Circuit Covers in Series-Parallel Mixed Graphs

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Abstract. A mixed graph is a graph that contains both edges and arcs. Given a nonnegative integer weight function p on the edges and arcs of a mixed graph M, we wish to decide whether (M, p) has a *circuit cover*, that is, if there is a list of circuits in M such that every edge (arc) e is contained in exactly $p(e)$ circuits in the list. When M is a directed graph or an undirected graph with no Petersen graph as a minor, good necessary and sufficient conditions are known for the existence of a circuit cover. For general mixed graphs this problem is known to be NP-complete. We provide necessary and sufficient conditions for the existence of a circuit cover of (M, p) when M is a *series-parallel* mixed graph, that is, the underlying graph of M does not have K_4 as a minor. We also describe a polynomial-time algorithm to find such a circuit cover, when it exists. Further, we show that p can be written as a nonnegative integer linear combination of at most m incidence vectors of circuits of M, where m is the number of edges and arcs. We also present a polynomial-time algorithm to find a minimum circuit in a series-parallel mixed graph with arbitrary weights. Other results on the fractional circuit cover and the circuit double cover problem are discussed.

1 Introduction

A **mixed graph** $M = (V, E, A)$ is a graph that contains both (undirected) edges and arcs. We denote by (M, p) a mixed graph M together with a weight function $p : E \cup A \to \mathbb{Q}_+$. We say that (M, p) has a **circuit cover** (or M has a circuit p-cover) if there is a vector of nonnegative integer coefficients $(\lambda_C : C \in \mathcal{C})$, where $\mathcal{C} := \mathcal{C}(M)$ denotes the set of circuits of M, such that $p = \sum_{C\in\mathcal{C}} \lambda_C \chi^C$. Here, for any subgraph H of M, χ^H denotes the {0,1}-incidence vector of the {edge,arc}-set of H. We also say that (M, p) has a **fractional circuit cover** if there is a vector of nonnegative rational coefficients $(\lambda_C : C \in \mathcal{C})$ such that $p = \sum_{C\in\mathcal{C}} \lambda_C \chi^C$.

We are interested in the following problems. Given (M, p), where p is a nonnegative *integer* (respectively, *rational*) weight function, decide whether (M, p) has a *circuit cover* (respectively, *fractional circuit cover*).

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These problems have been largely investigated when M is an undirected graph. In this case, the fractional version was solved by Seymour [16]. The integral version has attracted great interest as it is related to problems involving minimum circuit covers, the Chinese Postman Problem, perfect matchings and others (see [1] for further references). For directed graphs, these problems are easy and they are special cases of a more general problem solved by Hoffman [12].

For general mixed graphs, both problems are known to be NP-complete [3, 4]. In this note we shall be concerned with these problems and related ones in the special case M is a series-parallel graph.

This note is organized as follows. In Sect. 2 we establish the notation we shall use. In Sect. 3, we show a good characterization for the existence of a (fractional) circuit cover of (M, p) when M is series-parallel.

Section 4 is devoted to the study of Hilbert bases of circuits in series-parallel mixed graphs. There, we also describe a polynomial-time algorithm (relying on the ellipsoid method) that finds a circuit cover of (M, p), if it exists, when M is series-parallel.

In Sect. 5, we discuss an interesting special case of the circuit cover problem: the case where p is an integral constant vector. If M is a directed graph, the problem is easy as this is equivalent to characterizing eulerian directed graphs (for any constant vector p). For undirected graphs, there is a celebrated open problem. When $p = \mathbf{1}$ (or any odd number) the problem is easy: a graph has a circuit p-cover if, and only if, it is eulerian. When $p = \mathbf{2}$ we have the well-known Circuit Double Cover Conjecture. The cases $p = \mathbf{4}$ and $p = \mathbf{6}$ were solved by Jaeger [5, 13] and Fan [9]. These two results together settle the case where $p = \mathbf{2k}$ for any $k \geq 2$. For mixed graphs, the problem is settled only when p is odd [7, 10]. Using the result of Sect. 3, we give a solution for the case $p = \mathbf{2}$ (and hence for any even number) when M is series-parallel.

Finally, in Sect. 6 we describe a polynomial-time algorithm to find a minimum circuit in a weighted series-parallel mixed graph.

