SPLITTING NUMBER is NP-complete* (extended abstract)

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Abstract. We consider two graph invariants that are used as a measure of nonplanarity: the splitting number of a graph and the size of a maximum planar subgraph. The splitting number of a graph G is the smallest integer $k \geq 0$, such that a planar graph can be obtained from G by k splitting operations. Such operation replaces a vertex v by two nonadjacent vertices v_1 and v_2, and attaches the neighbors of v either to v_1 or to v_2. We prove that the SPLITTING NUMBER decision problem is NP-complete when restricted to cubic graphs. We obtain as a consequence that PLANAR SUBGRAPH remains NP-complete when restricted to cubic graphs. Note that NP-completeness for cubic graphs implies NP-completeness for graphs not containing a subdivision of K_5 as a subgraph.

1 Introduction

Applications in Computer Science are frequently modeled with nonplanar graphs. Graph visualization and VLSI projects many times require strategies of layout techniques. Layout algorithms are limited to special classes of graphs. For instance, there is a wealth of layout algorithms for planar graphs; however, these algorithms are useless for nonplanar graphs. One approach to handling nonplanarity in layout algorithms is to consider another topological invariant of the graph, the splitting number. The splitting number is a graph invariant that is used as a measure of nonplanarity in many applications such as graph drawing.

The *splitting number* $\sigma(G)$ of a graph G is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A *vertex splitting operation*, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 and P_2 and adds to $G \setminus v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a set of k splittings, we say that H is the *resulting graph* of this set of k splittings in G.

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Note that the resulting graph H can be obtained either by splitting only vertices of G, or by splitting vertices of G and vertices created by former splittings.

Two aspects of the study of splitting numbers have been considered recently by Eades and Mendonça [2, 3, 12]: they established the NP-completeness of a related problem — ELIGIBLE SPLIT SET —, and they successfully used splitting numbers in layout algorithm design. The splitting number has been computed for the class of complete graphs [6] and for the class of complete bipartite graphs [8]. For a recent survey on splitting numbers, see [10].

The knowledge of the value of nonplanarity invariants for the smallest nonplanar member in a class of graphs can help to find the values or bounds for this invariant for every member in the class. For instance, we have recently established [4] that the splitting number of the 4-cube is 4. We also showed that this result implies that the splitting number of the n-cube is in fact $\Theta(2^n)$.

Liu and Geldmacher [11] proved that the PLANAR SUBGRAPH decision problem is NP-complete. Note that, for a fixed value of k, PLANAR SUBGRAPH is easily seen to be polynomial whereas the number of all possible splittings for a vertex in a graph G being of order $\Omega(2^{|V(G)|})$ may indicate that SPLITTING NUMBER, even for a fixed value of k, is not a polynomial problem.

In this paper we prove that SPLITTING NUMBER is NP-complete. We obtain as a consequence that SPLITTING NUMBER remains NP-complete when restricted to graphs with maximum degree 3 or to graphs with no subdivision of K_5. We also prove that SPLITTING NUMBER remains NP-complete when restricted to cubic graphs. This result is used in turn to prove that PLANAR SUBGRAPH remains NP-complete when restricted to cubic graphs. We obtain as a consequence that PLANAR SUBGRAPH remains NP-complete when restricted to graphs with no subdivision of K_5. These variants of PLANAR SUBGRAPH had been open since 1979 [11].

2 Preliminaries

A *graph* G is an ordered triple $G = (V(G), E(G), \psi(G))$ consisting of a nonempty set $V(G)$ of *vertices*, a set of *edges* $E(G)$ disjoint from $V(G)$ and an *incidence function* $\psi(G)$ that associates to each edge of $E(G)$ an unordered pair of distinct vertices of $V(G)$. A graph with *multiple edges* is a graph that admits two edges associated to the same pair of vertices. We shall omit the incidence function of a graph by writing only $G = (V(G), E(G))$. We say that a graph $G' = (V(G'), E(G'))$ is a subgraph of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

A graph is *planar* when it admits a plane drawing, that is, a drawing in the plane such that no edges cross. There are efficient, linear-time algorithms for testing whether a graph G has $\sigma(G) = 0$, that is, for testing whether a graph is planar [7]. We use strongly the characterization of Kuratowski [9]: a graph is planar if and only if it does not contain a subdivision of K_5 or of $K_{3,3}$ as a subgraph, in particular the nonplanarity of the subdivisions of $K_{3,3}$.

In this way, for a better understanding of our proof it is important that the reader familiarizes himself with the special drawing of $K_{3,3}$ defined in Fig. 1a.

We say that a graph G is a $K_{3,3} \setminus \{e\}$ *linked to* the vertices v and w if G is defined by the drawing in Fig. 1b. This graph is an important *tool* in our proof, and the main property used is that a graph is not planar, if it contains as subgraph a $K_{3,3} \setminus \{e\}$ linked to vertices v and w and a path P joining v to w, where P is disjoint in vertices of this $K_{3,3} \setminus \{e\}$, because this is a subdivision of $K_{3,3}$.

