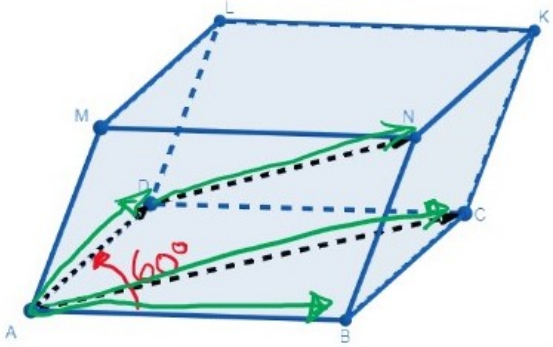
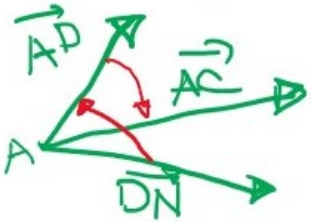
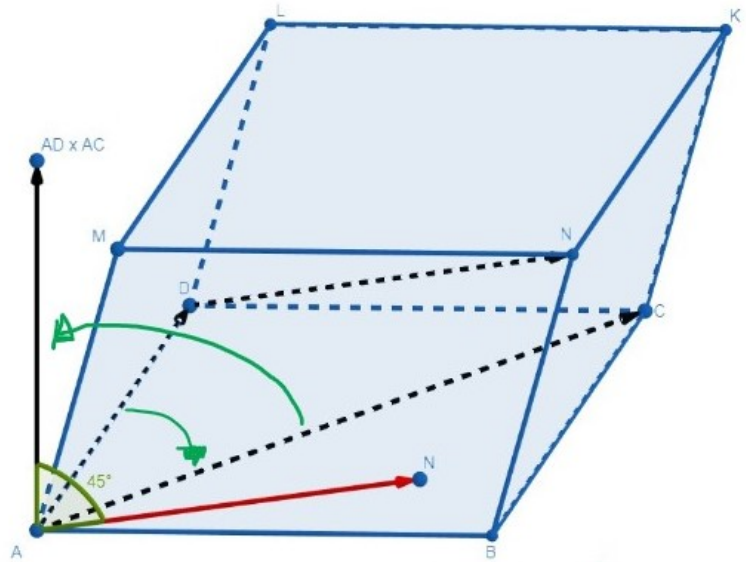
Considere um paralelepípedo qualquer cuja base é paralelogramo ABCD (paralelogramo em cima é MNKL). A orientação do espaço V^3 é definida pela escolha da base $E = \{\overrightarrow{DN}, \overrightarrow{AD}, \overrightarrow{AC}\}$. Assume que $|AD| = 1$, $|AC| = 2$, $|DN| = 2$, a medida angular entre $\overrightarrow{AB}$ e $\overrightarrow{AD}$ é $\frac{\pi}{3}$ e a medida angular entre $\overrightarrow{DN}$ e $\overrightarrow{AD} \wedge \overrightarrow{AC}$ é $\frac{\pi}{4}$.

Base positiva e satisfaz a regra da mão esquerda







$\{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AD} \times \overrightarrow{AC}\}$ é base positiva

$\Rightarrow$ satisfaz a regra da mão esquerda.

1. *(1 ponto)* Quantas bases tem entre seguintes conjuntos:

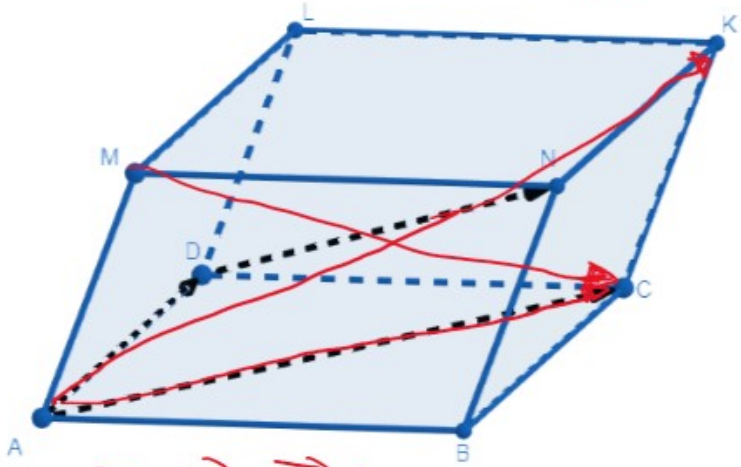
$\{\overrightarrow{AC}, \overrightarrow{MC}, \overrightarrow{AK}\}$, $\{\overrightarrow{DB}, \overrightarrow{AN}, \overrightarrow{DB} \wedge \overrightarrow{DC}\}$, $\{\overrightarrow{BC} \wedge \overrightarrow{BA}, \overrightarrow{ML}, \overrightarrow{NL} \wedge \overrightarrow{NK}\}$?