2 Basic terminology

Most of the concepts defined for undirected and directed graphs (see [6]) can be extended in a natural way to mixed graphs. We assume the reader is familiar with them. A **mixed graph** is a triplet $M = (V, E, A)$ where V is a finite set of vertices, E is a finite set of edges and A is a finite set of arcs. When $E = \emptyset$ we say that M is a **directed graph**, and when $A = \emptyset$ we say that M is an **undirected graph**.

We use the capital letters D, G and M for directed, undirected and mixed graphs, respectively. Sometimes, it is useful to see M as the union of a directed graph and an undirected graph. We indicate this by $M = D \sqcup G$. In a context referring to a mixed graph M, we may denote by $V(M), E(M)$ and $A(M)$ the set of vertices, edges and arcs of M, respectively.

For a directed graph D, we denote by $\delta_D^+(X)$ the set of arcs with tail in X and head in $V - X$, and we write $\delta_D^-(X) := \delta_D^+(V - X)$. For an undirected graph G, we denote by $\delta_G(X)$ the set of edges with an endnode in X and the other in $V-X$. For a mixed graph $M = D \sqcup G$, we write $\nabla_M(X) := \delta_G(X) \cup \delta^-_D(X) \cup \delta^+_D(X)$. We often omit the subscript if this does not cause any confusion; for example, $\nabla(X) := \nabla_M(X)$. We may also denote a unitary set $\{e\}$ simply by e.

A nonempty subset $F \subseteq E(M) \cup A(M)$ is a **cut** if there is a subset $S \subset V(M)$ such that $F = \nabla(S)$. A **bridge** is a cut of cardinality one. A mixed graph with no bridges is called **bridgeless**.

A **path** is a sequence $(v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1})$, where $v_1, \ldots, v_{k+1}$ are pairwise distinct vertices, and for each i $(1 \leq i \leq k)$, either e_i is an edge with endnodes v_i, v_{i+1} or e_i is an arc with tail v_i and head v_{i+1}. A **circuit** is a sequence with the same properties, except that $v_1 = v_{k+1}$. A mixed graph is **strongly connected** if for each pair u, v of vertices there is a path from u to v. A mixed graph is **eulerian** if its {edge,arc}-set can be partitioned into circuits. Two arcs are **anti-parallel** if they form a circuit of length two.

If $M = (V, E, A)$ is a mixed graph and $e \in E \cup A$ then $M - e$ (respectively, M/e) denotes the graph obtained from M by deleting (respectively, contracting) e. A graph H is a **minor** of a graph G (or G has an H-minor) if a graph isomorphic to H can be obtained from a subgraph of G by a sequence of edge contractions. If H is a cubic graph then H is a minor of G if and only if some subgraph of G is homeomorphic to H. We say that M is **series-parallel** if the underlying graph of M does not have a K_4-minor.

We recall that a mixed graph $M = (V, E, A)$ together with a weight function $p : E \cup A \to \mathbb{Q}_+$ is denoted by (M, p). We use the convention that $p(F)$ means $\sum_{e \in F} p(e)$, for any $F \subseteq E \cup A$.

3 Circuit covers in mixed graphs

Let us recall the two problems of our interest:

Circuit Cover Problem: Given a nonnegative integer weighted mixed graph (M, p), decide whether (M, p) has a circuit cover.

Fractional Circuit Cover Problem: Given a nonnegative rational weighted mixed graph (M, p), decide whether (M, p) has a fractional circuit cover.

For directed graphs both problems are equivalent in the sense that if (D, p) (with p integral) has a fractional circuit cover then it also has a circuit cover. This is an easy particular case of a more general result, due to Hoffman [12], related to circulations in directed graphs. A vector $p : A(D) \to \mathbb{Q}_+$ is called a **circulation** in D if $p(\delta^-(X)) = p(\delta^+(X))$ for every $X \subseteq V(D)$. Clearly, p is a circulation in D if and only if (D, p) has a fractional circuit cover. If, in addition, p is integral, then (D, p) has also a circuit cover.

Theorem 3.1 (Hoffman, 1960). *Let D be a directed graph with capacities $l, u : A(D) \to \mathbb{Q}_+$ such that $l \leq u$. Then there is a circulation p in D such that $l \leq p \leq u$ if and only if $l(\delta^-(X)) \leq u(\delta^+(X))$ for all $X \subseteq V(D)$. Furthermore, we can choose p integral if l, u are integral.* $\square$

For undirected graphs, the fractional circuit cover problem was solved by Seymour [16].