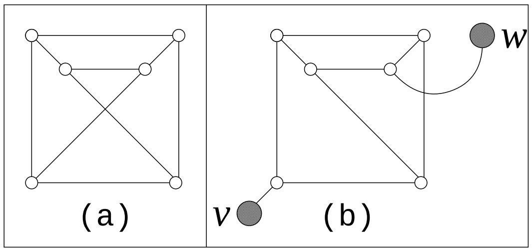


Fig. 1. $K_{3,3}$ and $K_{3,3} \setminus \{e\}$ linked to the vertices v and w.

A *simple drawing* $D(G)$ is a drawing of the graph G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. In what follows all drawings are assumed to be simple. Let $D(G)$ be a simple drawing of G and v be a vertex in $V(G)$, with degree $d(v)$. Because $D(G)$ is simple and in a simple drawing edges incident to the same vertex cannot share crossings, $D(G)$ defines for each vertex v an *ordered adjacency list* in the clockwise direction $\vec{Adj}(v) = (v_1, v_2, \ldots, v_{d(v)})$, where $\{v_1, v_2, \ldots, v_{d(v)}\}$ is the neighborhood of v. Thus, each such ordered adjacency list $\vec{Adj}(v)$ is a circular permutation of the set of edges incident to v. An example of the set of ordered adjacency lists with respect to a drawing $D(G)$ is shown in Fig. 2. Note that we may have multiple edges.

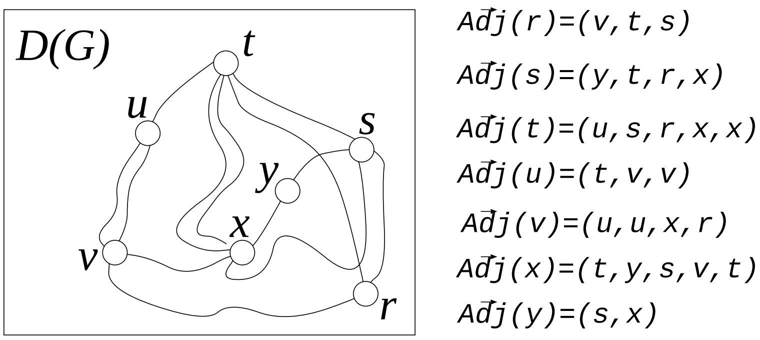


Fig. 2. The ordered adjacency lists of G with respect to $D(G)$.

3 The NP-completeness of SPLITTING NUMBER

In this section we prove that SPLITTING NUMBER is NP-complete, by reducing the NP-complete problem 3–SATISFIABILITY [1] to SPLITTING NUMBER. These problems are defined as follows:

3–SATISFIABILITY (3SAT)
INSTANCE: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ literals.
QUESTION: Is there a truth assignment for U such that each clause in C has at least one true literal?

SPLITTING NUMBER (SN)
INSTANCE: Graph $G = (V(G), E(G))$ and an integer $k \geq 0$.
QUESTION: Is $\sigma(G) \leq k$?

The strategy to reduce 3SAT to SN is to construct an integer $k \geq 0$ and a graph G from a generic instance (U, C) of 3SAT, such that C is satisfiable if and only if $\sigma(G) \leq k$. The graph G is composed of two types of subgraphs: Truth–Setting subgraphs corresponding to the variables of U and Satisfaction–Testing subgraphs corresponding to the clauses of C. The definition of the Satisfaction–Testing subgraphs requires some topological properties of a certain class $\mathcal{A}$ of graphs that we are about to define and study.

A graph G is a member of the class $\mathcal{A}$ if G has two subgraphs P_G and Q_G, such that $V(P_G) \cup V(Q_G) = V(G)$ and $V(P_G) \cap V(Q_G) = \{f_1, f_2, \ldots, f_6, g_1, g_2, \ldots, g_6\}$ with P_G and Q_G satisfying:

- The subgraph P_G is defined by the drawing in Fig. 3. In this figure the subset $\Sigma = \{a_1, a_2, \ldots, a_6, d_1, d_2, \ldots, d_6\}$ is depicted with black vertices. There are exactly $q \geq 2$ edges linking two adjacent vertices of $P_G \setminus \Sigma$, and a single edge linking a white vertex of $P_G \setminus \Sigma$ to a vertex of Σ. Note that we draw between two adjacent vertices in $P_G \setminus \Sigma$ only two edges: one drawn with a continuous line, and one drawn with a dashed line, the other $(q-2)$ edges are omitted, but considered drawn in the region without vertices bounded by those two edges.
- The subgraph Q_G is a connected planar graph not drawn in Fig. 3, such that Q_G admits a plane drawing within the exterior region defined by the drawing of P_G depicted in Fig. 3.

The following four lemmas consider how a planar graph can be obtained from $G \in \mathcal{A}$ by a set Z of splittings only in vertices of Σ. The full details and proofs are in the technical report [5].

