ON ANALYTIC METHODS IN COMBINATORICS

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2008-04-15; AofA08, Maresias, Brasil

The Target

Given a sequence of analytic in $|z| < 1$ functions

$$F_n(z) = \sum_{m=0}^{\infty} M_{mn} z^m \qquad (\to\ F(z), \quad \textit{not necessary})$$

$n = 1, 2, \ldots$ find

$$M_{nn} \ \sim ?$$

or

$$M_{mn} \ \sim$$

$(m, n) \in D$, some region, or $m = m(n) \to \infty$ as $n \to \infty$.

If

$$\sum_{n=1}^{\infty} F_n(z) y^n$$

is too cumbersome, the Tauber theorems for double sums fail.

For probabilistic problems,

$$F_n(z,t) = \sum_{m=0}^{\infty} M_{mn}(t) z^m \qquad (t \in T_n,\ a\ parameter(s))$$

find

$$M_{nn}(t) \sim$$

or

$$M_{mn}(t) \sim$$

$(m, n) \in D$, some region, or $m = m(n) \to \infty$ as $n \to \infty$ uniformly in $t \in T_n$.

Typical Problems

- Enumeration of combinatorial structures of order n with component sizes constraints depending on n;

- Asymptotic value distribution problems for sequences of mappings defined on combinatorial structures;

- Hypothetically! (Do not ask me) Evaluation of the total cost if the price at each step of an algorithm depends on the size of data;

First on the two problems, for permutations only.

Notation

Let σ be a permutation of an n set, $\mathbf{S}_n$ be the symmetric group, and

$$\sigma = \kappa_1 \cdots \kappa_w, \quad w = w(\sigma),$$

be the decomposition into the product of independent cycles. Let $k_j(\sigma)$ be the number of cycles of length j in the decomposition, $1 \leq j \leq n$,

$$\bar{k}(\sigma) = (k_1(\sigma), \ldots, k_n(\sigma))$$

be the structure vector.

Denote

$$\ell(\bar{k}) = 1k_1 + \cdots + nk_n, \quad \bar{k} \in \mathbf{Z}_+^{\mathbf{n}}.$$

Then

$$\ell(\bar{k}(\sigma)) = n.$$

Define

$$\nu_n(\dots)\colon = \frac{1}{n!}|\{\sigma \in \mathbf{S}_n \colon \ ...\}|.$$

The probability of permutations with a given structure vector $\bar{k} \in \mathbf{Z}_+^{\mathbf{n}}$ is

$$\nu_n\big(\bar{k}(\sigma) = \bar{k}\big) = \mathbf{1}\{\ell(\bar{k}) = n\} \prod_{j=1}^{n} \frac{1}{j^{k_j} k_j!} = P\big(\bar{\xi} = \bar{k} \,\big|\, \ell(\bar{\xi}) = n\big).$$

Here ξ_j, $1 \le j \le n$, are i. Poisson r.vs, $\mathbf{E}\xi_j = 1/j$.

Ewens probability

If $\Theta > 0$, $\Theta_{(n)} : = \Theta(\Theta+1)\cdots(\Theta+n-1)$, and

$$\nu_{n,\Theta}(\{\sigma\}) : = \Theta^{w(\sigma)}/\Theta_{(n)}, \qquad \sigma \in \mathbf{S}_n,$$

for the class of $\sigma \in \mathbf{S}_n$, we obtain

$$\nu_{n,\Theta}\left(\bar{k}(\sigma) = \bar{k}\right) = \mathbf{1}\{\ell(\bar{k}) = n\} \frac{n!}{\Theta_{(n)}} \prod_{j=1}^{n} \left(\frac{\Theta}{j}\right)^{k_j} \frac{1}{k_j!}.$$

The Ewens probability on the partitions $\ell(\bar{k}) = n$.

- $\mathbf{S}_n$ with respect to $\nu_{n,\Theta}$ is a clue to combinatorial structures having the generating function

$$G(z) \sim A(1-z)^{-\Theta}, \quad A \in \mathbf{R}, \;\; z \to 1.$$

Permutations missing long cycles

Let

$$\psi(n,m) = \Big|\{\sigma \in \mathbf{S}_n : \ k_j(\sigma) = 0 \ \ \forall\, m < j \le n\}\Big|$$

the number of permutations missing cycles with lengths in $(m, n]$.