a) 1; b) 2; c) 3

Base

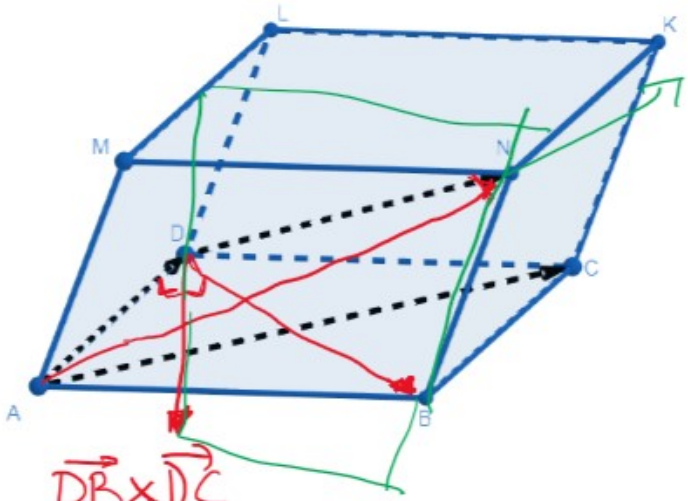
Não é base

Não é base



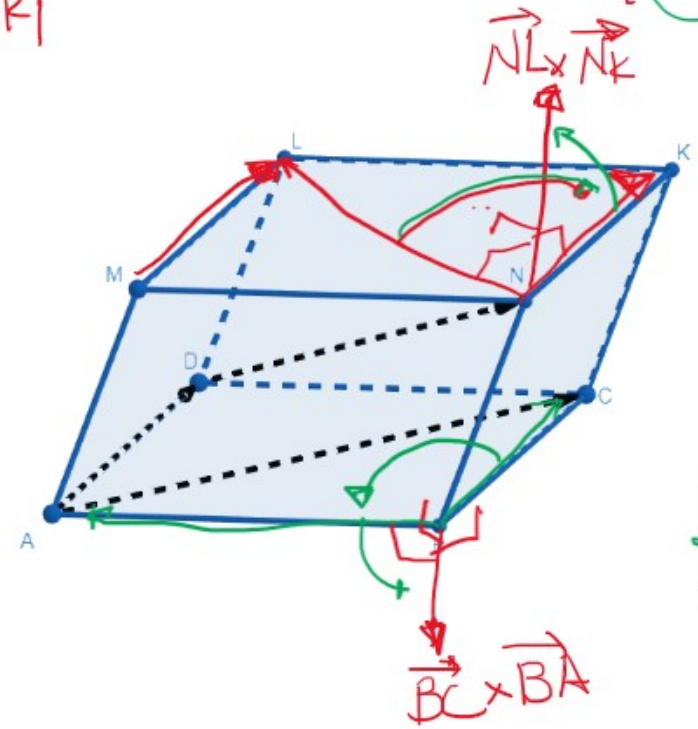
$\{\overrightarrow{AC}, \overrightarrow{MC}, \overrightarrow{AK}\}$

$\overrightarrow{MC} = \alpha \overrightarrow{AC} + \beta \overrightarrow{AK}$



$\overrightarrow{DB} \times \overrightarrow{DC}$

$\{\overrightarrow{DB}, \overrightarrow{AN}, \overrightarrow{DB} \times \overrightarrow{DC}\}$.



$\overrightarrow{BC} \times \overrightarrow{BA}$ e $\overrightarrow{NL} \times \overrightarrow{NK}$ são paralelos.

$\{\overrightarrow{BC}, \overrightarrow{BA}, \overrightarrow{BC} \times \overrightarrow{BA}\}$

Base positiva

(1 ponto) Seja $\vec{x} = \overrightarrow{AB} \wedge \overrightarrow{BD}$. Considere seguintes afirmações

2. *(1 ponto)* Seja $\vec{v} = \vec{AB} \wedge \vec{BD}$. Considere seguintes afirmações