Lemma 1. *Let G be a graph in $\mathcal{A}$. If H is a planar graph obtained from G by a set Z of splittings in vertices of Σ, then $|Z| \geq 6$.* □

Lemma 2. *Let G be a graph in $\mathcal{A}$. Let $i \in \{1, 2, 3\}$ be a fixed index and let $M_i = \{a_1, a_2, a_3, a_4, a_5, a_6, d_{2i-1}, d_{2i}\}$. If H is obtained from G by a set Z of splittings, with $|Z| = 8$, such that there is one splitting of Z in each vertex in the set M_i, then H is nonplanar.* □

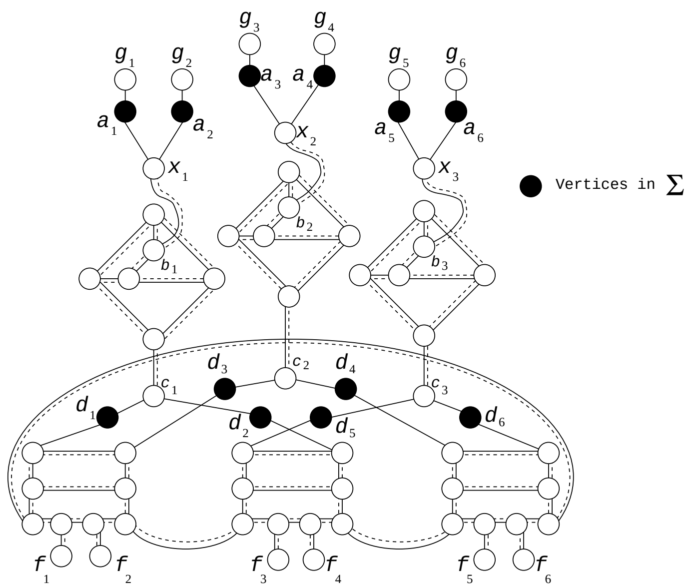


Fig. 3. A drawing for the subgraph P_G of G in class $\mathcal{A}$.

Lemma 3. *Let G be a graph in $\mathcal{A}$. Let Z be a nonempty set of splittings in the set $\{a_1, a_2, \ldots, a_6\}$, such that at most one splitting is done in each set $\{a_1, a_2\}$, $\{a_3, a_4\}$ and $\{a_5, a_6\}$ yielding a resulting graph G' from G. If H is a planar graph obtained in turn from G' by a set Z' of splittings in vertices of Σ, then $|Z'| \geq 5$ and there is such a set Z' satisfying $|Z'| = 5$. In addition, a drawing $D(H)$ can be constructed such that, the nonsplit vertices of G in H have the same ordered adjacency lists with respect to $D(H)$ and with respect to the drawing in Fig. 3.* □

Lemma 4. *Let G be a graph in $\mathcal{A}$. Let $l \in \{1, 2, 3\}$ be a fixed index. Let G' be the graph obtained from G by a set Z of splittings in vertices of $\{a_1, a_2, \ldots, a_6\}$, with $|Z| \geq 2l$, such that $2l$ splittings are in vertices of l of the pairs: $\{a_1, a_2\}$, $\{a_3, a_4\}$, $\{a_5, a_6\}$, and such that $3 - l$ of the pairs: $\{a_1, a_2\}$, $\{a_3, a_4\}$, $\{a_5, a_6\}$ have each one at most one splitting of Z. If H is a planar graph obtained in turn from G' by a set Z' of splittings in vertices of Σ, then $|Z'| \geq 5 - l$.* □

Theorem 1. SN *is NP-Complete.*

Proof. It is easy to see that SN is in NP, because once a non–deterministic algorithm guesses a set of splittings, we need only to check in linear time [7] whether the resulting graph is planar. We reduce 3SAT to SN as follows. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be an instance of 3SAT. We construct in polynomial time a graph G and an integer $k \geq 0$, such that $\sigma(G) \leq k$ if and only if C is satisfiable. In order to define G we construct first an auxiliary graph G^*.

Construction of G^*. The graph G^* is made up of two types of subgraphs: *Truth–Setting* subgraphs and *Satisfaction–Testing* subgraphs, and of a set of edges used to connect these subgraphs. We shall define G^* by ordered adjacency lists. We need to give drawings corresponding to each one of the two types of subgraphs in order to define their corresponding ordered adjacency lists. The two drawings we are about to describe, have the following common strategy: First, we use the drawings in Fig. 4c and 5 respectively to define each one of the subgraphs. Second, in Fig. 4c and 5 we partition the vertices of the subgraphs of G^* into white, black and striped vertices, such that every black vertex has degree 2 and every white vertex has degree 3. The striped vertices are linking vertices between subgraphs and may have degree 2 or 3. The vertices $e_i[1], e_i[2]$ in Fig. 4c and the vertices $f_j[1], f_j[6]$, $b_j[1], b_j[2], b_j[3]$ in Fig. 5, have an incident edge with a missing endpoint. These edges will be used later and indicate striped vertices that necessarily have degree 3 in G^*. Third, the edges of G^* in each one of the subgraphs are defined by continuous lines. Observe that in Fig. 4c and 5, for each continuous edge linking two vertices of degree 3 there is also a dashed edge. This dashed edge is not used in the construction of G^*, it should be ignored in the construction of G^*, because it will be used later only in the construction of G. Now we describe the two types of subgraphs used to construct G^*.

