BIORTHOGONALITY IN BANACH SPACES AND THE P-IDEAL DICHOTOMY

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Abstract. We analyse the impact of the P-ideal dichotomy to the existence of uncountable biorthogonal sistems in nonseparable Banach spaces. Our main result states that under the P-ideal dichotomy, any nonseparable Banach space of density strictly smaller than $\mathfrak{b}$ with weak* sequentially compact dual unit ball has an uncountable biorthogonal system. As a consequence, we get that the following are equivalent to the cardinal assumption $\mathfrak{b} = \omega_2$ assuming the PID: Every nonseparable Asplund space has an uncountable biorthogonal systems; Every nonseparable Asplund space has a nonseparable quotient with monotone Schauder basis. We add to this list few other statements of this kind related to the renorming of a given Asplund space with an equivalent norm with the Mazur intersection property.

1. Introduction

Given an uncountable set S, a P-ideal in S is a collection $\mathcal{I}$ of countable infinite subsets of S which form an ideal under inclusion and for every sequence $(I_n)_{n\in\omega}$ in $\mathcal{I}$ there is $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \omega$.

The second author introduced the following principle in [18]:

Definition 1 (P-Ideal Dichotomy)**.** *For every P-ideal $\mathcal{I}$ of countable infinite subsets of some (uncountable) set S, either*

(i) there is an uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathcal{I}$; or
(ii) $S = \bigcup_{n\in\omega} S_n$ where for every $n \in \omega$ and any $I \in \mathcal{I}$, $|S_n \cap I| < \omega$.

It has been proven in [18] that the P-Ideal Dichotomy (or simply PID) follows from the Proper Forcing Axiom and is consistent with the Generalized Continuum Hypothesis (relative to the consistency of a supercompact cardinal). When the index set S is equal to ω_1, the corresponding principle PID is consistent just relative to ZFC, see [1]. Many applications of PID have been obtained since it was first introduced, see [20]. In particular, several recent results assume PID to establish

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the equivalence between cardinal inequalities of the form $\mathfrak{t} > \omega_1$ where $\mathfrak{t}$ is a characteristic of the continuum and some mathematical statement concerning objects of higher order (i.e., object that are not necessarily reals), see e.g., [1, 3, 14, 18].

The present work goes in this direction, establishing, under the PID, the equivalence between one of the most basic cardinal assumptions of this kind and the existence of some uncountable structures in nonseparable Asplund spaces. For example, we show that, under PID, every nonseparable Aslund space has an uncountable biorthogonal system if and only if $\mathfrak{b} > \omega_1$, where $\mathfrak{b}$ stands for the bounding number, i.e. the minimal cardinality of an unbounded family in ω^ω with respect to the eventual dominance preorder $\leq^*$. One of the important influences of PID on this standard cardinal invariant of the continuum is that $\mathfrak{b} \in \{\omega_1, \omega_2\}$; see [20]. Hence, the following assertions are equivalent under PID:

(a_1) $\mathfrak{b} > \omega_1$.
(a_2) $\mathfrak{b} = \omega_2$.

and therefore in all the equivalences that we have, the cardinal inequality $\mathfrak{b} > \omega_1$ can, in fact, be replaced by the cardinal equality $\mathfrak{b} = \omega_2$. It would be interesting to see if this phenomenon is also true for other inequalities of the form $\mathfrak{t} > \omega_1$ which PID connects to some standard problems in mathematics.

In a different direction, there has been a lot of research to understand the existence of certain uncountable structures in nonseparabale Banach spaces. Given $0 \leq \varepsilon < 1$, an ε-biorthogonal system in a Banach space X is a family of pairs $(x_\alpha, f_\alpha)_{\alpha\in\kappa}$ of $X \times X^*$ such that $f_\alpha(x_\alpha) = 1$ and $|f_\alpha(x_\beta)| \leq \varepsilon$ for all $\alpha \neq \beta$ in κ. It is a biorthogonal system in the particular case when $\varepsilon = 0$. A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha\in\kappa}$ is fundamental if $\overline{span}\{x_\alpha : \alpha \in \kappa\} = X$.

It is well known that every infinite dimensional Banach space X admits an infinite biorthogonal system and if X is separable, a biorthogonal system $(x_n, f_n)_{n\in\omega}$ can be chosen so that $\overline{span}\{x_n : n \in \omega\} = X$ and $\overline{span}^{w^*}\{f_n : n \in \omega\} = X^*$.

Let us consider the following pairs of assertions:

(b_1) Every Banach space of density ω_1 has an uncountable ε-biorthogonal system, for every $\varepsilon \in (0, 1)$.
(b_2) Every nonseparable Banach space has an uncountable ε-biorthogonal system, for every $\varepsilon \in (0, 1)$.

(c_1) Every Banach space of density ω_1 has an uncountable biorthogonal system.
(c_2) Every nonseparable Banach space has an uncountable biorthogonal system.

(d_1) Every Banach space of density ω_1 has a fundamental biorthogonal system.
(d_2) Every Banach space of density ω_1 has a nonseparable quotient with a monotone Schauder basis.

