PROBABILISTIC ANALYSIS OF AN EXHAUSTIVE SEARCH ALGORITHM IN RANDOM GRAPHS

Hsien-Kuei Hwang

Academia Sinica, Taiwan
(joint work with Cyril Banderier, Vlady Ravelomanana, Vytas Zacharovas)

AofA 2008, Maresias, Brazil
April 14, 2008

MAXIMUM INDEPENDENT SET

Independent set

An independent (or stable) set in a graph is a set of vertices no two of which share the same edge.

MIS $= \{1, 3, 5, 7\}$

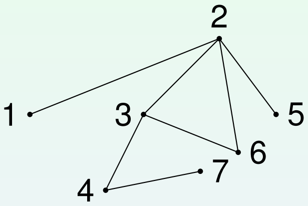
Maximum independent set (MIS)

The MIS problem asks for an independent set with the largest size.

NP hard!!

Independent set

An independent (or stable) set in a graph is a set of vertices no two of which share the same edge.



MIS $= \{1, 3, 5, 7\}$

Maximum independent set (MIS)

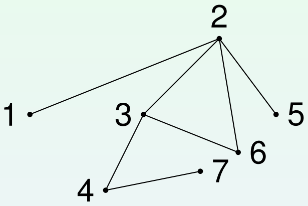
The MIS problem asks for an independent set with the largest size.

NP hard!!

MAXIMUM INDEPENDENT SET

Independent set

An independent (or stable) set in a graph is a set of vertices no two of which share the same edge.



MIS $= \{1, 3, 5, 7\}$

Maximum independent set (MIS)

The MIS problem asks for an independent set with the largest size.

NP hard!!

MAXIMUM INDEPENDENT SET

Equivalent versions

The same problem as MAXIMUM CLIQUE on the complementary graph (clique = complete subgraph).

Since the complement of a vertex cover in any graph is an independent set, MIS is equivalent to MINIMUM VERTEX COVERING. (*A vertex cover is a set of vertices where every edge connects at least one vertex.*)

Among Karp's (1972) original list of 21 NP-complete problems.

Equivalent versions

The same problem as MAXIMUM CLIQUE on the complementary graph (clique = complete subgraph).

Since the complement of a vertex cover in any graph is an independent set, MIS is equivalent to MINIMUM VERTEX COVERING. (*A vertex cover is a set of vertices where every edge connects at least one vertex.*)

Among Karp's (1972) original list of 21 NP-complete problems.

MAXIMUM INDEPENDENT SET

Equivalent versions

The same problem as MAXIMUM CLIQUE on the complementary graph (clique $=$ complete subgraph).

Since the complement of a vertex cover in any graph is an independent set, MIS is equivalent to MINIMUM VERTEX COVERING. (*A vertex cover is a set of vertices where every edge connects at least one vertex.*)

Among Karp's (1972) original list of 21 NP-complete problems.

THEORETICAL RESULTS

Random models: Erdős-Rényi's $G_{n,p}$

Vertex set $= \{1, 2, \ldots, n\}$ and all edges occur independently with the same probability p.

The cardinality of an MIS in $G_{n,p}$

Matula (1970), Grimmett and McDiarmid (1975), Bollobas and Erdős (1976), Frieze (1990): If $pn \to \infty$, then ($q := 1 - p$)

$$|\text{MIS}_n| \sim 2\log_{1/q} pn \quad \text{whp},$$

where $q = 1 - p$; and $\exists k = k_n$ such that

$$|\text{MIS}_n| = k \text{ or } k+1 \quad \text{whp}.$$

THEORETICAL RESULTS

Random models: Erdős-Rényi's $G_{n,p}$

Vertex set $= \{1, 2, \dots, n\}$ **and all edges occur independently with the same probability** p**.**

The cardinality of an MIS in $G_{n,p}$

Matula (1970), Grimmett and McDiarmid (1975), Bollobas and Erdős (1976), Frieze (1990): If $pn \to \infty$**, then** ($q := 1 - p$)

$$|\mathbf{MIS}_n| \sim 2\log_{1/q} pn \quad \textbf{whp},$$

where $q = 1 - p$**; and** $\exists k = k_n$ **such that**

$$|\mathbf{MIS}_n| = k \text{ or } k + 1 \quad \textbf{whp}.$$

A GREEDY ALGORITHM

Adding vertices one after another whenever possible

The size of the resulting IS:

$$S_n \stackrel{d}{=} 1 + S_{n-1-\mathbf{Binom}(n-1;p)} \qquad (n \geqslant 1),$$

with $S_0 \equiv 0$.

Equivalent to the length of the right arm of random digital search trees.

Adding vertices one after another whenever possible

The size of the resulting IS:

$$S_n \stackrel{d}{=} 1 + S_{n-1-\mathbf{Binom}(n-1;p)} \qquad (n \geqslant 1),$$

with $S_0 \equiv 0$**.**

Equivalent to the length of the right arm of random digital search trees.