- **Truth–Setting Subgraph.** For each variable $u_i \in U$, there is a Truth–Setting subgraph T_i defined by the drawing in Fig. 4c. The subgraph T_i is obtained from a $K_{3,3}$ (Fig. 4a) by replacing two edges and subdividing a third one as shown in Fig. 4b. Note that we have two additional vertices $e_i[1]$ and $e_i[2]$ (Fig. 4b and 4c). The two replaced edges give place to two graphs called R_{u_i} and $R_{\bar{u}_i}$. Let p be the positive integer that satisfies $2^p > 5m > 2^{p-1}$. Graphs R_{u_i} and $R_{\bar{u}_i}$ are complete binary trees, respectively with roots $r_{u_i[1]}, r_{u_i[2]}$, and $r_{\bar{u}_i[1]}, r_{\bar{u}_i[2]}$, linked by their leaves through vertices $u_i[1], u_i[2], \ldots, u_i[2^p], \bar{u}_i[1], \bar{u}_i[2], \ldots, \bar{u}_i[2^p]$ as shown in Fig. 4c. Note that the greatest level in each one of these trees has $O(m)$ vertices.
- **Satisfaction–Testing Subgraph.** For each clause $c_j \in C$, there is a Satisfaction–Testing subgraph S_j consisting of the graph defined by Fig. 5.

There is a set of edges connecting Truth–Setting subgraphs to Satisfaction–Testing subgraphs:

$$E' = \bigcup_{i=1}^{n-1} \{(e_i[2], e_{i+1}[1])\} \cup \{(e_n[2], f_1[1])\} \cup \bigcup_{j=1}^{m-1} \{(f_j[6], f_{j+1}[1])\} \cup \{(f_m[6], e_1[1])\}.$$

The only part in the construction of G^* that depends on which literals occur in which clauses is the following collection of edges produced sequentially when j grows from 1 until m. Let x_{i_1}, x_{i_2} and x_{i_3}, with $i_1, i_2, i_3 \in \{1, 2, \ldots, n\}$ be the three literals in clause c_j. We have the following sets of edges emanating of the subgraphs T_i and S_j: $E''_j = \{(b_j[1], x_{i_1}[l_1]), (b_j[2], x_{i_2}[l_2]), (b_j[3], x_{i_3}[l_3])\}$, where $l_s (s = 1, 2, 3)$ is the minimum number in the set $\{1, 2, \ldots, 2^p\}$ such that there is no vertex $b_{j'}[h], h \in \{1, 2, 3\}$ linked to $x_{i_s}[l_s]$ with $j' \leq j$.

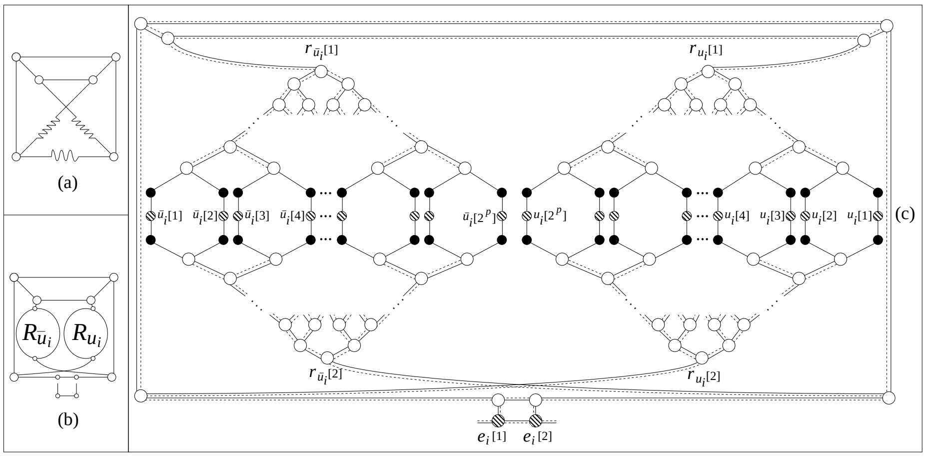


Fig. 4. The Truth–Setting Subgraph T_i.

The construction of G^* is completed by setting: $G^* = (V(G^*), E(G^*))$, where:

$$V(G^*) = (\bigcup_{i=1}^{n} V(T_i)) \cup (\bigcup_{j=1}^{m} V(S_j)),$$

$$E(G^*) = (\bigcup_{i=1}^{n} E(T_i)) \cup (\bigcup_{j=1}^{m} E(S_j)) \cup E' \cup (\bigcup_{j=1}^{m} E''_j).$$

Complexity of the construction of G^*. The size of T_i is in $\Theta(m)$ and therefore the size of G^* is in $\Theta(mn)$. The construction of the ordered adjacency lists for each subgraph S_j depends only on its drawing given in Fig. 5 and it is not dependent on the size of the instance of 3SAT. On the other hand, we can construct the ordered adjacency lists for each T_i in time $O(m)$ as follows. We obtain a total order of the vertices in a complete binary tree by using a Breadth First Search (BFS) from the root to the leaves and from the left side to the right side. The ordered adjacency lists are constructed in linear time by considering this total order restricted to the neighborhood of each vertex. Thus we can construct the ordered adjacency lists for a complete binary tree with 2^p vertices in the greatest level in time $O(m)$. Because of the tests for connecting the subgraphs S_j's and T_i's, we have total time of order $O(m^2n)$. Hence, it is possible to construct G^* in polynomial time in the size of the instance of 3SAT.