Theorem (A corollary from E.M., 1992). *If* $1 \le u := n/m \le Cm/\log m$, *then*

$$\frac{\psi(m,n)}{n!} = \rho(u)\left(1 + O\left(1 + \frac{u\log u}{m}\right)\right),$$

where $\rho(u)$ *is the Dickman function, a continuous solution to* $x\rho'(x) + \rho(x-1) = 0$ *with* $\rho(x) = 1$ *for* $0 \le x \le 1$.

For other regions of u, more complicated formulas.

The Saddle Point Method

Start with Cauchy:

$$\frac{\psi(m,n)}{n!} = \frac{1}{2\pi i} \int_{|z|=\alpha} \exp\Big\{ \sum_{j\le m} \frac{z^j}{j} \Big\} \frac{dz}{z^{n+1}},$$

where $\alpha = \alpha(m,n)$ satisfies

$$\sum_{j\le m} \alpha^j = n.$$

This is affordable to obtain asymptotical formulae for α and for the integral as well.

The number of permutations missing lengths in $J = J_n \subset \{1, \ldots, n\}$, an arbitrary set, remains mysterious.

An excursion

Let $J = J_n \subset \{1, \ldots, n\}$ be arbitrary. Then

$$\nu_n(J) := \nu_n\big(k_j(\sigma) = 0 \ \ \forall\, j \in J\big) \ll \exp\Big\{ -\sum_{j \in J} \frac{1}{j} \Big\}$$

with an absolute constant in $\ll$, an analogue of $O(\cdot)$.

The dependence on n changes the picture in the lower estimates.

Let

$$\mu_n(K) := \min_J \nu_n(J),$$

where the minimum is taken over J satisfying

$$\sum_{j \in J} \frac{1}{j} \leq K.$$

Theorem (E.M., 2001). *For all $K \geq 0$,*

$$\liminf_{n \to \infty} \mu_n(K) \geq \exp\{-\mathrm{e}^{7K}\}.$$

If $J = (m, n] \cap \mathbf{N}$, $K \sim \log \frac{n}{m}$, then

$$\nu_n(J) \sim \exp\{-(1 + o(1))K\mathrm{e}^K\}.$$

Conjectured as the smallest frequency in terms of K.

For the Ewens frequency,

$$\mu_{n,\Theta}(K) := \min_J \nu_{n,\Theta}(J),$$

where the minimum is taken over J satisfying

$$\sum_{j\in J} \frac{1}{j^{1\wedge\Theta}} \le K.$$

Theorem (E.M., 2002). *For all $K \ge 0$,*

$$\liminf_{n\to\infty} \mu_{n,\Theta}(K) \ge c_1 \exp\{-\mathrm{e}^{CK}\}, \quad c, C > 0.$$

There is no minimum $1 \wedge \Theta$ in the upper estimates.

Unsatisfactory, for $0 < \Theta < 1$.

A Value Distribution Problem

Given $a_{jn} \in \mathbf{R}$, $1 \le j \le n$, $n \in \mathbf{N}$, examine the distribution functions

$$V_n(x) := \nu_n\Big(a_{1n}k_1(\sigma) + \cdots + a_{nn}k_n(\sigma) - \alpha(n) < x\Big), \quad \alpha(n) \in \mathbf{R}$$

as $n \to \infty$.

If $a_{jn} = 1$ for $j \in J_n \subset \{1, 2, \ldots, n\}$ and $a_{jn} = 0$ otherwise,

the sum gives the number of cycles with lengths in J_n.

Test your methods if they are capable to deal with the sequences of generating functions and preserve the uniformity in parameters!

Theorem (E. M., 2005). *Let* $a_{jn} \in \{0, 1\}$ *and* $\alpha(n) = 0$. *The frequencies* $V_n(x)$ *weakly converge to the Poisson limit law with parameter* $\mu > 0$, *if and only if*

$$\sum_{\substack{j \le n \\ a_{jn}=1}} \frac{1}{j} = \mu + o(1)$$

and

$$\sum_{\substack{\varepsilon n < j \le n \\ a_{jn}=1}} \frac{1}{j} = o(1)$$

for each fixed $0 < \varepsilon < 1$.

(The influence of long cycles must be negligible).

More general results in E. M., *Acta Math. Univ. Ostraviensis*, 2005. Except of the degenerated at one point limit law (see E. M., *The Ramanujan J.*, to appear), the problem remains open.

