Effective resistance of random trees

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joint work with

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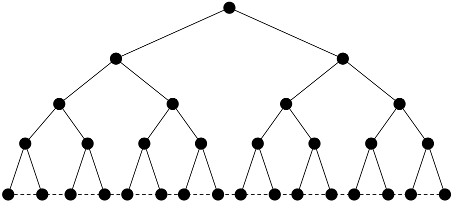


electric networks

An electric network is a connected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$.

Each edge $\mathbf{e} \in \mathbf{E}$ is equipped with a *resistance* $\mathbf{r_e} \geq \mathbf{0}$.

$\mathbf{c_e} = \mathbf{1}/\mathbf{r_e}$ is the *conductance* of $\mathbf{e}$.

Let $\mathbf{A}, \mathbf{B} \subset \mathbf{V}$ and assign "voltage" $\mathbf{U(u) = 1}$ to each $\mathbf{u} \in \mathbf{A}$ and $\mathbf{U(v) = 0}$ for all $\mathbf{v} \in \mathbf{B}$.

$\mathbf{U}$ can be extended, in a unique way, to all vertices according to *Ohm's law* and *Kirchhoff's node law*.

electric networks

Given $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ joined by $\mathbf{e} \in \mathbf{E}$, the *current* from $\mathbf{u}$ to $\mathbf{v}$ is $\mathbf{i}(\mathbf{u}, \mathbf{v})$.

By Ohm's law, $\mathbf{i}(\mathbf{u}, \mathbf{v})\mathbf{r}_{\mathbf{e}} = \mathbf{U}(\mathbf{u}) - \mathbf{U}(\mathbf{v})$.

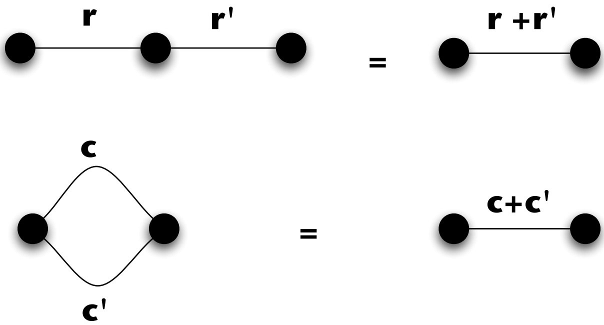
By Kirchhoff's node law, for any $\mathbf{u} \notin \mathbf{A} \cup \mathbf{B}$, $\sum_{\mathbf{v}:\mathbf{v}\sim\mathbf{u}} \mathbf{i}(\mathbf{u}, \mathbf{v}) = \mathbf{0}$.

The *effective conductance* between $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{C}(\mathbf{A} \leftrightarrow \mathbf{B}) = \sum_{\mathbf{u}\in\mathbf{A}} \sum_{\mathbf{v}:\mathbf{v}\sim\mathbf{u}} \mathbf{i}(\mathbf{u}, \mathbf{v}) \ .$$

The effective resistance between $\mathbf{A}$ and $\mathbf{B}$ is
$\mathbf{R}(\mathbf{A} \leftrightarrow \mathbf{B}) = \mathbf{1}/\mathbf{C}(\mathbf{A} \leftrightarrow \mathbf{B})$.

equivalent transformations



connection to random walks

A random walk on $\mathbf{G}$. At each vertex, transition probabilities are proportional to conductances.

Starting the walk at vertex $\mathbf{v}$, the probability that the walk reaches $\mathbf{A}$ before $\mathbf{B}$ is exactly $\mathbf{U(v)}$.

Helpful for studying transience/recurrence of random walks on infinite graphs.

Doyle and Snell (1984), Lyons and Peres (2008+).

random electrical networs

Benjamini and Rossignol (2007) study electrical networks with random resistances.

The $\mathbf{r_e}$ are i.i.d. taking values in $[\mathbf{a}, \mathbf{b}]$ with $\mathbf{0} < \mathbf{a} < \mathbf{b} < \infty$.

They study, in $\mathbb{Z}^{\mathbf{d}}$, the effective resistance between the origin and the boundary of a centered box of side $\mathbf{n}$.

random trees

We consider the rooted complete binary tree of height $\mathbf{n}$.

The resistance of an edge at depth $\mathbf{d}$ is $\mathbf{r_e = 2^d X_e}$ where the $\mathbf{X_e}$ are i.i.d. taking values in $\mathbf{[a, b]}$.

The scaling corresponds to the critical case.

See Pemantle (1988), Lyons (1990), Lyons and Pemantle (1992) for related models of random walks.

main results

Let $\mathbf{R_n}$ and $\mathbf{C_n}$ be the resistance and conductance between the root and the set of vertices at depth $\mathbf{n}$.