We shall see next that the graph G^* satisfies that C is satisfiable if and only if there exists a set of splittings in the black vertices of G^* of size $2^pn + 5m$ that obtains from G^* a planar graph. In order to conclude our reduction from 3SAT to SN, we want a graph G, such that C is satisfiable if and only if $\sigma(G) \leq 2^pn + 5m$. We show how to modify G^* in order to obtain a graph G, quite close to G^*, where the fact that a planar graph is obtained from G with at most

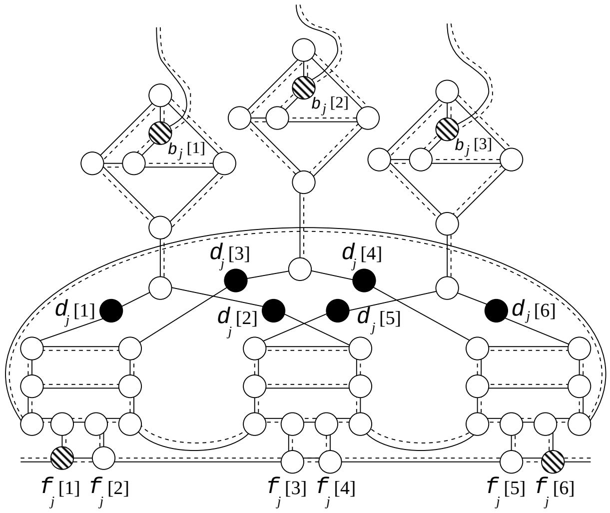


Fig. 5. The Satisfaction–Testing Subgraph S_j.

$2^p n + 5m$ splittings forces in G that these splittings occur close to the black vertices of G which in turn defines a truth assignment that satisfies C. Our strategy to define G is to modify the subgraph of G^* induced by vertices of degree 3 by replacing these vertices and corresponding edges by supervertices as we define next.

Construction of G. Let B be the subgraph of G^* induced by the vertices of degree 3. We shall exhibit a partition $V_1(B), V_2(B)$ for $V(B)$, showing that B is a bipartite graph and we shall use this partition to define G from G^*. To prove that B is in fact a bipartite graph it is enough to prove that each connected component of B is a bipartite graph.

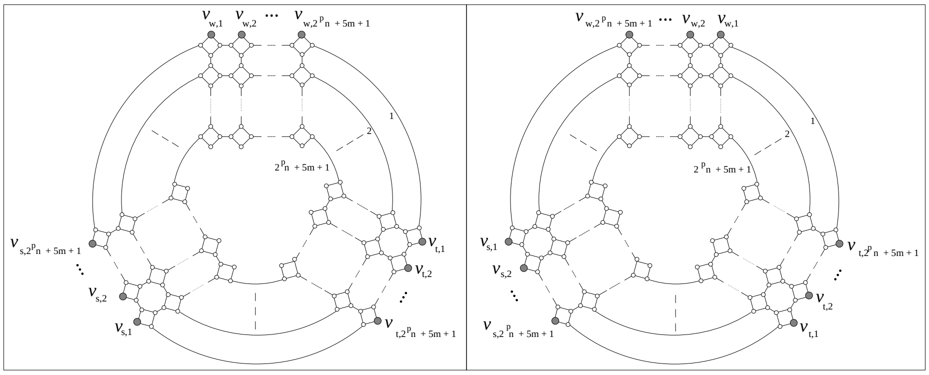
Observe first that there are exactly $3m + 1$ connected components in B, $3m$ of them are each isomorphic to $K_{3,3} \setminus \{e\}$ linked to two vertices, and the other one contains all edges of E'.

We define the partition of B into $V_1(B), V_2(B)$ in three steps:

- For each $h \in \{1, 2, 3\}$ and $j \in \{1, 2, \ldots, m\}$ take a Breadth First Search (BFS) from $b_j[h]$ in the connected component of B containing $b_j[h]$.
- For each $i \in \{1, 2, \ldots, n\}$, take a BFS from $e_i[1]$ in the subgraph of G^* induced by the set $V(T_i) \cap V(B)$. And, for each $j \in \{1, 2, \ldots, m\}$ take a BFS from $f_j[1]$ in the connected component of the subgraph of G^* containing $f_j[1]$ induced by the set $V(S_j) \cap V(B)$.
- For each one of the $n + 4m$ produced BFS–trees, add to $V_1(B)$ the vertices in the even level and add to $V_2(B)$ the vertices in the odd level.