Theorem 3.2 (Seymour, 1979). *Let G be a graph and $p : E(G) \to \mathbb{Q}_+$. Then (G,p) has a fractional circuit cover if and only if $p(\delta(X) - e) \geq p(e)$ for all $X \subseteq V(G)$, $e \in \delta(X)$.* □

In the undirected case the fractional version is quite different from the integral version. Clearly, the additional condition "$p(\delta(X))$ is even for every $X \subseteq V(G)$" is necessary for the existence of a circuit cover of (G,p). But it is not always sufficient as shows the following well-known counterexample: let $G = P_{10}$ be the Petersen graph and F a perfect matching of P_{10}; take $p_{10}(e) = 2$ for $e \in F$, and $p_{10}(e) = 1$, otherwise. Seymour [16] proved that this additional parity condition is sufficient for the existence of a circuit cover if G is planar. Alspach, Goddyn and Zhang [1] extended his result showing that, in a certain sense, (P_{10}, p_{10}) is a minimal counterexample.

Theorem 3.3 (Alspach, Goddyn and Zhang, 1994). *Let G be a graph with no P_{10}-minor and $p : E(G) \to \mathbb{Z}_+$. Then (G,p) has a circuit cover if and only if $p(\delta(X) - e) \geq p(e)$ for all $X \subseteq V(G)$, $e \in \delta(X)$, and $p(\delta(X))$ is even for all $X \subseteq V(G)$.* □

We turn now to the study of the (fractional) circuit cover problem in mixed graphs. In this case, as the following result due to Arkin and Papadimitriou [3, 4] shows, these problems are much harder than in the undirected or the directed case.

Theorem 3.4 (Arkin and Papadimitriou, 1986). *For mixed graphs M, the fractional circuit cover problem and the circuit cover problem are NP-complete, even if $p(e) = 2$ for $e \in E(M)$ and $p(a) = 1$ for $a \in A(M)$.* □

In view of this result, it is unlikely that we can find nice necessary and sufficient conditions for the existence of a circuit p-cover in an arbitrary mixed graph. We show that for series-parallel mixed graphs such a nice characterization exists.

Given a mixed graph $M = (V, E, A)$, the following conditions are clearly necessary for the existence of a fractional circuit cover of (M, p):

(a) For every $X \subseteq V$, $f_p(X) := p(\delta(X)) - |p(\delta^-(X)) - p(\delta^+(X))| \geq 0$;

(b) For every cut $\nabla(X)$ and $e \in \delta(X)$, $p(\nabla(X) - e) \geq p(e)$.

Further, if (M,p) is to have a circuit cover then it is necessary that p is nonnegative integral and the following holds:

(c) For every cut $\nabla(X)$, $p(\nabla(X))$ is even.

We say that (M,p) is **balanced** if it satisfies (b); **eulerian** if it satisfies conditions (a) and (c); and **fractionally admissible** if it satisfies (a) and (b). If p is integral, we say that (M,p) is **admissible** if it is eulerian and balanced (or, it is fractionally admissible and satisfies (c)). Sometimes, we prefer to say simply that p is balanced, eulerian, fractionally admissible or admissible (with respect to M).

The following simple example shows that (fractional) admissibility is not always a sufficient condition for the existence of a (fractional) circuit cover. Take the mixed graph MK_4 whose underlying graph is isomorphic to K_4: the directed part consists of a directed circuit D of length 4 and the undirected part consists of a 1-factor F; define $p_4(a) = 1$ for $a \in D$ and $p_4(e) = 2$ for $e \in F$. It is easy to see that (MK_4, p_4) is (fractionally) admissible and does not have a (fractional) circuit cover.

In the undirected case, the Petersen graph shows that natural conditions for the existence of a circuit cover do not always suffice. However, by Theorem 3.3, if it is excluded, these conditions become sufficient. We follow a similar approach. The main result of this section shows that the circuit cover problem has a solution when M is series-parallel. More precisely, we show that *if M is series-parallel then (M,p) has a circuit cover if and only if p is admissible*. Note that, due to the admissibility of (MK_4, p_4), this gives a complete characterization of mixed graphs for which the admissibility of (M,p) is equivalent to the existence of a circuit cover of (M,p).

To prove the main result we use the following two lemmas. The proofs are left to the reader.