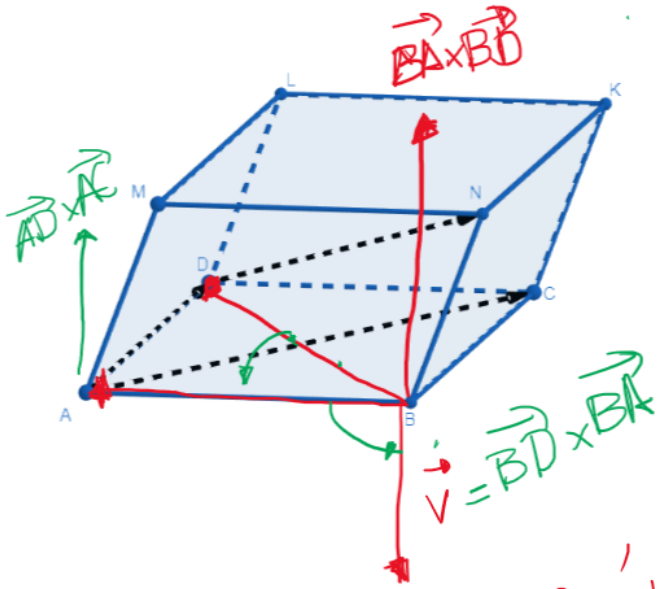
- $\vec{BD}, \vec{AB}, \vec{v}$ é uma base negativa;
- $\vec{v}$ tem a mesma direção com $\vec{AD} \wedge \vec{AC}$;
- $\vec{v}$ tem sentido oposto com $\vec{AD} \wedge \vec{AC}$.

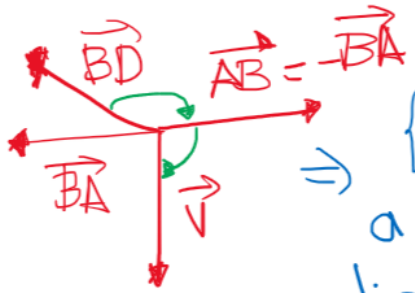
O número de afimações corretas é: a) 0; b) 1; c) 2; d) 3

$\vec{v} = -\vec{BA} \times \vec{BD} = \vec{BD} \times \vec{BA} \Rightarrow \vec{v}$ e $\vec{BA} \times \vec{BD}$ tem sentido oposto.

$\{\vec{BD}, \vec{BA}, \vec{BD} \times \vec{BA}\}$ é positiva.

$\{\vec{AB}, \vec{BD}, \vec{v}\}$ satisfaz a regra da mão esquerda $\Rightarrow$ é positiva





$\Rightarrow \{\vec{BD}, \vec{AB}, \vec{v}\}$ satisfaz a regra da mão direita

$\Rightarrow$ é base negativa.

Por definição $\{\vec{BA}, \vec{BD}, \vec{BA} \times \vec{BD}\}$ é base positiva $\Rightarrow$ satisfaz a regra da mão esquerda.

3. a) *(1 ponto)* Calcule a área do paralelogramo construído nos vetores $\overrightarrow{AB}$ e $\overrightarrow{BD}$;

$$\text{Área} = \|\overrightarrow{AB}\times\overrightarrow{BD}\| = \|\overrightarrow{AB}\times\overrightarrow{AD}\| = \underbrace{\|\overrightarrow{AB}\|}_{=?}\,\underbrace{\|\overrightarrow{AD}\|}_{=1}\,\operatorname{sen}\big(\underbrace{\widehat{\overrightarrow{AB},\overrightarrow{AD}}}_{=60^\circ}\big) = \left(\frac{\sqrt{13}-1}{2}\right)(1)\left(\frac{\sqrt{3}}{2}\right)$$

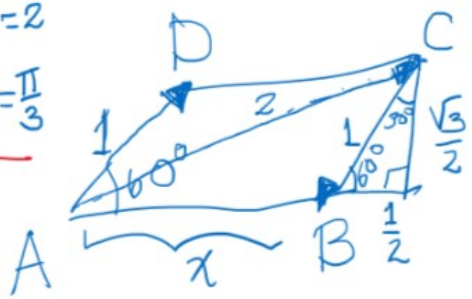
$$= \frac{\sqrt{3}(\sqrt{13}-1)}{4}$$

$$\boxed{\begin{aligned}\overrightarrow{BD} = \overrightarrow{BA}+\overrightarrow{AD} \Rightarrow \overrightarrow{AB}\times\overrightarrow{BD} &= \overrightarrow{AB}\times(\overrightarrow{BA}+\overrightarrow{AD})\\ &= \overrightarrow{AB}\times\overrightarrow{BA} + \overrightarrow{AB}\times\overrightarrow{AD}\\ &= \vec{0} + \overrightarrow{AB}\times\overrightarrow{AD}\\ &= \overrightarrow{AB}\times\overrightarrow{AD}.\end{aligned}}$$