The $3m$ components of B isomorphic each to $K_{3,3} \setminus \{e\}$ linked to two vertices are trivially bipartite graphs. To show that there is no conflict in the definition of the bipartition of B, it remains to analize the connected component of B containing the edges in E'. For note that $e_i[2] \in V_2(B)$ and $e_{i+1}[1] \in V_1(B)$, for $i \in \{1, 2, \ldots, (n-1)\}$; $e_n[2] \in V_2(B)$ and $f_1[1] \in V_1(B)$. And note that $f_j[6] \in V_2(B)$ and $f_{j+1}[1] \in V_1(B)$ for $j \in \{1, 2, \ldots, (m-1)\}$; $f_m[6] \in V_2(B)$ and $e_1[1] \in V_1(B)$.

Now we are ready to define the supervertices of G corresponding to the degree 3 vertices in G^*. For each such vertex v of G^* with ordered adjacency list $\vec{Adj}(v) = (w, t, s)$, we add to G one *Clockwise supervertex* N_v (Fig. 6a), if v is a vertex in partition $V_1(B)$ and one *Counterclockwise supervertex* N_v (Fig. 6b), if v is a vertex in partition $V_2(B)$. The $3(2^p n + 5m + 1)$ labeled vertices in each supervertex will be used later as endpoints of edges linking adjacent supervertices.



(a) N_v Clockwise (b) N_v Counterclockwise

Fig. 6. Non-Splitting subgraph N_v, with $\vec{Adj}(v) = (w, t, s)$.

Finally, we end the construction of the simple graph G as follows. For each edge $(v, w) \in E(B)$, let $E_{vw} = \{(v_{w,s}, w_{v,s}) | s \in \{1, 2, \ldots, 2^p n + 5m + 1\}\}$. For each edge $(v, w) \in E(G^*)$ where v has degree 2 in G^*, let $E_{vw} = \{(v, w_{v,1})\}$, if w has degree 3; and $E_{vw} = \{(v, w)\}$, if w has degree 2.

The construction of the instance (G, k) of SN is completed by setting $k = 2^p n + 5m$ and by setting $V(G) = (\bigcup_{v \in V(B)} V(N_v)) \cup (V(G^*) \setminus V(B))$ and $E(G) = (\bigcup_{v \in V(B)} E(N_v)) \cup (\bigcup_{(u,v) \in E(G^*)} E_{uv})$. Note that the simple graph G has a large number of vertices and edges. For N_v has $3 \times 4(k+1)^2 = 12(k+1)^2$ vertices and $3[(k+1)^2 + 4(k+1)^2 + k(k+1)] = 3(k+1)(6k+5)$ edges. In the set of T_i's there are $[(12(k+1)^2(10 + 4(2^p - 1))) + (6 \times 2^p)]n - 3m$ vertices and there are $[((k+1)(16 + 4(2^p - 2))) + (3(k+1)(6k+5)(10 + 4(2^p - 1))) + (8 \times 2^p)]n$ edges. In the set of S_j's there are $[(12(k+1)^2 54) + (6)]m$ vertices and $[((k+1)71) + 3(k+1)(6k+5)54 + (12)]m$ edges. And in the set of communication edges there are $(k+1)(m+n)$ edges. Figure 7 shows an example of the construction of an instance (G, k) of SN.

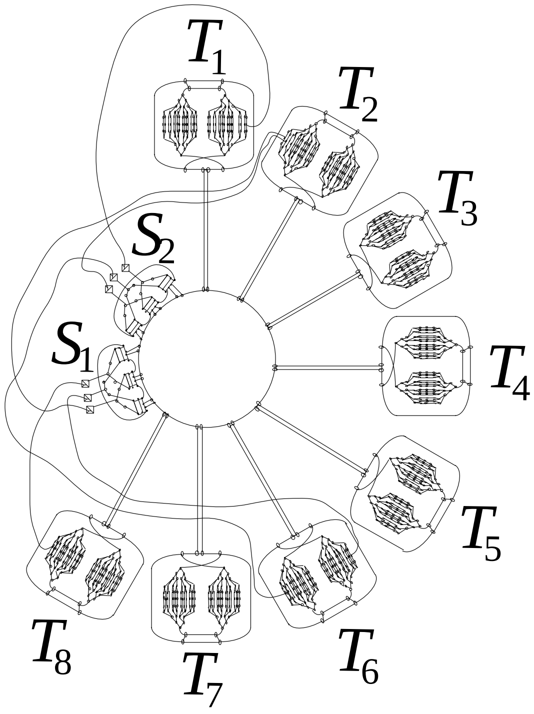


Fig. 7. SN instance obtained from 3SAT in which $U = \{u_1, u_2, \ldots, u_8\}$ and $C = \{(\bar{u}_2 \vee \bar{u}_6 \vee u_8), (\bar{u}_2 \vee u_6 \vee u_1)\}$. Here, graph G has 154.877.910 vertices, 232.139.388 edges and $k = 2^p n + 5m = 138$.

Complexity of the construction of G. As G^* can be constructed in polynomial time in the size of the instance of 3SAT and the size of G^* is $O((m^2n)^2)$, we have that G can be constructed in time $O((2^p n)^2.(m^2n)^2) = O((mn)^2.(m^2n)^2)$, which is polynomial in the size of the instance of 3SAT.