Lemma 3.5. *Let G be a series-parallel undirected graph, B a minimal cut of G and C a circuit of G. Then $|B \cap C| \leq 2$.* $\square$

Lemma 3.6. *Let (M,p) be an admissible pair. Consider a subset $X \subset V(M)$ and $e = uv \in \delta(X)$ such that $p(e) > 0$, $f_p(X) = 0$, $p(\delta^-(X)) - p(\delta^+(X)) > 0$, and $u \in X$. Let M' be the graph obtained from M by replacing the edge e by an arc (u,v); and let p' be the weight function on M', obtained from p by setting $p'(uv) := p(e)$. Then, the resulting pair (M',p') is admissible.* $\square$

The proof of the main theorem is inspired by the proof given by Seymour [16] in the special case of undirected planar graphs, and the proof of Theorem 3.3 of Alspach *et al.*

Theorem 3.7. *If M is series-parallel then (M,p) has a circuit cover if and only if p is admissible.*

Proof. (Outline) Let M be a series-parallel mixed graph. Clearly, if (M,p) has a circuit cover then p is admissible. Now suppose p is admissible and let us prove that (M,p) has a circuit cover. We can assume that $p(e) > 0$ for every element $e \in E \cup A$ since otherwise we could delete e from M. Applying Lemma 3.6 we can also assume that if $f_p(X) = 0$ then $\delta(X) = \emptyset$.

The proof is by induction on $p(E)$. If $p(E) = 0$ then M is a directed graph and p is a circulation in M. In this case, the result follows from Theorem 3.1. Suppose $p(E) > 0$. If there is a cut $\nabla(X)$ such that $|\nabla(X)| = |\delta(X)| = 2$, then we can contract one of the edges of this cut and apply the induction hypothesis. Thus, we can assume there are no such cuts.

If there is an edge e_0 with $p(e_0) = 1$ then we can assign an arbitrary orientation to e_0. Since $f_p(X) \geq 2$ for every $\emptyset \neq X \subset V$ such that $e_0 \in \delta(X)$, it follows that the resulting pair (M', p') is admissible. Using the induction hypothesis the result follows.

Let $e_0 = xy$ be an edge such that $p(e_0)$ is maximum. We can assume that $p(e_0) \geq 2$. Let $p' := p - 2\chi^{e_0}$. We claim that (M, p') is admissible (the proof is left to the reader).

Since (M, p') is admissible and $p'(E) = p(E) - 2$, by the induction hypothesis (M, p') has a circuit cover. Then there is a list L' (with possible repetitions) of circuits such that $p' = \sum_{C \in L'} \chi^C$. Our aim is to consider appropriate circuits in L' that do not contain e_0 and partition them into two paths, so that each path can be extended to a circuit containing the edge e_0.

For that, define an auxiliary undirected graph H as follows. Take $V(H) := V$ and for each circuit C in L' that do not contain e_0 construct in H a circuit C that is the underlying circuit of C. Label the edges of this circuit with "C". We claim there is a path from x to y in H. In fact, it is not difficult to prove, using Lemma 3.5, that if this does not hold we have a contradiction to the admissibility of (M, p).

Take a shortest path from x to y in H. For each section of this path corresponding to edges with the same label, take only one representative, and consider the sequence of such labels, say $(C_1, \ldots, C_k)$. Clearly, there are no repeated circuits and $V(C_i) \cap V(C_j) \neq \emptyset$ if and only if $|i - j| = 1$. We claim that $|V(C_i) \cap V(C_{i+1})| = 1$ for $i = 1, \ldots, k-1$. To prove this, one can show that if this does not happen then the graph induced by the edge e_0 together with the edges in these circuits contains a subgraph homeomorphic to K_4, a contradiction.

Considering how the circuits C_i $(1 \leq i \leq k)$ intersect each other, it is easy to see that $\bigcup C_i$ can be partitioned into a path from x to y and a path from y to x, say, P' and Q'. Let P (respectively, Q) be the circuit obtained from P' (respectively, Q') by adding the edge e_0. Now let $L := L' - \{C_1, \ldots, C_k\} \cup \{P, Q\}$. By construction, L contains only circuits of M, and furthermore $p = \sum_{C \in L} \chi^C$. This shows that (M, p) has a circuit cover. □

Corollary 3.8. *If M is series-parallel then (M, p) has a fractional circuit cover if and only if p is fractionally admissible.* □

To conclude this section we note that (the conditions (a), (b) and (c) of) admissibility is checkable in polynomial time. The reader can verify this. We only mention that checking condition (a) reduces to checking whether an appropriate directed graph D with capacities l, u satisfies Hoffman's condition in Theorem 3.1. This can be checked by a single max-flow computation (see [2], Sec. 6.7).