$\|\overrightarrow{AD}\| = 1$

$\|\overrightarrow{AC}\| = 2$

$\widehat{\overrightarrow{AB},\overrightarrow{AD}} = \frac{\pi}{3}$



Por pitágoras:

$$\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 2^2$$

$$\rightarrow x = \frac{-1\pm\sqrt{13}}{2} \Rightarrow \|\overrightarrow{AB}\| = \frac{\sqrt{13}-1}{2}$$

3. b) *(1 ponto)* Calcule o produto escalar de vetores $\overrightarrow{AD} \wedge \overrightarrow{AC}$ e $\overrightarrow{DN}$;

$$\overrightarrow{AD} \times \overrightarrow{AC} = \overrightarrow{AD} \times (\overrightarrow{AD} + \overrightarrow{DC}) = \vec{0} + \overrightarrow{AD} \times \overrightarrow{DC} = \overrightarrow{AD} \times \overrightarrow{AB}$$

$$\Rightarrow \langle \overrightarrow{AD} \times \overrightarrow{AC}, \overrightarrow{DN} \rangle = \langle \overrightarrow{AD} \times \overrightarrow{DC}, \overrightarrow{DN} \rangle = \|\overrightarrow{AD} \times \overrightarrow{AC}\| \, \|\overrightarrow{DN}\| \cos(\widehat{\overrightarrow{AD} \times \overrightarrow{AC}, \overrightarrow{DN}})$$

$$= \|\overrightarrow{AD} \times \overrightarrow{AB}\| \, (2) \cos(45^\circ)$$

$$= \frac{\sqrt{2}(\sqrt{13}-1)}{4} (\cancel{2}) \left(\frac{\sqrt{2}}{\cancel{2}}\right) = \frac{\cancel{2}(\sqrt{13}-1)}{\cancel{4}}$$

$$= \frac{\sqrt{13}-1}{2}$$

3. c) *(1 ponto)* Calcule a projeção ortogonal de vetor $\overrightarrow{DN}$ no $\overrightarrow{AD} \wedge \overrightarrow{AC}$;

$$\text{Proj}_{\overrightarrow{AD}\times\overrightarrow{AC}} \overrightarrow{DN} = \frac{\langle \overrightarrow{AD}\times\overrightarrow{AC}, \overrightarrow{DN}\rangle}{\|\overrightarrow{AD}\times\overrightarrow{AC}\|^2}\left(\overrightarrow{AD}\times\overrightarrow{AC}\right) = \frac{2(\sqrt{13}+1)}{9}\left(\overrightarrow{AD}\times\overrightarrow{AC}\right)$$

$$\frac{\langle \overrightarrow{AD}\times\overrightarrow{AC}, \overrightarrow{DN}\rangle}{\|\overrightarrow{AD}\times\overrightarrow{AC}\|^2} = \frac{\frac{\sqrt{13}-1}{2}}{\left[\frac{\sqrt{3}(\sqrt{13}-1)}{4}\right]^2} = \frac{\cancel{16}^{\,8}\cancel{(\sqrt{13}-1)}}{\cancel{2}_{\,1}(3)(\sqrt{13}-1)^{\cancel{2}}} = \frac{8}{3(\sqrt{13}-1)} = \frac{\cancel{8}^{\,2}(\sqrt{13}+1)}{3(\cancel{12}\,3)}$$

$$= \frac{2(\sqrt{13}+1)}{9}$$

3. d) *(1 ponto)* Usando o paralelepípedo dado dê exemplo de uma base ortonormal

$\{v_1, v_2, v_3\}$

$v_1 = \vec{AD}$, $\|\vec{AD}\| = 1$

$v_2 = \dfrac{u_2}{\|u_2\|}$, $u_2 = \vec{AB} - \text{Proj}_{\vec{AD}}\vec{AB} = \vec{AB} - \dfrac{\langle \vec{AB}, \vec{AD}\rangle}{\|AD\|^2}\vec{AD} = \vec{AB} - \langle \vec{AB}, \vec{AD}\rangle \vec{AD}$. $\|\vec{u_2}\| = ?$

?