It remains to prove that C is satisfiable if and only if $\sigma(G) \leq k$. See the technical report [5] for the proofs of Claims 1, 2 and 3.

Claim 1 *There is a drawing $D(G)$ for G such that:*
(i) For every $v \in V(B)$, every edge of the corresponding N_v is in no crossing;
(ii) For every $(u, v) \in E(B)$, there are no crossings between two edges linking vertices of N_u to vertices of N_v. □

Consider the drawing for a subgraph N_v of G depicted in Fig. 6. We define the 1–*meridian* of N_v to be the cycle contained in the exterior face of this drawing of N_v. Recursively, for $i = 1, 2, \ldots, k$ we remove the vertices of the exterior face (vertices of the i–meridian plus pendant vertices) obtaining a new drawing and define the $(i+1)$–*meridian* to be the current cycle contained in the exterior face of this drawing. Observe that by construction, if $i < j$, with $i, j \in \{1, 2, \ldots, k+1\}$, then the i–meridian and the j–meridian are disjoint in vertices.

Claim 2 *If G' is obtained from G by a set Z of splittings, where $|Z| \leq k$, then there is a subgraph B_c of G' contractible to B, such that B_c contains a meridian of N_v as subgraph, for all $v \in B$.* □

From now on we refer to the subgraphs of G corresponding to the subgraphs $T_i, S_j, R_{\bar{u}_i}$ and R_{u_i} of G^* by saying, respectively, $T_i, S_j, R_{\bar{u}_i}$ and R_{u_i}.

- If $\sigma(G) \leq k$, then C is satisfiable.
  Suppose there is a planar graph H obtained from G by a set Z, with $|Z| \leq k$ splittings. By Claim 2, H has a subgraph contractible to B. In order to make each T_i planar Z must admit a subset with 2^p splittings in the black or striped vertices of T_i, or in the supervertices N_v with vertices adjacent to some black vertex of T_i. Note that there can be no T_i in which Z has simultaneously 2^p splittings in R_{u_i} and 2^p splittings in $R_{\bar{u}_i}$ because: there are n disjoint subgraphs T_i's in G; each one of the T_i's requires at least 2^p splittings in Z; and H is obtained from G by the set Z of splittings with $|Z| \leq 2^p n + 5m$, where $5m < 2^p$. Let $i \in \{1, 2, \ldots, n\}$ be an index. Let L_i be the subgraph in the pair $R_{u_i}, R_{\bar{u}_i}$ of subgraphs of T_i, that contains at least 2^p splittings of Z. We denote by Z_i the subset of Z consisting of all splittings of Z in L_i. A truth assignment for U can be obtained by setting the variable $u_i = T$, if $L_i = R_{u_i}$. On the other hand we set the variable $u_i = F$, if $L_i = R_{\bar{u}_i}$. Note that this truth assignment can be obtained in polynomial time in the size of G, that is, in the size of the instance of 3SAT.

  Claim 3 *The following truth assignment satisfies C: set $u_i = T$, if $L_i = R_{u_i}$; set $u_i = F$, if $L_i = R_{\bar{u}_i}$.* □

- If C is satisfiable, then $\sigma(G) \leq k$.
  We shall define a set of size k of splittings that obtains a planar graph from G. Consider a truth assignment for U that satisfies C. If the literal u_i has value T, then split in G, the 2^p leaves of one of the two trees of R_{u_i}. If the literal u_i has value F, then split in G, the 2^p leaves of one of the two trees of $R_{\bar{u}_i}$. Let G' be the resulting graph obtained from G by this set of $2^p n$ splittings. Now consider $D(G)$, the drawing for G defined in Claim 1. Consider $D(G')$ a drawing for G' obtained from $D(G)$, such that all corresponding drawings for Truth–Setting subgraphs are plane. Thus in $D(G')$ the remaining crossings occur in edges linking vertices of some N_v in S_j to vertices not in this same N_v. As there is at least one literal with the value T for each clause in C, by applying Lemma 3 we define in polynomial time, for each graph S_j, a corresponding set of five splittings, such that we have no crossings in the edges of the resulting graph from S_j. Let G'' be the resulting graph obtained from G' by this set of $5m$ splittings. A plane drawing $D(G'')$ for G'' is obtained from $D(G')$, where each one of the three $K_{3,3} \setminus \{e\}$'s of each S_j, is located either inside a region of the plane drawing corresponding to S_j, or inside a region of the plane drawing corresponding to some T_i. Therefore, we obtain a planar graph from G with exactly $2^p n + 5m$ splittings. □

Corollary 1. SN *is NP-complete when restricted to cubic graphs.*

Proof. We use the strategy of Theorem 1 by modifying locally the graph G in Theorem 1 as follows. Consider the auxiliary graph C_v depicted in Fig. 8(a). For each vertex v of degree 2 in G, we add to G a copy of C_v, such that w_v is the vertex of C_v adjacent to v, as show in Fig. 8(b). □

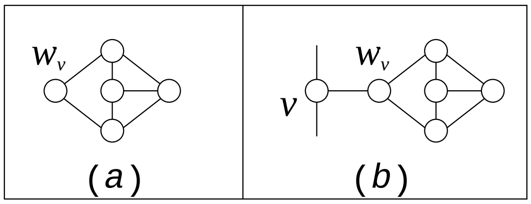


Fig. 8. Auxiliary graph for the proof of Corollary 1.