4 Hilbert bases and algorithmic aspects

Let H be a finite subset of $\mathbb{Q}^n$. Consider the following sets generated by H.

$$Cone(H) = \{\sum_{h \in H} \lambda_h h : \lambda_h \in \mathbb{Q}_+\},$$
$$Lat(H) = \{\sum_{h \in H} \lambda_h h : \lambda_h \in \mathbb{Z}\},$$
$$IntCone(H) = \{\sum_{h \in H} \lambda_h h : \lambda_h \in \mathbb{Z}_+\}.$$

They are called the **cone**, **lattice** and **integer cone** of H, respectively. Clearly, $IntCone(H) \subseteq Cone(H) \cap Lat(H)$. We say that a set H is a **Hilbert basis** if $IntCone(H) = Cone(H) \cap Lat(H)$ (see [14, 15] for a more detailed account of this subject). Here we are interested in the case where H is the set of (incidence vectors of) circuits of a mixed graph M. We denote the above sets by $Cone(\mathcal{C}), Lat(\mathcal{C})$ and $IntCone(\mathcal{C})$.

Using this terminology, the fractional circuit cover problem is equivalent to asking whether a given vector p belongs to the cone of the circuits of M; and the circuit cover problem asks whether p belongs to the integer cone. Corollary 3.8 says that if $M = (V, E, A)$ is series-parallel then $p \in Cone(\mathcal{C})$ if and only if p is fractionally admissible.

Furthermore, since for every $p \in Cone(\mathcal{C}) \cap Lat(\mathcal{C})$, the vector p must be admissible, it follows from Theorem 3.7 that the circuits of a series-parallel mixed graph M forms a Hilbert basis ($p \in IntCone(\mathcal{C})$ if and only if p is admissible). In view of Theorem 3.4, it is unlikely that we can find a complete characterization of $Cone(\mathcal{C})$ and $IntCone(\mathcal{C})$ for arbitrary mixed graphs.

A nice result in the theory of Hilbert bases is the following theorem of Sebő [15].

Theorem 4.1 (Sebő, 1990). *Let $H \subset \mathbb{Q}^n$ be a Hilbert basis. Then every vector of $IntCone(H)$ can be written as a nonnegative integer linear combination of at most $2n - 2$ vectors of H.* □

It has been conjectured that $2n-2$ can be replaced by n in the above theorem (note that this bound would be sharp). For Hilbert bases related to some well-known combinatorial problems this is true [15]. So is the case for Hilbert bases of circuits in a series-parallel mixed graph, as we show in the next theorem.

Theorem 4.2. *Let $M = (V, E, A)$ be a series-parallel mixed graph and $\mathcal{C}$ the set of circuits of M. Then every vector p of $IntCone(\mathcal{C})$ can be written as a nonnegative integer linear combination of at most $m = |E \cup A|$ circuits.* □

The proof of the theorem is by induction on the dimension of a minimal face of $Cone(\mathcal{C})$ that contains p. We leave it to the reader.

We turn now to the algorithmic aspect of the problem of finding a circuit cover of an admissible pair (M, p), where M is series-parallel. Unfortunately, our proof of Theorem 4.2 does not give directly a polynomial-time algorithm to find such a circuit cover. However, if we do not require that the circuit cover uses at most m circuits (but a polynomially bounded number of them) then we can design a polynomial-time algorithm that relies on the ellipsoid method. For that, we need a polynomial separation algorithm that is interesting in its own right: finding a minimum circuit in a series-parallel mixed graph.

We describe two algorithms to find a circuit cover. The first algorithm is based on the proof of Theorem 3.7 but it has pseudo-polynomial running time. The second algorithm is an elegant polynomial procedure that uses the first one and is based on the algorithm presented in [1].

CirCov1 Algorithm

Input: An admissible pair (M, p), where $M = (V, E, A)$ is series-parallel.
Output: A circuit cover L' of (M, p).