Easy for people in this community

- **Mean:** $\mathbb{E}(S_n) \sim \log_{1/q} n$ + **a bounded periodic function.**
- **Variance:** $\mathbb{V}(S_n) \sim$ **a bounded periodic function.**
- **Limit distribution does not exist:** $\mathbb{E}\left(e^{(X_n - \log_{1/q} n)y}\right) \sim F(\log_{1/q} n; y)$, **where**

$$F(u; y) := \frac{1-e^y}{\log(1/q)} \left(\prod_{\ell \geqslant 1} \frac{1-e^y q^\ell}{1-q^\ell}\right) \sum_{j \in \mathbb{Z}} \Gamma\left(-\frac{y+2j\pi i}{\log(1/q)}\right) e^{2j\pi i u}.$$

Easy for people in this community

- **Mean:** $\mathbb{E}(S_n) \sim \log_{1/q} n$ + **a bounded periodic function.**
- **Variance:** $\mathbb{V}(S_n) \sim$ **a bounded periodic function.**
- **Limit distribution does not exist:** $\mathbb{E}\left(e^{(X_n - \log_{1/q} n)y}\right) \sim F(\log_{1/q} n; y)$, **where**

$$F(u; y) := \frac{1-e^y}{\log(1/q)} \left(\prod_{\ell \geqslant 1} \frac{1-e^y q^\ell}{1-q^\ell}\right) \sum_{j \in \mathbb{Z}} \Gamma\left(-\frac{y+2j\pi i}{\log(1/q)}\right) e^{2j\pi i u}.$$

Easy for people in this community

- **Mean:** $\mathbb{E}(S_n) \sim \log_{1/q} n$ + **a bounded periodic function.**
- **Variance:** $\mathbb{V}(S_n) \sim$ **a bounded periodic function.**
- **Limit distribution does not exist:** $\mathbb{E}\left(e^{(X_n - \log_{1/q} n)y}\right) \sim F(\log_{1/q} n; y)$, **where**

$$F(u; y) := \frac{1 - e^y}{\log(1/q)} \left(\prod_{\ell \geqslant 1} \frac{1 - e^y q^\ell}{1 - q^\ell}\right) \sum_{j \in \mathbb{Z}} \Gamma\left(-\frac{y + 2j\pi i}{\log(1/q)}\right) e^{2j\pi i u}.$$

A BETTER ALGORITHM?

Goodness of GREEDY IS

Grimmett and McDiarmid (1975), Karp (1976), Fernandez de la Vega (1984), Gazmuri (1984), McDiarmid (1984):

Asymptotically, the GREEDY IS is half optimal.

Can we do better?

Frieze and McDiarmid (1997, *RSA*), Algorithmic theory of random graphs, Research Problem 15:

Construct a polynomial time algorithm that finds an independent set of size at least $(\frac{1}{2}+\varepsilon)|MIS_n|$ whp or show that such an algorithm does not exist modulo some reasonable conjecture in the theory of computational complexity such as, e.g., $P \neq NP$.

A BETTER ALGORITHM?

Goodness of GREEDY IS

Grimmett and McDiarmid (1975), Karp (1976), Fernandez de la Vega (1984), Gazmuri (1984), McDiarmid (1984):

Asymptotically, the GREEDY IS is half optimal.

Can we do better?

Frieze and McDiarmid (1997, *RSA*), Algorithmic theory of random graphs, Research Problem 15:

Construct a polynomial time algorithm that finds an independent set of size at least $(\frac{1}{2} + \varepsilon)|MIS_n|$ whp or show that such an algorithm does not exist modulo some reasonable conjecture in the theory of computational complexity such as, e.g., $P \neq NP$.

A BETTER ALGORITHM?

Goodness of GREEDY IS

Grimmett and McDiarmid (1975), Karp (1976), Fernandez de la Vega (1984), Gazmuri (1984), McDiarmid (1984):

Asymptotically, the GREEDY IS is half optimal.

Can we do better?

Frieze and McDiarmid (1997, *RSA*), Algorithmic theory of random graphs, Research Problem 15:
Construct a polynomial time algorithm that finds an independent set of size at least $(\frac{1}{2} + \varepsilon)|MIS_n|$ whp or show that such an algorithm does not exist modulo some reasonable conjecture in the theory of computational complexity such as, e.g., $P \neq NP$.