Corollary 2. SN *is NP-complete when restricted to graphs not containing a subdivision of* K_5 *as a subgraph.*

Proof. It follows from Corollary 1 because a cubic graph does not have vertices of degree 4. □

As an application of the NP-completeness of SPLITTING NUMBER, consider another nonplanarity measure of a graph: the size of its maximum planar subgraph, and the decision problem:

PLANAR SUBGRAPH (PS)
INSTANCE: Graph $G = (V(G), E(G))$ and an integer $0 \leq p \leq |E(G)|$.
QUESTION: Is there a subset $E' \subseteq E(G)$ with $|E'| \geq p$ such that $G' = (V(G), E')$ is planar?

Liu and Geldmacher [11] proved that PLANAR SUBGRAPH is NP-complete, but it was not known until now whether this problem remains NP-complete when restricted to cubic graphs. Our final result shows how to use that SPLITTING NUMBER FOR CUBIC GRAPHS (SNΔ3) is NP-complete to prove that PLANAR SUBGRAPH FOR CUBIC GRAPHS (PSΔ3) is NP-complete.

Corollary 3. PSΔ3 *is NP-complete.*

Proof. PSΔ3 is in NP because PS is in NP. Let G, k be an instance for SNΔ3. We may assume $k \leq |E(G)|$. Consider the instance of PSΔ3 consisting of G and the integer $(|E(G)| - k)$. Note that any splitting in a graph of maximum degree 3 yields one or two leaves. In addition, a crossing in an edge incident to a leaf can always be removed by considering a different drawing in the plane. Thus, if L is the set of the leaves of G, then G has the same splitting number as $G \setminus L$.

Assume there exists a set Z of splittings, $|Z| \leq k$, obtaining a planar graph H from G. Define a subset L of $V(H)$, $|L| = |Z|$, such that L is obtained from Z by adding to L one leaf obtained in each splitting of Z. By construction, the graph $H \setminus L$ is isomorphic to a subgraph of G with $|E(H \setminus L)| \geq |E(G)| - k$, i.e., we have the answer yes to PSΔ3. Now suppose that G has a planar subgraph $G' = (V(G), E')$, with $|E'| \geq |E(G)| - k$. Consider the subset $Z = (E(G) \setminus E')$ of $E(G)$. A planar graph H is obtained from G by splitting, for each edge (u, v) of Z one of its endpoints, say v, with degree greater than 1, into v_1 and v_2, such that $\{u\}$ is the neighborhood of v_1. Thus, a set of splittings of size k or less is produced obtaining a planar graph H from G, as required. $\square$

References

1. S. A. Cook (1971). "The complexity of theorem–proving procedures", *Proceedings of the 3rd. ACM Symposium on Theory of Computing*, Association for Computing Machinery, New York, 151–158.
2. P. Eades and C. F. X. Mendonça (1993). "Heuristics for planarization by vertex splitting". *Proc. ALCOM Int. Workshop on Graph Drawing, GD'93*, 83–85.
3. P. Eades and C. F. X. Mendonça (1996). "Vertex splitting and tension–free layout". *Proc. GD'95*, *Lecture Notes in Computer Science* **1027**, 202–211.
4. L. Faria, C. M. H. Figueiredo and C. F. X. Mendonça (1998). "The splitting number of the 4–cube", *Proc. LATIN'98*, *Lecture Notes in Computer Science* **1380**, 141–150.
5. L. Faria, C. M. H. Figueiredo and C. F. X. Mendonça (1997). "Splitting Number is NP–Complete", Technical Report **ES-443/97**, COPPE/UFRJ, Brazil. Available at ftp://ftp.cos.ufrj.br/pub/tech_reps/es44397.ps.gz.
6. N. Hartfield, B. Jackson and G. Ringel (1985). "The splitting number of the complete graph", *Graphs and Combinatorics* **1**, 311–329.
7. J. E. Hopcroft and R. E. Tarjan (1974). "Efficient planarity testing", *J. ACM* **21**, 549–568.
8. B. Jackson and G. Ringel (1984). "The splitting number of complete bipartite graphs", *Arch. Math.* **42**, 178–184.
9. K. Kuratowski (1930). "Sur le problème des courbes gauches en topologie", *Fund. Math.* **15**, 271–283.
10. A. Liebers (1996). "Methods for Planarizing Graphs – A Survey and Annotated Bibliography". Available at ftp://ftp.informatik.uni-konstanz.de/pub/preprints/ /1996/preprint-012.ps.Z.
11. P. C. Liu and R. C. Geldmacher (1979). "On the deletion of nonplanar edges of a graph", *Cong. Num.* **24**, 727–738.
12. C. F. X. Mendonça (1994). "A Layout System for Information System Diagrams", PhD thesis, Univ. of Queensland, Australia.