1. Delete arcs and edges of weight 0. Contract any edge that is in a cut $\nabla(X)$ such that $|\nabla(X)| = |\delta(X)| = 2$.
2. If there is an edge e in a cut $\nabla(X)$ such that $f_p(X) = 0$ then assign an orientation to e according to Lemma 3.6.
3. If $p(E) = 0$ then return a circuit cover L' (for directed graphs this is trivial) and halt.
4. If there is an edge with weight 1 then assign an arbitrary orientation to it. Call CirCov1 recursively to find a circuit cover L' of the new graph, return L' and halt.
5. Let $e_0 = xy$ be an edge with maximum weight.
6. Call CirCov1 recursively to find a circuit cover L' of $(M, p - 2\chi^{e_0})$.
7. As in the proof of Theorem 3.7, find a shortest (x, y)-path in the auxiliary undirected graph H. Let $\{C_1, \ldots, C_k\}$ be the arc labels along this path.
8. Decompose $\bigcup C_i$ into an (x, y)-path P' and a (y, x)-path Q'.
   Let $P := P'(y, e_0, x)$ and $Q := Q'(x, e_0, y)$.
   Return the circuit cover $L := L - \{C_1, \ldots, C_k\} \cup \{P, Q\}$ and halt.

Steps 1 and 2 require $O(|E|)$ max-flow min-cut computations. Step 3 can be done in $O(|A|.|V|)$ time. The total number of calls of CirCov1 is bounded by $p(E)/2$ as the total weight of the edges in each successive pair (M, p) is reduced by 2. So CirCov1 is a pseudo-polynomial algorithm.

We discuss now how to obtain a polynomial algorithm from CirCov1. The idea is to formulate the circuit cover problem for (M, p) as an integer program and to solve its relaxation. Then we separate out the fractional part of the resulting solution to define a new weight p' (with relatively small entries), and use CirCov1 to solve the circuit cover problem for (M, p'). A circuit cover of (M, p) is obtained by adjoining the partial circuit cover corresponding to the integral part of the linear program solution and the circuit cover found by CirCov1.

In what follows, N denotes the circuit-{edge,arc} incidence matrix of M and $\mathbf{1}$ denotes the vector of $|\mathcal{C}|$ ones.

CirCov2 Algorithm

Input: An admissible pair (M, p), where $M = (V, E, A)$ is series-parallel.

Output: A circuit cover (L, μ) of (M, p), where L is a list containing at most $2|E \cup A| - 1$ circuits and μ is a multiplicity vector whose entries are bounded by $r := \max\{p(e) : e \in E \cup A\}$.

1. Find a basic feasible solution $\lambda = (\lambda_C)_{C \in \mathcal{C}}$ to the following linear program:
$$\max\{\lambda \mathbf{1} : \lambda N = p,\ \lambda \geq 0\}. \tag{1}$$
2. Let $\lfloor \lambda \rfloor := (\lfloor \lambda_C \rfloor)_{C \in \mathcal{C}}$ and $\{\lambda\} := \lambda - \lfloor \lambda \rfloor$ be the integral and the fractional parts of λ, and let $p' := \{\lambda\}N = p - \lfloor \lambda \rfloor N$. (Note that, since p' is a non-negative combination of circuits, (M, p') is fractionally admissible. As p and $\lfloor \lambda \rfloor N$ are eulerian, then p' is also eulerian. Thus (M, p') is admissible.)
3. Call CirCov1 with input (M, p') to obtain a circuit cover L' of (M, p').
4. Adjoin L' to the circuit cover $(S, \lfloor \lambda \rfloor)$, where $S := \{C \in \mathcal{C} : \lfloor \lambda_C \rfloor > 0\}$, and return the resulting circuit cover (L, μ). Halt.

Let us show that $|L|$ is polynomially bounded. As λ is a basic solution, we have $|S| \leq |E \cup A|$. Furthermore, $|L'| + \lfloor \lambda \rfloor \mathbf{1} \leq \lambda \mathbf{1} = \lfloor \lambda \rfloor \mathbf{1} + \{\lambda\}\mathbf{1}$, and so $|L'| \leq \{\lambda\}\mathbf{1}$. Since each nonzero entry in $\{\lambda\}$ is less than 1 we have $\{\lambda\}\mathbf{1} < |E \cup A|$ and therefore $|L'| \leq |E \cup A| - 1$. Thus $|L| \leq |S| + |L'| \leq 2|E \cup A| - 1$. As $\max\{p'(e) : e \in E \cup A\} \leq |L'| < |E \cup A|$, we conclude that Step 3 can be done in polynomial time. It remains to show how to solve Step 1 in time bounded by a polynomial in $|E \cup A| \log(r)$, despite the exponential number of variables λ_C. For that, consider the dual linear program of (1):

$$\min\{px : Nx \geq \mathbf{1}\}. \tag{2}$$

The *separation problem* for (2) is the following: *Given a rational vector x, either certify that x satisfies $Nx \geq \mathbf{1}$, or find a violated inequality (a circuit in (M, x) having weight less than 1).*

A theorem of Grötschel, Lovász and Schrijver (see [11]) implies that a basic optimal solution of (1) can be found via the ellipsoid method in time polynomially bounded by $|E \cup A|$ and the input length of p, provided that we can solve the separation problem for (2) in time polynomially bounded by $|E \cup A|$ and the input length of x. For that, we can use a polynomial-time algorithm that finds a minimum circuit in the weighted series-parallel mixed graph (M, x). In the last section we describe such an algorithm.

