reducible, then also the other one would be, since reflection at $H_{\alpha,m}$ maps $H_{\beta,k}$ to $H_{\gamma,\ell}$. By Lemma 8.8(iii) we have $\Gamma_{E_{\beta,k}} E_{\gamma,\ell} = \Gamma_{E'_{\beta,k}} E'_{\gamma,\ell}$ and $E_{\beta,k}$, $E_{\gamma,\ell}$ could be replaced in our argument by their components $E'_{\beta,k}$, $E'_{\gamma,\ell}$ as they have only to satisfy $\langle (\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,m}}, \Gamma_{E_{\beta,k}} E_{\gamma,\ell} \rangle \neq 0$. However, $(\Gamma_{E'_{\beta,k}} E_{\alpha,i})_{E_{\delta,n}} \neq 0$ would then be in contradiction to Lemma 8.8(ii).

If $i - m$ is even or (α, i) corresponds to the vertex in the middle of the diagram of M (of $\tilde{C}_2$-type) then there exists due to Remark 8.9 an integer ℓ_1 such that H_{γ,ℓ_1} passes through the intersection point of $H_{\alpha,m}$ and $H_{\delta,n}$. Reflection at H_{γ,ℓ_1} maps x_m to x_i, $H_{\alpha,m}$ to $H_{\delta,n}$, $H_{\gamma,\ell}$ to a parallel focal line H_{γ,ℓ_2} through x_i, and preserves $H_{\beta,k}$. This implies $\Gamma_{E_{\beta,k}} E_{\gamma,\ell_2} \neq 0$ in contradiction to $\Gamma_{E_{\alpha,i}} E_{\delta,n} \neq 0$ and Lemma 8.8(i). Note that $E_{\alpha,i}$, $E_{\beta,k}$, $E_{\delta,n}$ and E_{γ,ℓ_2} all belong to the slice centered at x_i, which is of type B_2 as $E_{\alpha,i}$ is irreducible by assumption and $E_{\beta,k}$ is irreducible by the above. Thus $i - m$ is odd and (α, i) corresponds to one of the extremal vertices of the diagram of M. Since $i - m$ is odd, (α, m) corresponds to the other extremal vertex of the diagram. We also see that the root corresponding to (α, i) in the slice centered at x_i must be short by Lemma 8.8 as $\Gamma_{E_{\alpha,i}} E_{\delta,n} \neq 0$. Hence an arrow points to the vertex corresponding to (α, i).

Finally, consider $E_{\alpha,m}$. If it is irreducible, and thus the slice centered at x_m is also of type B_2, then the root corresponding to (α, m) must be long again by Lemma 8.8, as $\Gamma_{E_{\beta,k}} E_{\gamma,\ell} \neq 0$. Hence in this case the diagram of M is $\overset{(\alpha,m)}{\circ}\!\Longrightarrow\!\circ\!\Longrightarrow\!\overset{(\alpha,i)}{\circ}$ where (α, m) and (α, i) correspond to the vertices indicated. On the other hand, if $E_{\alpha,m}$ is reducible, the diagram necessarily is $\overset{(\alpha,m)}{\circledcirc}\!\Longleftarrow\!\circ\!\Longrightarrow\!\overset{(\alpha,i)}{\circ}$ as the arrow between two vertices always points to that one which corresponds to a reducible eigenspace, if such a vertex occurs. $\square$

The next theorem is not necessary for the proof of the continuity of Γ, but it contains interesting information that sharpens Theorem 8.4.

Theorem 8.12. *Assume $E_{\alpha,i}$ is irreducible. Then*

(i)

$$\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_{\mathbf{0}}$$

if $i-j$ is divisible by 4 *or the Weyl group of M is not of type $\tilde{C}_n$ ($n \geq 2$).*

(ii)

$$\Gamma_{E_{\alpha,i}} E_{\alpha,j} \subset E_{\mathbf{0}} + E_{\alpha,\frac{i+j}{2}}$$

if $i-j$ is even.