A degenerate form of simulated annealing

Sequentially increase the clique (K) size by: (i) ***choose a vertex*** v ***u.a.r. from V;*** (ii) ***if*** $v \notin K$ ***and*** v ***connected to every vertex of*** K***, then add*** v ***to*** K***;*** (iii) ***if*** $v \in K$***, then*** v ***is subtracted from*** K ***with probability*** λ^{-1}***.***

He showed: $\forall \lambda \geqslant 1, \exists$ **an initial state from which the expected time for the Metropolis process to reach a clique of size at least** $(1+\varepsilon)\log_{1/q}(pn)$ **exceeds** $n^{\Omega(\log pn)}$**.**

$$n^{\log n} = e^{(\log n)^2}$$

JERRUM'S (1992) METROPOLIS ALGORITHM

A degenerate form of simulated annealing

Sequentially increase the clique (K) size by: (i) ***choose a vertex*** v ***u.a.r. from V;*** (ii) ***if*** $v \notin K$ ***and*** v ***connected to every vertex of*** K***, then add*** v ***to*** K***;*** (iii) ***if*** $v \in K$***, then*** v ***is subtracted from*** K ***with probability*** λ^{-1}***.***

He showed: $\forall \lambda \geqslant 1, \exists$ **an initial state from which the expected time for the Metropolis process to reach a clique of size at least** $(1+\varepsilon)\log_{1/q}(pn)$ **exceeds** $n^{\Omega(\log pn)}$**.**

$$n^{\log n} = e^{(\log n)^2}$$

POSITIVE RESULTS

Exact algorithms

A huge number of algorithms proposed in the literature; see Bomze et al.'s survey (in *Handbook of Combinatorial Optimization*, 1999).

Special algorithms

- **Wilf's (1986) *Algorithms and Complexity* describes a *backtracking* algorithms enumerating all independent sets with time complexity $n^{O(\log n)}$.**
- **Chvátal (1977) proposes *exhaustive* algorithms where almost all $G_{n,1/2}$ creates at most $n^{2(1+\log_2 n)}$ subproblems.**
- **Pittel (1982):**

$$\mathbb{P}\left(n^{\frac{1-\varepsilon}{4}\log_{1/q} n} \leqslant \text{Time}^{\text{used by}}_{\text{Chvátal's algo}} \leqslant n^{\frac{1+\varepsilon}{2}\log_{1/q} n}\right) \geqslant 1 - e^{-c\log^2 n}$$

POSITIVE RESULTS

Exact algorithms

A huge number of algorithms proposed in the literature; see Bomze et al.'s survey (in *Handbook of Combinatorial Optimization*, 1999).

Special algorithms

- **Wilf's (1986) *Algorithms and Complexity* describes a *backtracking* algorithms enumerating all independent sets with time complexity $n^{O(\log n)}$.**
- **Chvátal (1977) proposes *exhaustive* algorithms where almost all $G_{n,1/2}$ creates at most $n^{2(1+\log_2 n)}$ subproblems.**
- **Pittel (1982):**

$$\mathbb{P}\left(n^{\frac{1-\varepsilon}{4}\log_{1/q} n} \leqslant \text{Time}^{\text{used by}}_{\text{Chvátal's algo}} \leqslant n^{\frac{1+\varepsilon}{2}\log_{1/q} n}\right) \geqslant 1 - e^{-c\log^2 n}$$

POSITIVE RESULTS

Exact algorithms

A huge number of algorithms proposed in the literature; see Bomze et al.'s survey (in *Handbook of Combinatorial Optimization*, 1999).

Special algorithms

- **Wilf's (1986) *Algorithms and Complexity* describes a *backtracking* algorithms enumerating all independent sets with time complexity $n^{O(\log n)}$.**
- **Chvátal (1977) proposes *exhaustive* algorithms where almost all $G_{n,1/2}$ creates at most $n^{2(1+\log_2 n)}$ subproblems.**
- **Pittel (1982):**

$$\mathbb{P}\left(n^{\frac{1-\varepsilon}{4}\log_{1/q} n} \leqslant \text{Time}^{\text{used by}}_{\text{Chvátal's algo}} \leqslant n^{\frac{1+\varepsilon}{2}\log_{1/q} n}\right) \geqslant 1 - e^{-c\log^2 n}$$

AIM: A MORE PRECISE ANALYSIS OF THE EXHAUSTIVE ALGORITHM

MIS contains either v or not

$$X_n \stackrel{d}{=} X_{n-1} + X^*_{n-1-\mathbf{Binom}(n-1;p)} \qquad (n \geqslant 2),$$

with $X_0 = 0$ and $X_1 = 1$.

Special cases

- **If p is close to 1, then the second term is small, so we expect a *polynomial* time bound.**
- **If p is sufficiently small, then the second term is large, and we expect an *exponential* time bound.**

AIM: A MORE PRECISE ANALYSIS OF THE EXHAUSTIVE ALGORITHM

MIS contains either v or not

$$X_n \stackrel{d}{=} X_{n-1} + X^*_{n-1-\mathbf{Binom}(n-1;p)} \qquad (n \geqslant 2),$$

with $X_0 = 0$ and $X_1 = 1$.