$v_3 = \dfrac{u_3}{\|u_3\|}$ $u_3 = \vec{AB} \times \vec{AD}$, $\|u_3\| = \|\vec{AB} \times \vec{AD}\| = \dfrac{\sqrt{3}(\sqrt{13}-1)}{4}$

Item a)

$\langle \vec{AB}, \vec{AD}\rangle = \|\vec{AB}\|\,\|\vec{AD}\| \cos(\widehat{\vec{AB}, \vec{AD}})$

$= \dfrac{\sqrt{13}-1}{2}(1)\cos(60°)$

$= \dfrac{\sqrt{13}-1}{2}\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{13}-1}{4}$

$\|u_2\|^2 = \|\vec{AB}\|^2 - 2\langle \vec{AB}, \langle \vec{AB}, \vec{AD}\rangle \vec{AD}\rangle + \|\vec{AD}\|^2$

$= \|\vec{AB}\|^2 - 2(\langle \vec{AB}, \vec{AD}\rangle)^2 + \|AD\|^2$

$= \left(\dfrac{\sqrt{13}-1}{4}\right)^2 - 2\left(\dfrac{\sqrt{13}-1}{2}\right)^2 + 1$

3. e) *(1 ponto)* Calcule o volume do paralelepípedo dado.

$$\text{Volume} = |\langle \vec{DA} \times \vec{DC}, \vec{DN} \rangle| = |\langle -\vec{AD} \times \vec{AC}, \vec{DN} \rangle| = |\langle \vec{AD} \times \vec{AC}, \vec{DN} \rangle| = \frac{\sqrt{13}-1}{2}$$

$$\vec{DA} \times \vec{DC} = \vec{DA} \times (\vec{DA} + \vec{AC}) = \vec{DA} \times \vec{AC} = -\vec{AD} \times \vec{AC}$$

4. a) *(1 ponto)* Seja $F = \{\overrightarrow{LA}, \overrightarrow{KD}, \overrightarrow{KC}\}$. Verificar se base F é negativa ou positiva;

$E = \{\overrightarrow{DN}, \overrightarrow{AD}, \overrightarrow{AC}\}$

$\overrightarrow{KC} = \overrightarrow{KN} + \overrightarrow{ND} + \overrightarrow{DC} = -\overrightarrow{AD} - \overrightarrow{DN} + \overrightarrow{DA} + \overrightarrow{AC} = -\overrightarrow{DN} - 2\overrightarrow{AD} + \overrightarrow{AC}$

$\overrightarrow{KD} = \overrightarrow{KC} + \overrightarrow{CD} = -\overrightarrow{DN} - 2\overrightarrow{AD} + \cancel{\overrightarrow{AC}} + \cancel{\overrightarrow{CA}} + \overrightarrow{AD} = -\overrightarrow{DN} - \overrightarrow{AD}$

$\overrightarrow{LA} = \overrightarrow{LK} + \overrightarrow{KC} + \overrightarrow{CA} = \overrightarrow{DC} - \overrightarrow{DN} - 2\overrightarrow{AD} + \cancel{\overrightarrow{AC}} + \cancel{\overrightarrow{CA}} = \overrightarrow{DA} + \overrightarrow{AC} - \overrightarrow{DN} - 2\overrightarrow{AD}$

$= -\overrightarrow{DN} - 3\overrightarrow{AD} + \overrightarrow{AC}$

$$\begin{vmatrix} -1 & -1 & -1 \\ -3 & -1 & -2 \\ 1 & 0 & 1 \end{vmatrix} = -1 \Rightarrow F \text{ é negativa.}$$

4. b) *(1 ponto)* Se $\vec{v} = (1, -1, 2)_E$, achar as coordenadas de $\vec{v}$ na base F.

$$\vec{v} = \vec{DN} - \vec{AD} + 2\vec{AC}$$

$$\begin{cases} \vec{LA} = -\vec{DN} - 3\vec{AD} + \vec{AC} \\ \vec{KD} = -\vec{DN} - \vec{AD} \\ \vec{KC} = -\vec{DN} - 2\vec{AD} + \vec{AC} \end{cases}$$

$$\Rightarrow \vec{LA} - \vec{KC} = -\vec{AD} \Rightarrow \boxed{\vec{AD} = -\vec{LA} + \vec{KC}}$$