Proof. If $i-j$ is even or W is not of type $\tilde{C}_n$ then $(\Gamma_{E_{\alpha,2i-j}} E_{\alpha,j})_{E_{\alpha,i}} = (\Gamma_{E_{\alpha,i}} E_{\alpha,2j-i})_{E_{\alpha,j}} = 0$ by Proposition 8.10 (note that $(2i-j)-j = 2(i-j)$ is divisible by 4 if $i-j$ is even, and that E_{2i-j} is irreducible) and hence $(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,2i-j}} = (\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,2j-i}} = 0$ by Lemma 3.16. Thus (i) and (ii) follow from Theorem 8.6 and Proposition 8.10. □

Proposition 8.13. *Let i, $j \in \mathbb{Z}$ with $i-j$ even, $i \neq j$, and $m = \frac{i+j}{2}$. Let $\alpha \in \mathcal{A}$ with $E_{\alpha,i}$ reducible. Then*

(i) $(\Gamma_{E''_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,m}} = 0$.

(ii) $E_{\alpha,m}$ is reducible and $(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E'_{\alpha,m}} = 0$ if $i-j$ is divisible by 4.

Proof. In both cases we follow essentially the proof of Proposition 8.10.

(i) Suppose, to the contrary, that $(\Gamma_{E''_{\alpha,i}} E_{\alpha,j})_{E_{\alpha,m}} \neq 0$. Then we can find β, γ, $\delta \in \mathcal{A}$ and k, ℓ, $n \in \mathbb{Z}$ with $(\Gamma_{E_{\beta,k}} E_{\gamma,\ell})_{E_{\alpha,m}} \neq 0$, $(\Gamma_{E''_{\alpha,i}} E_{\beta,k})_{E_{\delta,n}} \neq 0$ and $(\Gamma_{E_{\alpha,j}} E_{\gamma,\ell})_{E_{\delta,n}} \neq 0$, which implies that $H_{\beta,k}$ and $H_{\gamma,\ell}$ are orthogonal, each of them makes an angle of $\pi/4$ with $H_{\alpha,m}$, and $H_{\delta,n}$ is orthogonal to $H_{\alpha,i}$. In particular $E_{\delta,n}$ is also reducible as the reflection on $H_{\beta,k}$ is an element of the affine Weyl group that maps $H_{\alpha,i}$ to $H_{\delta,n}$. However $(\Gamma_{E''_{\alpha,i}} E_{\beta,k})_{E_{\delta,n}} \neq 0$ yields $\Gamma_{E''_{\alpha,i}} E_{\delta,n} \neq 0$ and this contradicts Lemma 8.8(iii).

(ii) If $i-j$ is divisible by 4, then $i-m$ is divisible by 2, so there is an element in the affine Weyl group that maps $H_{\alpha,i}$ to $H_{\alpha,m}$ and this shows that $E_{\alpha,m}$ is reducible.

Suppose now that $(\Gamma_{E_{\alpha,i}} E_{\alpha,j})_{E'_{\alpha,m}} \neq 0$. Then we can find β, γ, $\delta \in \mathcal{A}$ and k, ℓ, $n \in \mathbb{Z}$ with $(\Gamma_{E_{\beta,k}} E_{\gamma,\ell})_{E'_{\alpha,m}} \neq 0$, $(\Gamma_{E_{\alpha,i}} E_{\beta,k})_{E_{\delta,n}} \neq 0$ and $(\Gamma_{E_{\alpha,j}} E_{\gamma,\ell})_{E_{\delta,n}} \neq 0$. However the first inequality is a contradiction to Lemma 8.8(ii) as $H_{\beta,k}$ and $H_{\gamma,\ell}$ make an angle of $\pi/4$ with $H_{\alpha,m}$ and thus $E_{\beta,k}$, $E_{\gamma,\ell}$ are irreducible. Note that in a finite dimensional rank 2 slice of type $(BC)_2$ exactly one pair of orthogonal focal lines corresponds to irreducible eigenspaces. □

Using the same idea as in the proof of Theorem 8.12, one gets from Theorem 8.6 and Proposition 8.13 the statements (i)-(iv) of Theorem E. Thus Theorems D and E follow from Theorem 8.6 and the last two results.

The following result is a simple application of Theorems D and E.