Analysis

The number of permutations missing the cycles of length $j \in J = J_n$.

$$\sum_{n=0}^{\infty} \nu_n(J) z^n = \frac{1}{1-z} \exp\left\{ -\sum_{j \in J} \frac{z^j}{j} \right\} =: \frac{1}{1-z} \exp\left\{ A_n(z) \right\}.$$

If

- J is fixed (does not depend on n) and

$$\sum_{j \in J} \frac{1}{j} < \infty,$$

- $n^{-1} \big| \{ j \le n : \; j \in J \} \big| \to d(J)$ exists.

Tauber theorems (see the recent book by A. L. Jakymiv, 2005).

Transfer method (Flajolet-Odlyzko) if you can control $A_n(z)$) in some region outside $|z| \le 1$.

In the value distribution problem,

$$\varphi_n(y) := \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \prod_{j=1}^{n} y^{a_{jn} k_j(\sigma)}, \qquad a_{jn} \in \mathbf{R},\ |y| \le 1.$$

$$\sum_{n=0}^{\infty} \varphi_n(y) z^n = \exp\left\{ \sum_{j=1}^{\infty} \frac{y^{a_{jn}}}{j} z^j \right\}.$$

On the right, the variable $y \in \mathbf{C}$ is "deeply" hidden and the sequence parameter n is present!

We have the sequence of analytic in $|z| < 1$ functions, depending also on $|y| \le 1$. That is the only information we begin with!

Some Ideas

(See, E. M. (1996,...), V. Zakharovas (2001,...)). Let

$$F_n(z) = \sum_{m=0}^{\infty} M_{mn} z^m := \exp\left\{ \sum_{j \le n} \frac{b_{jn}}{j} z^j \right\} = \mathrm{e}^{A_n(z)}, \quad |b_{jn}| \le 1.$$

Omitting the extra index n. Estimate $M_n (= M_{nn}) = [z^n] F(z)$.

Use

$$M_n = \frac{1}{n} \sum_{k=0}^{n-1} M_k b_{n-k}, \quad M_0 = 1.$$

and Parseval's equality.

For $1/n \le \varepsilon < 1$,

$$|M_n| \le \varepsilon + \frac{1}{\varepsilon n^{3/2}} \sum_{k=0}^{n} k^2 |M_k|^2.$$

Hence we obtain

$$|M_n| \le \varepsilon + \frac{1}{\varepsilon n} \left(\frac{1}{2\pi n} \int_{-\pi}^{\pi} |F(\mathrm{e}^{it})|^2 |A'(\mathrm{e}^{it})|^2 dt \right)^{1/2}.$$

Extract the $\max |F(\mathrm{e}^{it})|$ and use again Parseval for the remaining integral.

Theorem . *If $|b_j| \le 1$ and $n \ge 1$, then*

$$|M_n| \le 14 \exp\left\{ -\frac{1}{2} \min_{|t|\le\pi} \sum_{j\le n} \frac{1-\Re\bigl(b_j \mathrm{e}^{-ijt}\bigr)}{j} \right\}.$$

Uniformity in all parameters is preserved.

To derive an asymptotic formula for M_n, use the integral Cauchy formula on $|z| = \mathrm{e}^{-1/n}$ and, similarly, apply Parseval's equality in the strip $K/n \le |argz| \le \pi$. Here $1 \le K \le n$ is a parameter.

Nothing outside $|z| < 1$ is needed!

For $|argz| \le K/n$, apply standard analysis.

Disadvantage: Some information about the logarithmic derivative is needed.

For General Decomposable Structures

Find a formula for M_n defined via

$$M(z) = \sum_{n\geq 0} M_n z^n := \exp\left\{\sum_{j\geq 1} \frac{d_j b_j z^j}{j}\right\} \cdot H(z),$$

where $d_j \geq 0$, $b_j = b_j(n,t) \in \mathbf{C}$, $|b_j| \leq 1$, and $H(z) = H_n(z)$ is a "better" function than the series under the exponent, say, analytic in $|z| < 1$ and smooth on $|z| = 1$.

The numbers d_j appear from the structure definitions. Actually, we have them or D_n in

$$\sum_{n\geq 0} D_n z^n := \exp\left\{\sum_{j\geq 1} \frac{d_j z^j}{j}\right\}.$$

The safety vest: it is sufficient to know $M_n/D_n \sim ?$

To get rid of $H(z)$ is not difficult. Further just $H(z) = 1$.