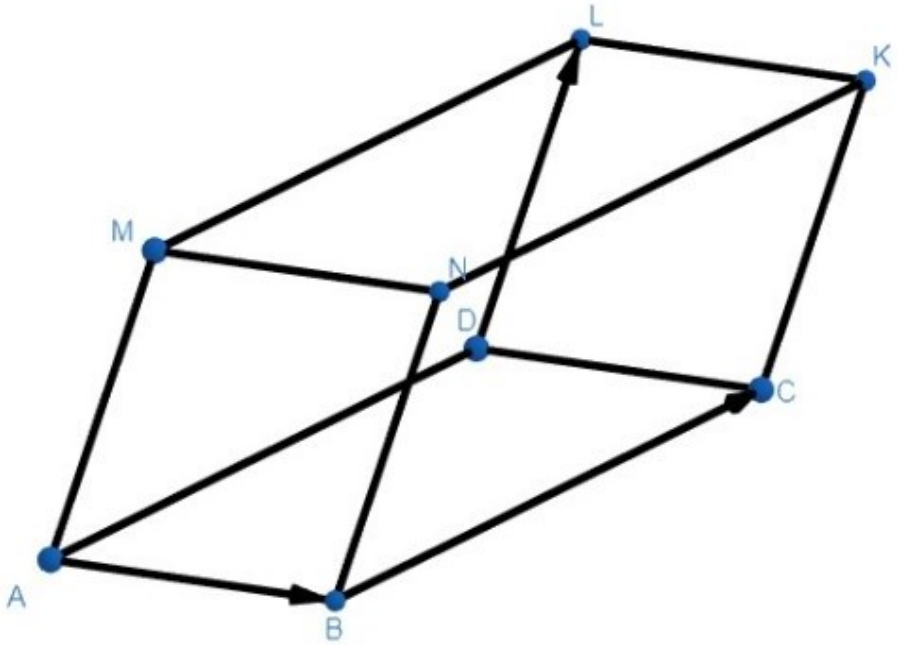
$$\vec{DN} = -\vec{KD} - \vec{AD}$$
$$= -\vec{KD} + \vec{LA} - \vec{KC}$$

$$\vec{AC} = \vec{DN} + 2\vec{AD} + \vec{KC} = -\vec{KD} + \vec{LA} - \cancel{\vec{KC}} - 2\vec{LA} + 2\vec{KC} + \cancel{\vec{KC}}$$
$$= 2\vec{KC} - \vec{LA} - \vec{KD}$$

$$\vec{v} = -\vec{KD} + \cancel{\vec{LA}} - \vec{KC} + \cancel{\vec{LA}} - \vec{KC} + 4\vec{KC} - 2\cancel{\vec{LA}} - 2\vec{KD} = 2\vec{KC} - 3\vec{KD}$$

$$\Rightarrow \vec{v} = (0, -3, 2)_F \qquad F = \{\vec{LA}, \vec{KD}, \vec{KC}\}$$

Considere um paralelepípedo qualquer cuja base é paralelogramo ABCD (paralelogramo em cima é MNKL). A orientação do espaço V^3 é definida pela escolha da base $E = \{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{DL}\}$. Assume que $|AB| = 1$, $|AD| = 2$, $|AL| = 2$, a medida angular entre $\overrightarrow{AB}$ e $\overrightarrow{AD}$ é $\frac{\pi}{3}$ e a medida angular entre $\overrightarrow{AL}$ e $\overrightarrow{AB} \wedge \overrightarrow{AC}$ é $\frac{\pi}{3}$.



1. *(1 ponto)* Quantas bases tem entre seguintes conjuntos:

$\{\overrightarrow{AC}, \overrightarrow{NK}, \overrightarrow{NC}\}$, $\{\overrightarrow{DL}, \overrightarrow{NK}, \overrightarrow{DB} \wedge \overrightarrow{DL}\}$, $\{\overrightarrow{BC} \wedge \overrightarrow{KC}, \overrightarrow{CL}, \overrightarrow{BN} \wedge \overrightarrow{BK}\}$?

a) 1; b) 2; c) 3

$\overrightarrow{BN} \times \overrightarrow{BK} = \overrightarrow{CK} \times (\overrightarrow{BC} + \overrightarrow{CK}) = \overrightarrow{CK} \times \overrightarrow{BC} + \overrightarrow{CK} \times \overrightarrow{CK}$

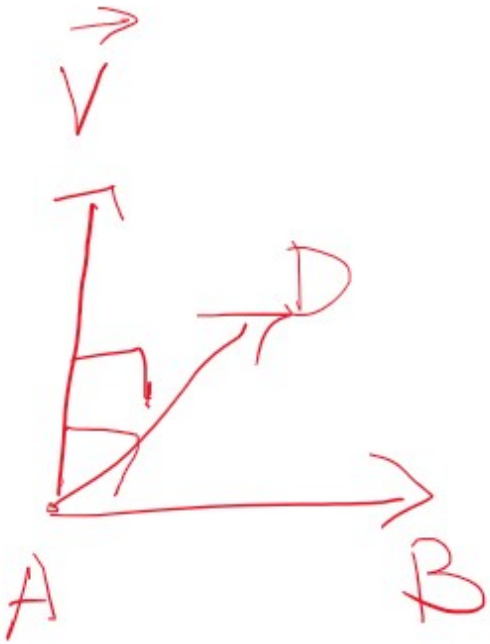
$= -\overrightarrow{KC} \times \overrightarrow{BC} + \vec{0} = \overrightarrow{BC} \times \overrightarrow{KC} + 0\,\overrightarrow{CL}$

