Polar actions on compact rank one symmetric spaces are taut

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B R A S I L

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- A polar representation of a compact Lie group in an Euclidean space is orbit equivalent to the isotropy representation of a symmetric space (Dadok, 1985).

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- Recall that L_q is a Morse function if and only if q is not a focal point of M, and the index of a critical point $p \in M$ of L_q is the sum of the multiplicities of the focal points along the geodesic segment $\overline{pq}$ (Morse index thm).

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- If L_q is Morse, the weak Morse inequalities say that $\mu_k(L_q|M^c) \geq \beta_k(M^c; F)$ for all k and $c > 0$; L_q is called F-perfect if these are equalities.

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- E_q is a Morse function iff q is not a focal point of M, γ is a critical point of E_q iff it is a geodesic perpendicular to M at $\gamma(0)$, and the Morse index thm holds.
- E_q is bounded below and satisfies the Palais-Smale condition, so for a Morse function E_q, $\mu_k(E_q|\mathcal{M}^c)$ is finite and the weak Morse inequalities hold; E_q is F-perfect if $\mu_k(E_q|\mathcal{M}^c) = \beta_k(\mathcal{M}^c; F)$ for all k and $c > 0$.

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- **Theorem (Biliotti-G., 2005)** A polar action of a compact Lie group on a compact rank one symmetric space is $\mathbf{Z}_2$-taut.
- There are nonpolar, taut actions on Euclidean space (G.-Thorbergsson, 2000) and on compact rank one symmetric spaces (Biliotti-G., 2005).

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- **Theorem (Biliotti-G., 2005)** A polar action of a compact Lie group on a compact rank one symmetric space is $\mathbf{Z}_2$-taut.
- There are nonpolar, taut actions on Euclidean space (G.-Thorbergsson, 2000) and on compact rank one symmetric spaces (Biliotti-G., 2005).
- A hyperpolar action of a compact Lie group on a complete Riemannian manifold is $\mathbf{Z}_2$-taut (Bott-Samelson and Conlon).

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- CROSS's are S^n, $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$ and $\mathbf{Ca}P^2$. Polar actions on these spaces have been classified by Podestà and Thorbergsson (1999), but we do not need to use their classification in full.

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- We will only use that a polar action on $\mathbf{Ca}P^2 = \mathbf{F}_4/\mathbf{Spin}(9)$ of cohomogeneity greater than one is given by one of the following subgroups of $\mathbf{F}_4$:

$$\mathbf{Spin}(8), \quad S^1 \cdot \mathbf{Spin}(7), \quad \mathbf{SU}(2) \cdot \mathbf{SU}(4), \quad \mathbf{SU}(3) \cdot \mathbf{SU}(3).$$

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- **Proposition** Let $\pi : \hat{X} \to X$ be a Riemannian submersion and a principal G-bundle. Let M be a properly embedded submanifold of X and $\hat{M} = \pi^{-1}(M)$. Then $\hat{M}$ is taut in $\hat{X}$ iff M is taut in X.

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- **Lemma** (Heintze, Liu, Olmos, 2000) Let $\pi : \hat{X} \to X$ be a Riemannian submersion, M a properly embedded submanifold of X and $\hat{M} = \pi^{-1}(M)$. Then a normal vector $\hat{\xi}$ is a multiplicity-m focal direction of $\hat{M}$ iff $\pi_* \hat{\xi}$ is a multiplicity-m focal direction of M.

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The map

$$\Phi : \mathcal{P}(X, M \times q) \times \mathcal{P}(G, G \times 1) \to \mathcal{P}(\hat{X}, \hat{M} \times \hat{q}),$$

given by $\Phi(x, g)(t) = \hat{x}(t)g(t)$, where $\hat{x}(t)$ is the horizontal lift of $x(t)$ with $\hat{x}(1) = \hat{q}$, is a diffeomorphism.

$\mathcal{P}(G, G \times 1)$ is contractible.

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Then there are homeomorphisms

$${}_cX/\bar{N} \to X/G \quad \text{and} \quad \bar{X}/\bar{G} \to X/G.$$

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Lemma Let M be a G-orbit, $q \in \bar{X}$ and $c > 0$. Suppose there exists a subgroup $L \subset H$, finitely iterated $\mathbf{Z}_2$-extension of the identity, such that the connected component of X^L containing q equals $\bar{X}$. Then

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$$\beta(\Omega^L) \leq \beta(\Omega^{H_{n-1}}) \leq \ldots \leq \beta(\Omega^{H_0}).$$

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