Theorem 8.14. *If M is infinite dimensional so is $E_{\mathbf{0}}$.*

Proof. Let $\alpha \in \mathcal{A}$. Then Corollary 8.7 implies that for all $j \in \mathbb{Z}\backslash\{0\}$, $\Gamma_{E_{\alpha,0}} E_{\alpha,j} \perp \Gamma_{E_{\alpha,0}} E_{\alpha,k}$ for all but finitely many $k \in \mathbb{Z}$. Starting with $j_1 = 4$ we may thus construct a monotone sequence $(j_k)_k \geq 1$ in $4\mathbb{Z}$ such that $\Gamma_{E_{\alpha,0}} E_{\alpha,j_k} \perp \Gamma_{E_{\alpha,0}} E_{\alpha,j_\ell}$ for all $k \neq \ell$. If $E_{\alpha,0}$ is irreducible (resp. reducible) so are the E_{α,j_k}, and $\Gamma_{E_{\alpha,0}} E_{\alpha,j_k} \subset E_{\mathbf{0}}$ (resp. $\Gamma_{E''_{\alpha,0}} E''_{\alpha,j_k} \subset E_{\mathbf{0}}$) by Theorem D (resp. Theorem E). Since those subspaces are pairwise orthogonal and never zero due to Corollary 5.9, the result follows. □

9. Continuity of Γ and rigidity

In this section, we collect results from previous sections to prove the continuity of Γ in complete generality (Theorem A). The rigidity theorem (Theorem B) is then a consequence.

Proposition 9.1. *Let $\alpha \in \mathcal{A}$, let i, $j \in \mathbb{Z}$ with $i \neq j$, and let $X_{\alpha,i} \in E_{\alpha,i}$, $Y_{\alpha,j} \in E_{\alpha,j}$. Then*

(i)

$$||\Gamma_{X_{\alpha,i}} Y_{\alpha,j}||^2 \leq ||v_{\alpha,i}||^2 ||X_{\alpha,i}||^2 ||Y_{\alpha,j}||^2 + 3||(\Gamma_{X_{\alpha,i}} Y_{\alpha \mathbf{j}})_{E_{\alpha,\frac{i+j}{2}}}||^2,$$

where the last term is to be omitted in case $i-j$ is odd.

(ii) If $i-j$ is not divisible by 2^k for some integer $k \geq 1$ then

$$||\Gamma_{X_{\alpha,i}} Y_{\alpha,j}|| \leq 2^{k-1} ||v_{\alpha,i}|| ||X_{\alpha,i}|| ||Y_{\alpha,j}||.$$

Proof. We combine Proposition 6.5 together with Theorem 8.6 to write

$$\begin{aligned} \frac{1}{2}||v_{\alpha,i}||^2||X_{\alpha,i}||^2||Y_{\alpha,j}||^2 &= ||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_\mathbf{0}}||^2 + 2||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,2i-j}}||^2 \\ &\quad + \frac{1}{2}||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,2j-i}}||^2 - ||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,\frac{i+j}{2}}}||^2. \end{aligned}$$

Multiplying through by 2 and adding $3||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,\frac{i+j}{2}}}||^2$ to both sides yields (i).

In the proof of (ii) we use induction on k. The case $k=1$ is contained in (i). Now we assume that (ii) holds for some $k \geq 1$ and that $i-j$ is not divisible by 2^{k+1}. Then $i - \frac{i+j}{2}$ is not divisible by 2^k. Therefore, for $Z = (\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,\frac{i+j}{2}}}$, we get

$$\begin{aligned} ||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,\frac{i+j}{2}}}||^2 &= \langle \Gamma_{X_{\alpha,i}} Y_{\alpha,j}, Z \rangle \\ &= |\langle Y_{\alpha,j}, \Gamma_{X_{\alpha,i}} Z \rangle| \\ &\leq ||Y_{\alpha,j}|| \, ||\Gamma_{X_{\alpha,i}} Z|| \\ &\leq 2^{k-1} \, ||Y_{\alpha,j}|| \, ||v_{\alpha,i}|| \, ||X_{\alpha,i}|| \, ||Z||, \end{aligned}$$

by the Cauchy-Schwarz inequality and the induction hypothesis, and thus

$$||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,\frac{i+j}{2}}}|| \leq 2^{k-1} \, ||v_{\alpha,i}|| \, ||X_{\alpha,i}|| \, ||Y_{\alpha,j}||.$$

The inequality in (ii) now follows from (i). □

We finally come to one of our main results.