Use Cauchy formula

$$M_n = \frac{1}{2\pi i}\left(\int_{\Delta_0} + \int_{\Delta}\right)\frac{M(z)}{z^{n+1}}\,\mathrm{d}z =: J_0 + J;$$

$$\Delta_0 = \{z = re^{i\tau} : |\tau| \le K/n\}, \qquad \Delta = \{z = re^{i\tau} : K/n < |\tau| \le \pi\},$$

and $r = e^{-1/n}$, $2 \le K \le n$.

Under $0 < \theta^- \le d_j \le \theta^+ < \infty$ and

$$\sum_{j \le n} \frac{d_j(1 - \Re b_j)}{j} \le L < \infty,$$

we obtain $J = O\big(D_n(K^{-c} + n^{-1/2})\big)$, where the constants implied depend on L, θ^-, and θ^+ only.

Very sensitive in θ^-.

Theorem (E.M., to appear). *If*

$$0 < \theta^- \le d_j \le \theta^+ < \infty,$$

$$\sum_{j \le n} \frac{d_j(1 - \Re b_j)}{j} \le L < \infty, \qquad (*)$$

and, for some $\mu_n = o(1)$,

$$\frac{1}{n} \sum_{j \le n} d_j |1 - b_j| \le \mu_n = o(1), \qquad (**)$$

then

$$\frac{M_n}{D_n} = \exp\left\{ \sum_{j \le n} \frac{d_j(b_j - 1)}{j} \right\} + O\big(\mu_n^{c_1} + n^{-c_2}\big).$$

The constant in $O(\cdot)$ depends at most on L, θ^-, and θ^+ while $c_1 = c_1(\theta^-, \theta^+) > 0$ and $c_2 = c_2(\theta^-, \theta^+) > 0$.

Condition (**) is unsatisfactory. If no dependence on n,
(*) $\Rightarrow$ (**).

Is the Approach Sharp?

Yes, under mild conditions. No, if you can integrate beyond the convergence disk.

Again for permutations:

$$F_n(z) = \sum_{m=0}^{\infty} M_{mn} z^m := \exp\left\{ \sum_{j \le n} \frac{b_{jn}}{j} z^j \right\} = \mathrm{e}^{A_n(z)}, \quad |b_{jn}| \le 1.$$

If the quantity

$$r(p) := \sum_{j \le n} \frac{|b_{jn} - 1|^p}{j}, \quad p > 1,$$

is small enough, one can obtain asymptotic expansion for M_{nn} with $N - 1 \ge 1$ terms with accuracy $O\big(r(p)^N + n^{-c}\big)$, $c > 0$.

Some Results

Again

$$V_n(x) := \nu_n\Big(a_{1n}k_1(\sigma) + \cdots + a_{nn}k_n(\sigma) - A(n) < x\Big),$$

where now

$$\sum_{j\le n} \frac{a_{jn}^2}{j} = 1, \qquad A(n) := \sum_{j\le n} \frac{a_{jn}}{j}.$$

Set

$$D_n := \sum_{\substack{j,k\le n\\ j+k>n}} \frac{a_{jn}a_{kn}}{jk}, \qquad E_n := \sum_{j\le n} \frac{|a_{jn}|^3}{j}.$$

Theorem (E.M., 1998). *Let $\Phi(x)$ be the distribution function of the standard normal law. Then*

$$V_n(x) - \Phi(x) - \frac{D_n x}{2\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \ll E_n$$

with an absolute constant in the symbol $\ll$.

Corollary. *We have*

$$V_n(x) - \Phi(x) \ll E_n^{2/3}$$

and the exponent 2/3 is optimal.

Further Reading

1) V. Zakharovas (PhD and *Lithuanian Math. J.*, 2002-04):

- Two terms in the Erdős-Turán problem, e.g. in the CLT for the group theoretical order of a random permutation;

- Mean value theorems and value distribution problems for mapping on the subset

$$\{\sigma \in \mathbf{S}_n : \ \sigma^r = \mathbf{1}\}, \quad r \in \mathbf{N}.$$

2) Generalizations for other combinatorial structures (assemblies, multisets, selections (see R.Arratia, A. Barbour and S. Tavaré, 2003, for definitions and examples)) under progress.

THE END