Special cases

- **If p is close to 1, then the second term is small, so we expect a *polynomial* time bound.**
- **If p is sufficiently small, then the second term is large, and we expect an *exponential* time bound.**
- **What happens for p in between?**

AIM: A MORE PRECISE ANALYSIS OF THE EXHAUSTIVE ALGORITHM

MIS contains either v or not

$$X_n \stackrel{d}{=} X_{n-1} + X^*_{n-1-\textbf{Binom}(n-1;p)} \qquad (n \geqslant 2),$$

with $X_0 = 0$ and $X_1 = 1$.

Special cases

- **If p is close to 1, then the second term is small, so we expect a *polynomial* time bound.**
- **If p is sufficiently small, then the second term is large, and we expect an *exponential* time bound.**
- **What happens for p in between?**

AIM: A MORE PRECISE ANALYSIS OF THE EXHAUSTIVE ALGORITHM

MIS contains either v or not

$$X_n \stackrel{d}{=} X_{n-1} + X^*_{n-1-\textbf{Binom}(n-1;p)} \qquad (n \geqslant 2),$$

with $X_0 = 0$ and $X_1 = 1$.

Special cases

- **If p is close to 1, then the second term is small, so we expect a *polynomial* time bound.**
- **If p is sufficiently small, then the second term is large, and we expect an *exponential* time bound.**
- **What happens for p in between?**

The expected value $\mu_n := \mathbb{E}(X_n)$ satisfies

$$\mu_n = \mu_{n-1} + \sum_{0\leqslant j<n} \binom{n-1}{j} p^j q^{n-1-j} \mu_{n-1-j}.$$

with $\mu_0 = 0$ and $\mu_1 = 1$.

Poisson generating function

Let $\tilde{f}(z) := e^{-z} \sum_{n\geqslant 0} \mu_n z^n/n!$. Then

$$\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}.$$

MEAN VALUE

The expected value $\mu_n := \mathbb{E}(X_n)$ satisfies

$$\mu_n = \mu_{n-1} + \sum_{0\leqslant j<n} \binom{n-1}{j} p^j q^{n-1-j} \mu_{n-1-j}.$$

with $\mu_0 = 0$ and $\mu_1 = 1$.

Poisson generating function

Let $\tilde{f}(z) := e^{-z} \sum_{n\geqslant 0} \mu_n z^n/n!$. Then

$$\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}.$$

Laplace transform

The Laplace transform of $\tilde{f}$

$$\mathscr{L}(s) = \int_0^\infty e^{-xs}\tilde{f}(x)\,\mathbf{d}x$$

satisfies

$$s\mathscr{L}(s) = \frac{1}{q}\mathscr{L}\left(\frac{s}{q}\right) + \frac{1}{s+1}.$$

Exact solutions

$$\mathscr{L}(s) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

Laplace transform

The Laplace transform of $\tilde{f}$

$$\mathscr{L}(s) = \int_0^\infty e^{-xs}\tilde{f}(x)\,\mathbf{d}x$$

satisfies

$$s\mathscr{L}(s) = \frac{1}{q}\mathscr{L}\left(\frac{s}{q}\right) + \frac{1}{s+1}.$$

Exact solutions

$$\mathscr{L}(s) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

Exact solutions

$$\mathscr{L}(s) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

Inverting gives $\tilde{f}(z) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{j!} z^{j+1} \int_0^1 e^{-q^j uz}(1-u)^j \, \mathrm{d}u$**.**

Thus $\mu_n = \sum_{1\leqslant j\leqslant n} \binom{n}{j}(-1)^j \sum_{1\leqslant \ell\leqslant j} (-1)^\ell q^{j(\ell-1)-\binom{\ell}{2}}$**, or**

$$\mu_n = n \sum_{0\leqslant j<n} \binom{n-1}{j} q^{\binom{j+1}{2}} \sum_{0\leqslant \ell<n-j} \binom{n-1-j}{\ell} \frac{q^{j\ell}(1-q^j)^{n-1-j-\ell}}{j+\ell+1}.$$

Neither is useful for numerical purposes for large n***.***

Exact solutions

$$\mathscr{L}(s) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

Inverting gives $\tilde{f}(z) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{j!} z^{j+1} \int_0^1 e^{-q^j uz}(1-u)^j \, du$**.**

Thus $\mu_n = \sum_{1\leqslant j\leqslant n} \binom{n}{j}(-1)^j \sum_{1\leqslant \ell\leqslant j}(-1)^\ell q^{j(\ell-1)-\binom{\ell}{2}}$**, or**

$$\mu_n = n \sum_{0\leqslant j<n} \binom{n-1}{j} q^{\binom{j+1}{2}} \sum_{0\leqslant \ell<n-j} \binom{n-1-j}{\ell} \frac{q^{j\ell}(1-q^j)^{n-1-j-\ell}}{j+\ell+1}.$$

Neither is useful for numerical purposes for large n.