Theorem 9.2. *For all $\mathbf{i} \in \mathbf{I}^*$ and $X \in E_\mathbf{i}$, Γ_X is continuous. More precisely, there exists a constant C such that $||\Gamma_X Y|| \leq C \, ||v_\mathbf{i}|| \, ||X|| \, ||Y||$ for all $\mathbf{i} \in \mathbf{I}^*$, $X \in E_\mathbf{i}$ and $Y \in T_xM$. In particular, the one-parameter groups F_X^t are smooth curves in the Banach-Lie group of isometries of V.*

Proof. Fix $(\alpha, i) \in \mathcal{A} \times \mathbb{Z}$ and $X_{\alpha,i} \in E_{\alpha,i}$. For the continuity of $\Gamma_{X_{\alpha,i}}$, in view of the discussion preceeding Lemma 6.4, Corollary 8.7 and Proposition 9.1, it is enough to show that

$$||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,m}}|| \leq 2 \, ||v_{\alpha,i}|| \, ||X_{\alpha,i}|| \, ||Y_{\alpha,j}|| \tag{9.3}$$

for all $j \in \mathbb{Z}$ with $j \neq i$ such that $j-i$ is divisible by 4 and all $Y_{\alpha,j} \in E_{\alpha,j}$, where $m = \frac{i+j}{2}$.

If $E_{\alpha,i}$ is irreducible or $E_{\alpha,i}$ is reducible and $X_{\alpha,i} \in E''_{\alpha,i}$, the left hand side of (9.3) is zero (Theorems 8.10 and 8.13(i)). Thus we may assume $E_{\alpha,i}$ reducible and $X_{\alpha,i} \in E'_{\alpha,i}$.

Let $Z = (\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,m}}$. Due to Proposition 8.13(ii), $Z \in E''_{\alpha,m}$. Thus

$$\begin{aligned} ||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,m}}||^2 &= \langle \Gamma_{X_{\alpha,i}} Y_{\alpha,j}, Z\rangle \\ &= |\langle Y_{\alpha,j}, (\Gamma_{X_{\alpha,i}} Z)_{E_{\alpha,j}}\rangle| \\ &\leq ||Y_{\alpha,j}||\, ||(\Gamma_{X_{\alpha,i}} Z)_{E_{\alpha,j}}|| \\ &= ||Y_{\alpha,j}||\, ||(\Gamma_Z X_{\alpha,i})_{E_{\alpha,j}}|| \left| \frac{v_{\alpha,i} - v_{\alpha,j}}{v_{\alpha,m} - v_{\alpha,j}} \right| \\ &\leq ||Y_{\alpha,j}||\, ||v_{\alpha,m}||\, ||Z||\, ||X_{\alpha,i}|| \left| \frac{v_{\alpha,i} - v_{\alpha,j}}{v_{\alpha,m} - v_{\alpha,j}} \right|, \end{aligned}$$

where we have used Cauchy-Schwarz, Codazzi (Proposition 3.15) and Proposition 9.1(i) (note that $(\Gamma_Z X_{\alpha,i})_{E_{\alpha,\frac{m+i}{2}}} = 0$ by Proposition 8.13(i)).

Recall that $v_{\alpha,k} = \frac{a}{b+k} v_0$ for a unit vector v_0 and a, $b \in \mathbb{R}$. This implies that

$$||v_{\alpha,m}|| \left| \frac{v_{\alpha,i} - v_{\alpha,j}}{v_{\alpha,m} - v_{\alpha,j}} \right| = 2||v_{\alpha,i}||$$

and hence

$$||(\Gamma_{X_{\alpha,i}} Y_{\alpha,j})_{E_{\alpha,m}}|| \leq 2||v_{\alpha,i}||\, ||X_{\alpha,i}||\, ||Y_{\alpha,j}||,$$

as we wished.