2. *(1 ponto)* Seja $\vec{v} = \overrightarrow{BA} \wedge \overrightarrow{DA}$. Considere seguintes afirmações

- $\overrightarrow{AB}, \overrightarrow{AD}, \vec{v}$ é uma base negativa; Falso
- $\vec{v}$ tem a mesma direção com $\overrightarrow{BD} \wedge \overrightarrow{BC}$; Verdadeiro
- $\vec{v}$ tem o mesmo sentido com $\overrightarrow{BD} \wedge \overrightarrow{BC}$. Falso

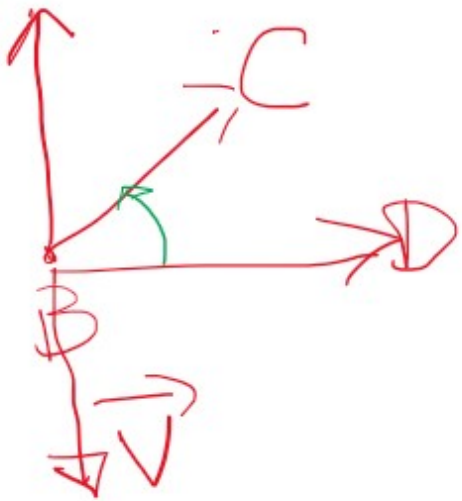
O número de afimações corretas é: a) 0; b) 1; c) 2; d) 3

$$\vec{V} = \overrightarrow{BA} \times \overrightarrow{DA} = -\overrightarrow{AB} \times -\overrightarrow{AD} = \overrightarrow{AB} \times \overrightarrow{AD}$$



$$\begin{cases} \vec{V} = \overrightarrow{BA} \times \overrightarrow{DA} = (\overrightarrow{BD} + \overrightarrow{DA}) \times \overrightarrow{DA} \\ = \overrightarrow{BD} \times \overrightarrow{DA} = \overrightarrow{BD} \times \overrightarrow{CB} \\ = - \overrightarrow{BD} \times \overrightarrow{BC} \end{cases}$$

$\overrightarrow{BD} \times \overrightarrow{BC}$



3. a) *(1 ponto)* Calcule a área do paralelogramo construído nos vetores $\overrightarrow{AB}$ e $\overrightarrow{AC}$;

b) *(1 ponto)* Calcule o produto escalar de vetores $\overrightarrow{AB} \wedge \overrightarrow{AC}$ e $\overrightarrow{AL}$;

c) *(1 ponto)* Calcule a projeção ortogonal de vetor $\overrightarrow{AL}$ no $\overrightarrow{AB} \wedge \overrightarrow{AC}$;

d) *(1 ponto)* Usando o paralelepípedo dado dê exemplo de uma base ortonormal

e) *(1 ponto)* Calcule o volume do paralelepípedo dado.

a) $\overrightarrow{AB} \times \overrightarrow{AC} - \overrightarrow{AB} \times (\overrightarrow{AB} + \overrightarrow{BC})$

$= \overrightarrow{AB} \times \overrightarrow{BC}$

$= \overrightarrow{AB} \times \overrightarrow{AD}$

$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \|\overrightarrow{AB} \times \overrightarrow{AD}\| = \|\overrightarrow{AB}\| \, \|\overrightarrow{AD}\| \operatorname{sen}(\widehat{\overrightarrow{AB}, \overrightarrow{AD}})$

$= (1)(2) \operatorname{sen}(60°) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$

b) $\langle \overrightarrow{AB} \times \overrightarrow{AC}, \overrightarrow{AL} \rangle = \|\overrightarrow{AB} \times \overrightarrow{AC}\| \, \|\overrightarrow{AL}\| \cos(\widehat{\overrightarrow{AB} \times \overrightarrow{AC}, \overrightarrow{AL}})$

$= \sqrt{3}\,(2) \cos(60°)$

$= \sqrt{3}\,(2)\frac{1}{2} = \sqrt{3}$

c) $\text{Proj}_{\vec{AB}\times\vec{AC}} \vec{AL} = \dfrac{\langle \vec{AB}\times\vec{AC}, \vec{AL}\rangle}{\|\vec{AB}\times\vec{AC}\|^2}\left(\vec{AB}\times\vec{AC}\right) = \dfrac{\sqrt{3}}{(\sqrt{3})^2}\left(\vec{AB}\times\vec{AC}\right)$