RESOLUTION OF THE RECURRENCE

Exact solutions

$$\mathscr{L}(s) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

Inverting gives $\tilde{f}(z) = \sum_{j\geqslant 0} \frac{q^{\binom{j+1}{2}}}{j!} z^{j+1} \int_0^1 e^{-q^j uz}(1-u)^j \, \mathrm{d}u$**.**

Thus $\mu_n = \sum_{1\leqslant j\leqslant n} \binom{n}{j}(-1)^j \sum_{1\leqslant \ell\leqslant j} (-1)^\ell q^{j(\ell-1)-\binom{\ell}{2}}$**, or**

$$\mu_n = n \sum_{0\leqslant j<n} \binom{n-1}{j} q^{\binom{j+1}{2}} \sum_{0\leqslant \ell<n-j} \binom{n-1-j}{\ell} \frac{q^{j\ell}(1-q^j)^{n-1-j-\ell}}{j+\ell+1}.$$

Neither is useful for numerical purposes for large n***.***

Back-of-the-envelope calculation

Take $q = 1/2$. Since Binom$(n-1; \frac{1}{2})$ has mean $n/2$, we roughly have

$$\mu_n \approx \mu_{n-1} + \mu_{\lfloor n/2 \rfloor}.$$

This is reminiscent of Mahler's partition problem. Indeed, if $\varphi(z) = \sum_n \mu_n z^n$, then

$$\varphi(z) \approx \frac{1+z}{1-z}\varphi(z^2) = \prod_{j \geqslant 0} \frac{1}{1 - z^{2^j}}.$$

So we expect that (de Bruijn, 1948; Dumas and Flajolet, 1996)

$$\log \mu_n \approx c\left(\log \frac{n}{\log_2 n}\right)^2 + c' \log n + c'' \log\log n + \textbf{Periodic}_n.$$

Back-of-the-envelope calculation

Take $q = 1/2$. Since Binom$(n-1; \frac{1}{2})$ has mean $n/2$, we roughly have

$$\mu_n \approx \mu_{n-1} + \mu_{\lfloor n/2 \rfloor}.$$

This is reminiscent of Mahler's partition problem. Indeed, if $\varphi(z) = \sum_n \mu_n z^n$, then

$$\varphi(z) \approx \frac{1+z}{1-z}\varphi(z^2) = \prod_{j \geqslant 0} \frac{1}{1-z^{2^j}}.$$

So we expect that (de Bruijn, 1948; Dumas and Flajolet, 1996)

$$\log \mu_n \approx c\left(\log \frac{n}{\log_2 n}\right)^2 + c' \log n + c'' \log\log n + \textbf{Periodic}_n.$$

Back-of-the-envelope calculation

Take $q = 1/2$. Since Binom$(n-1; \frac{1}{2})$ has mean $n/2$, we roughly have

$$\mu_n \approx \mu_{n-1} + \mu_{\lfloor n/2 \rfloor}.$$

This is reminiscent of Mahler's partition problem. Indeed, if $\varphi(z) = \sum_n \mu_n z^n$, then

$$\varphi(z) \approx \frac{1+z}{1-z}\varphi(z^2) = \prod_{j\geqslant 0} \frac{1}{1-z^{2^j}}.$$

So we expect that (de Bruijn, 1948; Dumas and Flajolet, 1996)

$$\log \mu_n \approx c\left(\log \frac{n}{\log_2 n}\right)^2 + c' \log n + c'' \log\log n + \textbf{Periodic}_n.$$

ASYMPTOTICS OF μ_n

Poisson heuristic (de-Poissonization, saddle-point method)

$$\mu_n = \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z\tilde{f}(z)\,\mathbf{d}z$$

$$\approx \sum_{j\geqslant 0}\frac{\tilde{f}^{(j)}(n)}{j!}\frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z(z-n)^j\,\mathbf{d}z$$

$$= \tilde{f}(n) + \sum_{j\geqslant 2}\frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n),$$

where $\tau_j(n) := n![z^n]e^z(z-n)^j = j![z^j](1+z)^n e^{-nz}$ **(Charlier polynomials). In particular,** $\tau_0(n) = 1$, $\tau_1(n) = 0$, $\tau_2(n) = -n$, $\tau_3(n) = 2n$, **and** $\tau_4(n) = 3n^2 - 6n$.

Poisson heuristic (de-Poissonization, saddle-point method)

$$\begin{aligned}\mu_n &= \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1} e^z \tilde{f}(z)\, \mathbf{d}z \\ &\approx \sum_{j\geqslant 0} \frac{\tilde{f}^{(j)}(n)}{j!} \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1} e^z (z-n)^j\, \mathbf{d}z \\ &= \tilde{f}(n) + \sum_{j\geqslant 2} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n),\end{aligned}$$

where $\tau_j(n) := n![z^n]e^z(z-n)^j = j![z^j](1+z)^n e^{-nz}$ **(Charlier polynomials). In particular,** $\tau_0(n) = 1$**,** $\tau_1(n) = 0$**,** $\tau_2(n) = -n$**,** $\tau_3(n) = 2n$**, and** $\tau_4(n) = 3n^2 - 6n$**.**