It would be interesting to find a combinatorial (polynomial) algorithm for this case. It is not difficult to design such an algorithm when M is an *undirected* series-parallel graph using Seymour's proof of Theorem 3.2, and the technique in the proof of Theorem 4.2.

5 Eulerian mixed graphs and circuit double covers

Recall that a mixed graph M is **eulerian** if its {edge,arc}-set can be partitioned into circuits, that is, $(M, \mathbf{1})$ has a circuit cover. We saw in Sect. 3 a characterization of eulerian series-parallel mixed graphs. In fact, the admissibility of $(M, \mathbf{1})$ is a necessary and sufficient condition for any mixed graph M to be eulerian, according to the next theorem (see [10, 7]). In what follows, we use the notation $f_M(X) := |\delta(X)| + |\delta^+(X)| - |\delta^-(X)|$.

Theorem 5.1. *A mixed graph M is eulerian if and only if $f_M(X) \geq 0$ and $f_M(X)$ is even for every $X \subseteq V(M)$.* □

We can try to extend this result studying circuit p-covers where p is an integral constant vector. The problem is to characterize which mixed graphs M have a circuit p-cover. Note that the condition "$f_M(X) \geq 0$ for every $X \subseteq V(M)$" is clearly a necessary one.

If p is odd it is easy to prove (using Theorem 5.1) that M has a circuit p-cover if and only if it is eulerian. The interesting case is when p is even. We say that M has a **circuit double cover** if $(M, \mathbf{2})$ has a circuit cover, or in other words, there is a list of circuits of M such that each edge and each arc appears in exactly two of them. It is not known a complete characterization for the existence of a circuit double cover (p-cover) in an arbitrary mixed graph.

This is trivial for directed graphs, as a directed graph has a circuit p-cover (for any p) if and only if it is eulerian. For undirected graphs, the case p odd is equivalent to characterizing eulerian graphs, which is well-known (and also a particular case of Theorem 5.1). The case $p = \mathbf{4}$ and the case $p = \mathbf{6}$ were solved by Jaeger [13, 5] and Fan [9], respectively. These results together settle the case $p = \mathbf{2k}$ for any $k \geq 2$.

Theorem 5.2. *Every bridgeless undirected graph has a circuit $\mathbf{4}$-cover and a circuit $\mathbf{6}$-cover.* □

Corollary 5.3. *For any $k \geq 2$, every bridgeless undirected graph has a circuit $\mathbf{2k}$-cover.* □

The case $p = \mathbf{2}$ is the famous Circuit Double Cover Conjecture.

Conjecture. *Every bridgeless undirected graph has a circuit double cover.*

This conjecture was solved for some special classes of graphs, as for example, graphs with no P_{10}-minor (Theorem 3.3). For further references on this topic, see [1].

For mixed graphs, Theorem 3.7 gives a characterization for the series-parallel graphs.

Theorem 5.4. *A bridgeless series-parallel mixed graph M has a circuit double cover (or a $\mathbf{2k}$-cover) if and only if $f_M(X) \geq 0$ for every $X \subseteq V(M)$.* □

In view of the preceding result, a natural question arises here. *Is it possible to drop the series-parallelness hypothesis in Theorem 5.4?* Note that an affirmative answer to this question would settle the Circuit Double Cover Conjecture.

A natural relaxation of this problem is the following. Suppose that M is an arbitrary bridgeless mixed graph such that $f_M(X) \geq 0$ for every $X \subseteq V(M)$. *Is it true that* $(M, \mathbf{1})$ *has a fractional circuit cover?* In the undirected case it is easy to show that $\mathbf{1} \in Cone(\mathcal{C}(G))$ for every bridgeless undirected graph G. But here we do not have a characterization of $Cone(\mathcal{C}(M))$ and we could not find a proof of that inclusion for mixed graphs, neither a counterexample.