Our discussion so far shows that there exists a constant C such that

$$||\Gamma_X Y|| \leq C\, ||v_{\mathbf{i}}||\, ||X||\, ||Y||$$

for all $\mathbf{i} \in \mathbf{I}^*$, $X \in E_{\mathbf{i}}$ and $Y \in E_{\mathbf{0}}^{\perp}$. Assume now that $Y \in E_{\mathbf{0}}$. Since $\Gamma_{X_{\mathbf{i}}} E_{\mathbf{0}} \perp E_{\mathbf{0}}$, we can find a sequence $Z_n \in \sum_{\mathbf{j} \in \mathbf{I}^*} E_{\mathbf{j}}$ such that $Z_n \to \Gamma_{X_{\mathbf{i}}} Y$. Then

$$\langle \Gamma_{X_{\mathbf{i}}} Y, Z_n\rangle = -\langle Y, \Gamma_{X_{\mathbf{i}}} Z_n\rangle \leq ||Y||\, ||\Gamma_{X_{\mathbf{i}}} Z_n|| \leq C\, ||v_{\mathbf{i}}||\, ||X_{\mathbf{i}}||\, ||Y||\, ||Z_n||,$$

which yields in the limit as $n \to \infty$ the desired inequality. □

Corollary 9.4. *Γ is continuous as a bilinear mapping, that is, there exists $C > 0$ such that $||\Gamma_X Y|| \leq C\, ||X||\, ||Y||$ for all X, $Y \in T_x M$ with $X \perp E_{\mathbf{0}}$.*

Proof. If $X = \sum_{\mathbf{i} \in \mathbf{I}^*} X_{\mathbf{i}}$ with $X_{\mathbf{i}} \in E_{\mathbf{i}}$ and $Y \in \sum_{\mathbf{i} \in \mathbf{I}} E_{\mathbf{i}}$ then

$$\begin{aligned} ||\Gamma_X Y|| &\leq \sum_{\mathbf{i} \in \mathbf{I}^*} ||\Gamma_{X_{\mathbf{i}}} Y|| \\ &\leq \sum_{\mathbf{i} \in \mathbf{I}^*} C\, ||v_{\mathbf{i}}||\, ||X_{\mathbf{i}}||\, ||Y|| \\ &\leq C\, ||Y|| \left(\sum_{\mathbf{i} \in \mathbf{I}^*} ||v_{\mathbf{i}}||^2 \sum_{\mathbf{i} \in \mathbf{I}^*} ||X_{\mathbf{i}}||^2 \right)^{1/2} \\ &\leq C'\, ||X||\, ||Y||, \end{aligned}$$

where $C' = C \left(\sum_{\mathbf{i} \in \mathbf{I}^*} ||v_{\mathbf{i}}||^2 \right)^{1/2} < +\infty$. □

The continuity of Γ is essential in the proof of the following rigidity theorem.

Theorem 9.5. *For any point $x \in M$, α_x and $(\nabla \alpha)_x$ determine M completely.*

Proof. Let $\tilde{M}$ be a second connected complete full irreducible isoparametric submanifold of V with $x \in M \cap \tilde{M}$, $T_xM = T_p\tilde{M}$, $\alpha_x = \tilde{\alpha}_x$ and $(\nabla\alpha)_x = (\nabla\tilde{\alpha})_x$. Owing to Theorem 4.3, $\Gamma_x = \tilde{\Gamma}_x$.

It follows from $T_xM = T_x\tilde{M}$ and $\alpha_x = \tilde{\alpha}_x$ that M and $\tilde{M}$ have the same normal spaces and the same curvature spheres at x. Moreover, for each $\mathbf{i} \in \mathbf{I}^*$ and $X \in E_{\mathbf{i}}(x)$, the one-parameter groups F_X^t and $\tilde{F}_X^t$ coincide since they have the same infinitesimal generators, defined on the whole of V by Theorem 9.2 (notice that it is at this point that the continuity of Γ_X and $\tilde{\Gamma}_X$ is crucial since otherwise the self-adjoint infinitesimal generators of $(F_X^t)_*$ and $(\tilde{F}_X^t)_*$ might not coincide). Therefore for all $y \in S_{\mathbf{i}}(x)$, we have by equivariance that $y \in M \cap \tilde{M}$, $T_yM = T_y\tilde{M}$, $\alpha_y = \tilde{\alpha}_y$ and $\Gamma_y = \tilde{\Gamma}_y$. Proceeding by induction we now see that M and $\tilde{M}$ coincide along $Q_x = \tilde{Q}_x$. Since Q_x and $\tilde{Q}_x$ are dense in M and $\tilde{M}$, respectively, the desired result follows. □

If M is finite dimensional and homogeneous, the continuity of Γ is of course obvious and the proof of Theorem 9.5 also applies. Even in this case the result seems to be new.

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