Asymptotics of $\tilde{f}(x)$

Let $\rho = 1/\log(1/q)$ **and** $R \log R = x/\rho$**. Then**

$$\tilde{f}(x) \sim \frac{R^{\rho+1/2} e^{(\rho/2)(\log R)^2} G(\rho \log R)}{\sqrt{2\pi\rho \log R}} \left(1 + \sum_{j\geqslant 1} \frac{\phi_j(\rho \log R)}{(\rho \log R)^j}\right),$$

as $x \to \infty$**, where** $G(u) := q^{(\{u\}^2+\{u\})/2} F(q^{-\{u\}})$**,**

$$F(s) = \sum_{-\infty<j<\infty} \frac{q^{j(j+1)/2}}{1+q^j s} s^{j+1},$$

and the $\phi_j(u)$**'s are bounded,** 1**-periodic functions of** u **involving the derivatives** $F^{(j)}(q^{-\{u\}})$**.**

A MORE EXPLICIT ASYMPTOTIC APPROXIMATION

$R = x/\rho/W(x/\rho)$, Lambert's W-function

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + \frac{(\log\log x)^2 - 2\log\log x}{2(\log x)^2} + \cdots .$$

So that

$$\tilde{f}(x) \sim \frac{x^{\rho+1/2} G\left(\rho \log \frac{x/\rho}{\log(x/\rho)}\right)}{\sqrt{2\pi}\,\rho^{\rho+1/2}\log x} \exp\left(\frac{\rho}{2}\left(\log \frac{x/\rho}{\log(x/\rho)}\right)^2\right).$$

Method of proof: a variant of the saddle-point method

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} e^{sz}\mathscr{L}(s)\,\mathrm{d}s$$

$R = x/\rho/W(x/\rho)$, Lambert's W-function

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + \frac{(\log\log x)^2 - 2\log\log x}{2(\log x)^2} + \cdots.$$

So that

$$\tilde{f}(x) \sim \frac{x^{\rho+1/2} G\left(\rho \log \frac{x/\rho}{\log(x/\rho)}\right)}{\sqrt{2\pi}\rho^{\rho+1/2}\log x} \exp\left(\frac{\rho}{2}\left(\log\frac{x/\rho}{\log(x/\rho)}\right)^2\right).$$

Method of proof: a variant of the saddle-point method

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} e^{sz}\mathscr{L}(s)\,\mathrm{d}s$$

$R = x/\rho/W(x/\rho)$, Lambert's W-function

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + \frac{(\log\log x)^2 - 2\log\log x}{2(\log x)^2} + \cdots .$$

So that

$$\tilde{f}(x) \sim \frac{x^{\rho+1/2} G\left(\rho \log \frac{x/\rho}{\log(x/\rho)}\right)}{\sqrt{2\pi}\rho^{\rho+1/2}\log x} \exp\left(\frac{\rho}{2}\left(\log \frac{x/\rho}{\log(x/\rho)}\right)^2\right).$$

Method of proof: a variant of the saddle-point method

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} e^{sz}\mathscr{L}(s)\,\mathbf{d}s$$

Four properties are sufficient

The following four properties are enough to justify the Poisson-Charlier expansion.

- $\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}$;
- $F(s) = sF(qs)$ ($F(s) = \sum_{i\in\mathbb{Z}} q^{j(j+1)/2} s^{j+1}/(1+q^j s)$);
- $\dfrac{\tilde{f}^{(j)}(x)}{\tilde{f}(x)} \sim \left(\dfrac{\rho \log x}{x}\right)^j$;
- $|f(z)| \leqslant f(|z|)$ **where** $f(z) := e^z \tilde{f}(z)$.

Thus ($\rho = 1/\log(1/q)$)

$$\mu_n \sim \frac{n^{\rho+1/2} G\left(\rho \log \frac{n/\rho}{\log(n/\rho)}\right)}{\sqrt{2\pi}\rho^{\rho+1/2}\log n} \exp\left(\frac{\rho}{2}\left(\log \frac{n/\rho}{\log(n/\rho)}\right)^2\right).$$

JUSTIFICATION OF THE POISSON HEURISTIC

Four properties are sufficient

The following four properties are enough to justify the Poisson-Charlier expansion.

- $\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}$;
- $F(s) = sF(qs)$ ($F(s) = \sum_{i\in\mathbb{Z}} q^{j(j+1)/2} s^{j+1}/(1+q^j s)$);
- $\dfrac{\tilde{f}^{(j)}(x)}{\tilde{f}(x)} \sim \left(\dfrac{\rho \log x}{x}\right)^j$;
- $|f(z)| \leqslant f(|z|)$ **where** $f(z) := e^z \tilde{f}(z)$.