We note that to answer these two questions, we can restrict ourselves to mixed graphs $M = (V, E, A)$ such that (V, A) is eulerian and (V, E) is acyclic. In view of the lack of a solution for the fractional circuit cover problem and Theorem 3.4, answering these problems seems to be very difficult.

6 Minimum circuits in series-parallel graphs

In this section, we investigate the problem of finding a minimum circuit in a weighted series-parallel mixed graph (M, w). This is needed to solve the separation problem mentioned in Sect. 4. Here, without loss of generality we assume M is strongly connected, has at least one circuit and does not have cut vertices.

As w can have negative entries, we must deal with negative circuits. It is known that for arbitrary mixed graphs the problem of detecting negative circuits is NP-complete [4]. We show here how to compute in polynomial time a minimum circuit in a series-parallel mixed graph (M, w) for any weight w.

The next result is often used in this section. The proof follows from the following classical theorem due to Dirac [8]: *If G is a simple 2-connected graph with minimum degree at least 3 then G contains a subgraph homeomorphic to K_4.*

Lemma 6.1. *If G is a 2-connected series-parallel graph with $|V(G)| \geq 3$ then G has a vertex with exactly two neighbours.* □

We say that a vertex with exactly two neighbours is a **special vertex**.

Since we can compute in polynomial time a minimum circuit of length two in M, if we knew how to find a minimum circuit of length at least three in M we could find a minimum circuit in M simply by taking the lightest of the two solutions. To solve the latter problem, we can restrict ourselves to directed graphs: we replace each edge e of M with two anti-parallel arcs, each one with weight $w(e)$. Thus, it suffices to show how to solve the following problem:

MC3P(D,w): Given a series-parallel directed graph (D, w), find, if it exists, a minimum directed circuit C of length at least 3.

To describe our algorithm we need to consider the following two operations on a directed graph (D, w). It is easy to see that at least one of these operations can always be carried over a weighted series-parallel directed graph (D, w).

Parallel Arc Deletion (PAD)

1. Let a and b be two parallel arcs;
2. Set $D' := D - b$ and $w(a) := \min\{w(a), w(b)\}$;
3. Let w' be the restriction of w to D'.

Special Vertex Elimination (SVE) (assuming D does not have parallel arcs)

Let v be a special vertex and x and y its neighbours;

1. If $(x,v),(v,x),(y,v),(v,y) \in A(D)$ then
   set $D' := D/\{(y,v),(v,y)\}$;
   $w(x,v) := w(x,v) + w(v,y)$;
   $w(v,x) := w(v,x) + w(y,v)$;
   let w' be the restriction of w to D';
2. If $(x,v),(v,x),(v,y) \in A(D)$ and $(y,v) \notin A(D)$ then
   set $D' := (D - (v,x))/(v,y)$;
   $w(x,v) := w(x,v) + w(v,y)$;
   let w' be the restriction of w to D';
3. If $(x,v),(v,y) \in A(D)$ and $(v,x),(y,v) \notin A(D)$ then
   set $D' := D/(v,y)$;
   $w(x,v) := w(x,v) + w(v,y)$;
   let w' be the restriction of w to D';

Denote by $\gamma(D,w)$ the weight of a minimum circuit of length at least 3 in (D,w). Clearly, if (D',w') is obtained from (D,w) by a PAD operation then $\gamma(D,w) = \gamma(D',w')$. On the other hand, if (D',w') comes from a SVE operation then $\gamma(D,w) = \min\{w(C), \gamma(D',w')\}$ where C is a minimum circuit of length 3 containing v, x and y (if there is no such a circuit, set $w(C) = +\infty$).

As D is series-parallel and does not have cut vertices it has a special vertex by Lemma 6.1. Moreover, operations PAD and SVE preserve series-parallelness and does not create cut vertices. So, a simple polynomial-time algorithm that solves MC3P(D,w) consists of successive applications of these two operations. This gives us immediately a polynomial algorithm to find a minimum circuit in a series-parallel mixed graph.

7 Concluding remarks

Several interesting questions concerning circuit covers in mixed graphs still remain open. Among them we mention the (fractional) circuit cover problem in planar graphs, the circuit double cover problem and its fractional relaxation. As far as the series-parallel mixed graphs, now that we know that there is a polynomial-time (ellipsoid-based) algorithm for the circuit cover problem, it would be interesting to find a combinatorial one.

The proof of Theorem 3.7 presented here was only outlined and need be worked out in more detail. This, as well as the proofs not given here and other related results will be presented elsewhere.

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