Let $\mu = \mathbb{E}\mathbf{X_e}$ and $\sigma^2 = \mathrm{var}(\mathbf{X_e})$.

$$\mathbb{E}\mathbf{R_n} = \mu\mathbf{n} - \frac{\sigma^2}{\mu}\ln\mathbf{n} + O(1) \quad \text{and} \quad \mathrm{var}(\mathbf{R_n}) = O(1)$$

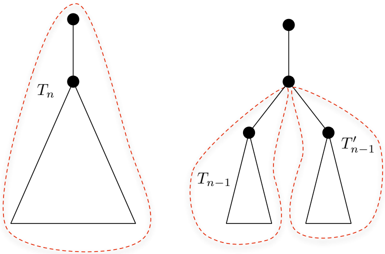
$$\mathbb{E}\mathbf{C_n} = \frac{1}{\mu\mathbf{n}} + \frac{\sigma^2}{\mu^3}\cdot\frac{\ln\mathbf{n}}{\mathbf{n}^2} + O(\mathbf{n}^{-2}) \quad \text{and} \quad \mathrm{var}(\mathbf{C_n}) = O(\mathbf{n}^{-4})$$

proof ideas

The key is to establish the concentration results first, starting with the conductance.

Decompose the tree into two parallel subtrees.

The subtrees are rooted at an edge instead of a vertex.



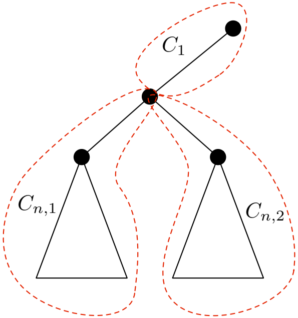
Efron-Stein inequality

If $X_1, \ldots, X_n$ are independent random variables and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ then

$$\begin{aligned}
&\mathrm{var}(f(X_1, \ldots, X_n)) \\
&\quad \leq \quad \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\left[\big(f(X_1, \ldots, X_n) - f(X_1, \ldots, X_i', \ldots, X_n)\big)^2\right]
\end{aligned}$$

the variance of the conductance

We write the conductance as a function of **3** independent random variables.



$$\mathbf{C_n = \frac{C_1(C_{n,1} + C_{n,2})}{C_1 + C_{n,1} + C_{n,2}}}$$

Now use Efron-Stein.

This leads to the recursion

$$\mathbf{var(C_n) \leq \frac{1}{2} var(C_{n-1}) + \frac{K}{(n-1)^4}}$$

the variance of the resistance

We have $\mathbf{var}(C_n) = O(n^{-4})$.

$$
\begin{aligned}
\mathbb{P}\left\{\left|R_n - \frac{1}{\mathbb{E}C_n}\right| > t\right\} &= \mathbb{P}\left\{\left|\frac{\mathbb{E}C_n}{C_n} - 1\right| > t\mathbb{E}C_n\right\} \\
&\leq \mathbb{P}\left\{|C_n - \mathbb{E}C_n| > \frac{t}{b^2n^2}\right\} \\
&\leq \frac{K}{t^2}
\end{aligned}
$$

This implies

$$
\mathbf{var}(R_n) \leq \mathbb{E}\left[\left(R_n - \frac{1}{\mathbb{E}C_n}\right)^2\right] = O(1)
$$

the expected conductance and resistance

Once again we work with the recursion

$$\mathbf{C_n} = \frac{\mathbf{C_1}(\mathbf{C_{n,1}} + \mathbf{C_{n,2}})}{\mathbf{C_1} + \mathbf{C_{n,1}} + \mathbf{C_{n,2}}}$$

The variance bound is crucial.

flows

A *unit flow* is a function Θ over the edges $\{(u, v) : u \sim v\}$ which is

- antisymmetric: $\Theta(u, v) = -\Theta(v, u)$,
- $\sum_{v:v\sim u} \Theta(u, v) = 0$ for any $u \notin A \cup B$,
- $$\sum_{u\in A} \sum_{v\notin A:v\sim u} \Theta(u, v) = \sum_{v\in B} \sum_{u\notin B:u\sim v} \Theta(u, v) = 1 .$$

Thomson's principle:

$$R(A \leftrightarrow B) = \inf_{\Theta} \sum_{e\in E} r_e \Theta(e)^2$$

The unique flow Θ^* which attains the infimum is the current $i(u, v)$.

an alternative approach

By Efron-Stein we get

$$\mathbf{var}(R_n) \le \frac{(b-a)^2}{2} \sum_{e \in E(T_n)} 2^{2d(e)} \mathbb{E}\left[\Theta^*(e)^4\right].$$

If $e_0, \ldots, e_{n-1}$ are the edges on the leftmost branch,

$$\mathbf{var}(R_n) \le \frac{(b-a)^2}{2} \sum_{i=0}^{n-1} 2^{3i} \mathbb{E}\left[\Theta^*(e_i)^4\right],$$

Also,

$$\Theta^*(e_i) = \Theta^*(e_{i-1}) \cdot \frac{R^r_{n,i}}{R^\ell_{n,i} + R^r_{n,i}}.$$

With this technique and generalizations of Efron-Stein we can prove

$$\mathbb{E}\left[(R_n - \mathbb{E}R_n)^k\right] = O(1) \quad \text{for all } k.$$

questions

- More general trees?
- Galton-Watson trees. Scale by expected number of offspring $\mathbb{E}\mathbf{B}$.
- If $\mathbf{Z_n}$ is the number of vertices at depth $\mathbf{n}$ and $\mathbf{W = \lim_n Z_n/(\mathbb{E}B)^n}$, is it true that

$$\frac{\mathbf{R_n}}{\mathbf{n}} \longrightarrow \frac{\boldsymbol{\mu}}{\mathbf{W}}\mathbf{?}$$