Remark 2. *$(b_1) \Leftrightarrow (b_2)$ and $(c_1) \Leftrightarrow (c_2)$ hold in ZFC because of Hahn-Banach Theorem. $(d_1) \Leftrightarrow (d_2)$ holds in ZFC thanks to a result by Plichko,* [13] *- see also* [7, Theorem 4.15].

From now on, we refer to (b), (c) and (d) meaning any of the two equivalent corresponding statements above, and also to (a) to refer to any of the assertions (a_1) and (a_2), whenever we are working under the PID.

The main purpose of this paper is to prove the following result.

Main Theorem. *The following statements are equivalent assuming the P-ideal dichotomy:*

- (a) $\mathfrak{b} = \omega_2$.
- (b') *Every nonseparable Asplund space has an uncountable ε-biorthogonal system, for every $\varepsilon \in (0, 1)$.*
- (c') *Every nonseparable Asplund space has an uncountable biorthogonal system.*
- (d') *Every Asplund space of density ω_1 has a fundamental biorthogonal system.*
- (e') *The dual of every Asplund space of density ω_1 has a fundamental biorthogonal system.*

Notice that the statements $(b')-(d')$ correspond to $(b)-(d)$ restricted to Asplund spaces. Asplund spaces, also called strong differentiability spaces, are those Banach spaces X such that any convex continuous function on X is Fréchet-differentiable in a dense set. This is equivalent to say that all the separable subspaces of X have separable duals. Hence, subspaces of Asplund spaces are Asplund and (b_1) (respectively $(c_1), (d_1)$) remains equivalent to (b_2) (respectively $(c_2), (d_2)$) when "Banach space" is replaced by "Asplund space". One interesting feature of our Main Theorem is that it connects phenomena in two different parts of mathematics, a connection that would have been difficult to guess without PID at disposal. However note that this theorem connects also two known phenomena (b') and (c') from the geometry of Asplund spaces. In fact, in the literature (see, e.g., [8],[10]) one finds the question if (b) and (c) are equivalent restrictions on a given Banach space. For example, it is shown in [8] that if a nonseparable Banach space X has an equivalent norm $\|\cdot\|'$ with the Mazur intersection property (stating that every closed convex subset of X is the intersection of closed $\|\cdot\|'$-balls) then for every $\varepsilon > 0$ the space X has an uncountable ε-biorthogonal system and asked if one can get also an uncountable biorthogonal system. The reason for this question is reinforced by another result from [8] saying that if a Banach space X has a biorthogonal system $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$ such that the functionals $(f_\gamma)_{\gamma\in\Gamma}$ span the dual space X^*, then X has a renorming with the Mazur intersection property. On the other hand, it is shown in [10] that there is a generic Asplund space X such that for all $\varepsilon > 0$ there is an uncountable ε-biorthogonal system in X but X has no uncountable biorthogonal system. Thus, some assumption like PID is necessary in our Main Theorem.

The Main Theorem gives a partial answer to the following question, raised by the second author:

Question 3 (Todorcevic, Question 26.7, [20]). *Under the PID, are the following statements equivalent?*

- (a) $\mathfrak{b} > \omega_1$.
- (c) *Every nonseparable Banach space contains an uncountable biorthogonal system.*

This question was motivated, on one side, by the result establishing the consistency that every nonseparable Banach space has an uncountable biorthogonal system. This result was first obtained by the second author in [19] using Martin's Maximum (MM) and then reformulated in [20] by replacing MM by the conjunction of the PID and the cardinal assumption $\mathfrak{p} > \omega_1$.

On the other hand, the maximal cardinality of a biorthogonal system in a given Banach space X and its relation to the density of X have been studied by several authors but have not yet been completely understood. Since the 80's several consistent examples of nonseparable Banach spaces with no uncountable biorthogonal systems have been obtained under assumptions like the Continuum Hypothesis or the combinatorial principles $\diamondsuit$ or $\clubsuit$, see [7], and by forcing, like those of [4, 10]. For a further discussion on this subject see [9]. Among those examples, the one constructed by the second author in [16, Theorem 2.4] under the cardinal assumption $\mathfrak{b} = \omega_1$ is of great importance to us here:

Theorem 4 (Todorcevic, [16], [17])**.** *Assuming $\mathfrak{b} = \omega_1$, there is a compact Hausdorff scattered space K of weight ω_1 whose finite powers K^n are all hereditarily separable.*

The relevance of this result here can briefly be explained as follows. First of all, since the compact space K is scattered, the function space $C(K)$ is Asplund. The fact that finite powers of K are all hereditarily separable is equivalent to the fact that the function space $C(K)$ is hereditarily Lindelöf with respect to the weak topology and so, in particular, it has no ε-biorthogonal system for any $\varepsilon \geq 0$. All this is quite standard and can be found in the references given in the following sketch of proof of the Main Theorem.

Sketch of the proof of the Main Theorem: The implications $(d') \Rightarrow (c') \Rightarrow (b')$ are immediate.