$= \dfrac{\sqrt{3}}{3}\left(\vec{AB}\times\vec{AC}\right)$

e) $\text{Volume} = \langle \vec{AB}\times\vec{AD}, \vec{AM}\rangle = \langle \vec{AB}\times\vec{AD}, \vec{AL}\rangle - \cancel{\langle \vec{AB}\times\vec{AD}, \vec{AD}\rangle} \to 0$

$= \langle \vec{AB}\times\vec{AD}, \vec{AL}\rangle = \sqrt{3}$

$\vec{AL} = \vec{AM} + \vec{AD} \Rightarrow \vec{AM} = \vec{AL} - \vec{AD}$

Lembre: $\vec{u} \perp \vec{v} \Leftrightarrow \langle \vec{u}, \vec{v}\rangle = 0$

d) $\{v_1, v_2, v_3\}$ base ortonormal

$v_1 = \overrightarrow{AB}$

$v_2 = \dfrac{\vec{u}_2}{\|\vec{u}_2\|}$ $\quad \vec{u}_2 = \overrightarrow{AD} - \mathrm{Proj}_{\overrightarrow{AB}} \overrightarrow{AD} = \overrightarrow{AD} - \dfrac{\langle \overrightarrow{AD}, \overrightarrow{AB} \rangle}{\|AB\|^2} \overrightarrow{AB} = \overrightarrow{AD} - \overrightarrow{AB} \Rightarrow$

$|AB| = 1$

$\langle \overrightarrow{AD}, \overrightarrow{AB} \rangle = |\overrightarrow{AD}||\overrightarrow{AB}| \cos(\widehat{\overrightarrow{AD}, \overrightarrow{AB}})$

$= (2)(1)\cos(60^\circ) = (2)\left(\frac{1}{2}\right) = 1.$

$\vec{u}_2 = \overrightarrow{AD} - \overrightarrow{AB}$

$\|\overrightarrow{AD} - \overrightarrow{AB}\|^2 = \|AD\|^2 - 2\langle \overrightarrow{AD}, \overrightarrow{AB} \rangle + \|AB\|^2$

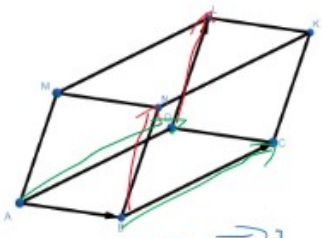
$= 4 - 2(1) + 1 = 3 \Rightarrow \boxed{\|\vec{u}_2\| = \sqrt{3}}$

$v_2 = \dfrac{u_2}{\|u_2\|} = \dfrac{u_2}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} u_2$

$v_3 = \dfrac{\overrightarrow{AB} \times \overrightarrow{AD}}{\|\overrightarrow{AB} \times \overrightarrow{AD}\|} = \dfrac{1}{\sqrt{3}} \overrightarrow{AB} \times \overrightarrow{AD}$

a) *(1 ponto)* Seja $F = \{\overrightarrow{AN}, \overrightarrow{BN}, \overrightarrow{BD}\}$. Verificar se base F é negativa ou positiva;

b) *(1 ponto)* Se $\vec{v} = (1, -1, 2)_E$, achar as coordenadas de $\vec{v}$ na base F.



$E = \{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{DL}\}$

$\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN} = \overrightarrow{AB} + \overrightarrow{DL} = (1, 0, 1)$

$\overrightarrow{BN} = \overrightarrow{DL} \longrightarrow = (0, 0, 1)$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\overrightarrow{AB} + \overrightarrow{BC} \longrightarrow (-1, 1, 0)$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1$$

$\vec{v} = (1, -1, 2)_E = \overrightarrow{AB} - \overrightarrow{BC} + 2\overrightarrow{DL}$

$\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{DL}$

$\overrightarrow{BN} = \overrightarrow{DL}$

$\overrightarrow{BD} = -\overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{DL} = \overrightarrow{BN} \Rightarrow \overrightarrow{AB} = \overrightarrow{AN} - \overrightarrow{DL}$

$\overrightarrow{AB} = \overrightarrow{AN} - \overrightarrow{BN}$

$\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AB} = \overrightarrow{BD} + \overrightarrow{AN} - \overrightarrow{BN}$

$\vec{v} = \cancel{\overrightarrow{AN}} - \cancel{\overrightarrow{BN}} - \overrightarrow{BD} - \cancel{\overrightarrow{AN}} + \cancel{\overrightarrow{BN}} + 2\overrightarrow{BN} = 2\overrightarrow{BN} - \overrightarrow{BD} = (0, 2, -1)_F$

$F = \{\overrightarrow{AN}, \overrightarrow{BN}, \overrightarrow{BD}\}$