Thus ($\rho = 1/\log(1/q)$)

$$\mu_n \sim \frac{n^{\rho+1/2} G\left(\rho \log \frac{n/\rho}{\log(n/\rho)}\right)}{\sqrt{2\pi}\rho^{\rho+1/2}\log n} \exp\left(\frac{\rho}{2}\left(\log \frac{n/\rho}{\log(n/\rho)}\right)^2\right).$$

VARIANCE OF X_n

$\sigma_n := \sqrt{\mathbb{V}(X_n)}$

$$\sigma_n^2 = \sigma_{n-1}^2 + \sum_{0\leqslant j<n} \pi_{n,j}\sigma_{n-1-j}^2 + T_n, \quad \pi_{n,j} := \binom{n-1}{j} p^j q^{n-1-j},$$

where $T_n := \sum_{0\leqslant j<n} \pi_{n,j}\Delta_{n,j}^2$, $\Delta_{n,j} := \mu_j + \mu_{n-1} - \mu_n$.

Asymptotic transfer: $a_n = a_{n-1} + \sum_{0\leqslant j<n} \pi_{n,j} a_{n-1-j} + b_n$

If $b_n \sim n^\beta (\log n)^\kappa \tilde{f}(n)^\alpha$, **where** $\alpha > 1$, $\beta, \kappa \in \mathbb{R}$, **then**

$$a_n \sim \sum_{j\leqslant n} b_j \sim \frac{n}{\alpha\rho \log n} b_n.$$

VARIANCE OF X_n

$\sigma_n := \sqrt{\mathbb{V}(X_n)}$

$$\sigma_n^2 = \sigma_{n-1}^2 + \sum_{0 \leqslant j < n} \pi_{n,j} \sigma_{n-1-j}^2 + T_n, \quad \pi_{n,j} := \binom{n-1}{j} p^j q^{n-1-j},$$

where $T_n := \sum_{0 \leqslant j < n} \pi_{n,j} \Delta_{n,j}^2$, $\Delta_{n,j} := \mu_j + \mu_{n-1} - \mu_n$.

Asymptotic transfer: $a_n = a_{n-1} + \sum_{0 \leqslant j < n} \pi_{n,j} a_{n-1-j} + b_n$

If $b_n \sim n^\beta (\log n)^\kappa \tilde{f}(n)^\alpha$, **where** $\alpha > 1$, $\beta, \kappa \in \mathbb{R}$, **then**

$$a_n \sim \sum_{j \leqslant n} b_j \sim \frac{n}{\alpha \rho \log n} b_n.$$

ASYMPTOTICS OF THE VARIANCE

Asymptotics of T_n: by elementary means

$$T_n \sim q^{-1} p \rho^4 n^{-3} (\log n)^4 \tilde{f}(n)^2.$$

Applying the asymptotic transfer

$$\sigma_n^2 \sim C n^{-2} (\log n)^3 \tilde{f}(n)^2.$$

where $C := p\rho^3/(2q)$.

$$\frac{\textbf{Variance}}{\textbf{Mean}^2} \sim C \frac{(\log n)^3}{n^2}$$

Asymptotics of T_n: by elementary means

$$T_n \sim q^{-1} p\rho^4 n^{-3} (\log n)^4 \tilde{f}(n)^2.$$

Applying the asymptotic transfer

$$\sigma_n^2 \sim C n^{-2} (\log n)^3 \tilde{f}(n)^2.$$

where $C := p\rho^3/(2q)$**.**

$$\frac{\textbf{Variance}}{\textbf{Mean}^2} \sim C \frac{(\log n)^3}{n^2}$$

ASYMPTOTICS OF THE VARIANCE

Asymptotics of T_n: by elementary means

$$T_n \sim q^{-1} p\rho^4 n^{-3} (\log n)^4 \tilde{f}(n)^2.$$

Applying the asymptotic transfer

$$\sigma_n^2 \sim C n^{-2} (\log n)^3 \tilde{f}(n)^2.$$

where $C := p\rho^3/(2q)$.

$$\frac{\textbf{Variance}}{\textbf{Mean}^2} \sim C \frac{(\log n)^3}{n^2}$$

ASYMPTOTIC NORMALITY OF X_n

Convergence in distribution

The distribution of X_n is asymptotically normal

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathscr{N}(0, 1),$$

with convergence of all moments.

Proof by the method of moments

- **Derive recurrence for $\mathbb{E}(X_n - \mu_n)^m$.**
- **Prove by induction (using the asymptotic transfer) that**

$$\mathbb{E}(X_n - \mu_n)^m \begin{cases} \sim \frac{(m)!}{(m/2)!2^{m/2}}\,\sigma_n^m, & \text{if } 2 \mid m, \\ = o(\sigma_n^m), & \text{if } 2 \nmid m, \end{cases}$$

ASYMPTOTIC NORMALITY OF X_n

Convergence in distribution

The distribution of X_n is asymptotically normal

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathscr{N}(0,1),$$

with convergence of all moments.

Proof by the method of moments

- **Derive recurrence for $\mathbb{E}(X_n - \mu_n)^m$.**
- **Prove by induction (using the asymptotic transfer) that**

$$\mathbb{E}(X_n - \mu_n)^m \begin{cases} \sim \dfrac{(m)!}{(m/2)!2^{m/2}}\,\sigma_n^m, & \text{if } 2 \mid m, \\ = o(\sigma_n^m), & \text{if } 2 \nmid m, \end{cases}$$

A STRAIGHTFORWARD EXTENSION

$b = 1, 2, \ldots$

$$X_n \overset{d}{=} X_{n-b} + X^*_{n-b-\mathbf{Binom}(n-b;p)},$$

with $X_n = 0$ for $n < b$ and $X_b = 1$.

For example, MAXIMUM TRIANGLE PARTITION:

$$X_n \overset{d}{=} X_{n-3} + X^*_{n-3-\mathbf{Binom}(n-3;p^3)},$$

The same tools we developed apply

X_n asymptotically normally distributed with mean and variance of the same order as the case $b = 1$.

A STRAIGHTFORWARD EXTENSION

$b = 1, 2, \ldots$

$$X_n \stackrel{d}{=} X_{n-b} + X^*_{n-b-\mathbf{Binom}(n-b;p)},$$

with $X_n = 0$ for $n < b$ and $X_b = 1$.

For example, MAXIMUM TRIANGLE PARTITION:

$$X_n \stackrel{d}{=} X_{n-3} + X^*_{n-3-\mathbf{Binom}(n-3;p^3)},$$

The same tools we developed apply

X_n asymptotically normally distributed with mean and variance of the same order as the case $b = 1$.

A STRAIGHTFORWARD EXTENSION

$b = 1, 2, \ldots$

$$X_n \overset{d}{=} X_{n-b} + X^*_{n-b-\mathbf{Binom}(n-b;p)},$$

with $X_n = 0$ for $n < b$ and $X_b = 1$.

For example, MAXIMUM TRIANGLE PARTITION:

$$X_n \overset{d}{=} X_{n-3} + X^*_{n-3-\mathbf{Binom}(n-3;p^3)},$$

The same tools we developed apply

X_n asymptotically normally distributed with mean and variance of the same order as the case $b = 1$.

A NATURAL VARIANT

What happens if $X_n \stackrel{d}{=} X_{n-1} + X^*_{\text{uniform[0,n-1]}}$?

$$\mu_n = \mu_{n-1} + \frac{1}{n} \sum_{0 \leqslant j < n} \mu_j,$$

satisfies $\mu_n \sim cn^{-1/4} e^{2\sqrt{n}}$**. Note:** $\mu_n \approx \mu_{n-1} + \mu_{n/2}$ **fails.**

Limit law not Gaussian (by method of moments)

$$\frac{X_n}{\mu_n} \xrightarrow{d} X,$$

where $g(z) := \sum_{m \geqslant 1} \mathbb{E}(X^m) z^m / (m \cdot m!)$ **satisfies**

$$z^2 g'' + zg' - g = zgg'.$$

A NATURAL VARIANT

What happens if $X_n \stackrel{d}{=} X_{n-1} + X^*_{\text{uniform[0,n-1]}}$?

$$\mu_n = \mu_{n-1} + \frac{1}{n} \sum_{0 \leqslant j < n} \mu_j,$$

satisfies $\mu_n \sim cn^{-1/4} e^{2\sqrt{n}}$**. Note:** $\mu_n \approx \mu_{n-1} + \mu_{n/2}$ **fails.**

Limit law not Gaussian (by method of moments)

$$\frac{X_n}{\mu_n} \xrightarrow{d} X,$$

where $g(z) := \sum_{m \geqslant 1} \mathbb{E}(X^m) z^m / (m \cdot m!)$ **satisfies**

$$z^2 g'' + zg' - g = zgg'.$$

A NATURAL VARIANT

What happens if $X_n \stackrel{d}{=} X_{n-1} + X^*_{\text{uniform[0,n-1]}}$?

$$\mu_n = \mu_{n-1} + \frac{1}{n} \sum_{0 \leqslant j < n} \mu_j,$$

satisfies $\mu_n \sim cn^{-1/4} e^{2\sqrt{n}}$. **Note:** $\mu_n \approx \mu_{n-1} + \mu_{n/2}$ **fails.**

Limit law not Gaussian (by method of moments)

$$\frac{X_n}{\mu_n} \xrightarrow{d} X,$$

where $g(z) := \sum_{m \geqslant 1} \mathbb{E}(X^m) z^m / (m \cdot m!)$ **satisfies**

$$z^2 g'' + zg' - g = zgg'.$$

Random graph algorithms:
a rich source of interesting recurrences

Obrigado!

Random graph algorithms:
a rich source of interesting recurrences

Obrigado!