The contrapositive of $(b') \Rightarrow (a)$ follows from Theorem 4: the construction in [16, Theorem 2.4] yields, under $\mathfrak{b} = \omega_1$, a compact Hausdorff scattered space K of weight ω_1 whose finite powers K^n are all hereditarily separable. By a result of Namioka and Phelps [11], from the fact that K is scattered, we can conclude that the Banach space $C(K)$ of all scalar valued continuous functions defined on K, with the supremum norm, is an Asplund space (of density ω_1). As also mentioned above, the fact that K is scattered and has hereditarily separable finite powers implies that $C(K)$ is hereditarily Lindelöf with respect to the weak topology (see e.g. [7, Theorem 4.38]). This in turn guarantees that it has no uncountable ε-biorthogonal systems for any $\varepsilon \in [0, 1)$. In particular, it has no uncountable biorthogonal systems, nor fundamental biorthogonal systems.

The rest of the paper is mostly devoted to prove that, under PID, $(a) \Rightarrow (d')$, to conclude the proof. □

Notice that PID wasn't used in the previous sketch. The main use of PID can be illustrated by the following simple lemma, which gives typical useful P-ideals under $\omega_1 < \mathfrak{b}$.

Lemma 5. *If a family $\mathcal{F}$ of subsets of a set S is such that $|\mathcal{F}| < \mathfrak{b}$, then*

$$\mathcal{I} = \{I \in [S]^\omega : (\forall F \in \mathcal{F}) \quad |F \cap I| < \omega\}$$

is a P-ideal and is denoted by $\mathcal{F}^\perp$. □

The proofs go along similar lines as those contained in [2, 19, 20], by weakening the cardinal assumption. We prove that $(a) \Rightarrow (c')$ in Section 2, where we also prove the crucial Theorem 6, which will be used several times. Then, we prove that $(a) \Rightarrow (d')$ in Section 3 and Section 4 is devoted to analyse the impact of these results in the dual of an Asplund space. Our notation is fairly standard and follows [7].

2. Uncountable biorthogonal systems

Our main purpose in this section is to prove the following result.

Theorem 6. *Assume the PID and $\mathfrak{b} > \omega_1$. If X is a Banach space with weak* sequentially compact dual unit ball which contains a $\mathbb{Q}$-linear dense subspace D of cardinality ω_1, then there is an uncountable bounded sequence $(f_\alpha : \alpha < \omega_1)$ of (distinct) functionals in X^* such that*

$$\forall x \in X \quad (f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1) \tag{1}$$

and

$$\forall x \in D \quad (f_\alpha(x) : \alpha < \omega_1) \in \ell_1(\omega_1). \tag{2}$$

This theorem will be crucial to get all subsequent constructions of biorthogonal systems, whose functionals will be obtained by refining the sequence of functionals satisfying properties (1) and (2) given above. For instance, we easily get the following corollary, which fulfills the proof that (a) implies (c'):

Corollary 7. *Assume PID and $\mathfrak{b} > \omega_1$. Every nonseparable Asplund space X contains an uncountable biorthogonal system.*

Proof. Stegall showed in [15] that Asplund spaces have weak* sequentially compact dual unit balls. Hence, given an Asplund space X of density ω_1, we can apply Theorem 6, and get for a dense $\mathbb{Q}$-linear subspace D of X of cardinality ω_1, a bounded sequence $(f_\alpha)_{\alpha\in\omega_1}$ of (distinct) functionals in X^* sastifying properties (1) and (2). For each $\alpha \in \omega_1$, let $x_\alpha \in D$ be such that $f_\alpha(x_\alpha) \neq 0$. It follows from [6, Proposition 11] that X has an uncountable biorthogonal system. □

The proof of Theorem 10 goes in two steps, first to obtain property (1) and then to obtain property (2). We first need the following easy lemma.

Lemma 8. *Let K be a topological Hausdorff space of weight strictly smaller than $\mathfrak{b}$. If a sequence $(x_n)_{n\in\omega} \subseteq K$ converges to some $x \in K$ and for each $n \in \omega$, $(y^n_k)_{k\in\omega} \subseteq K$ is a sequence converging to x_n, then there is an increasing sequence $(k_n)_{n\in\omega}$ such that $(y^n_{k_n})_{n\in\omega}$ converges to x. Consequently, the sequential closure $Seq(S)$ of any set S in K is the set of all limits of sequences of elements of S.* □

The following proposition guarantees the existence of a sequence of functionals satisfying property (1):

Proposition 9. *Assume the PID and $\mathfrak{b} > \omega_1$. If X is a Banach space of density ω_1 whose dual unit ball is weak* sequentially compact, then there is an uncountable bounded sequence $(f_\alpha)_{\alpha\in\omega_1}$ of (distinct) functionals in X^* such that*

$$\forall x \in X \quad (f_\alpha(x) : \alpha \in \omega_1) \in c_0(\omega_1).$$

Proof. Fix D a dense $\mathbb{Q}$-linear subspace of X of cardinality ω_1. It is easy to construct an increasing chain $(X_\alpha)_{\alpha<\omega_1}$ of separable subspaces of X such that $dim(X_{\alpha+1}/X_\alpha) = 1$, if $\lambda < \omega_1$ is a limit ordinal, then $X_\lambda = \overline{span}\bigcup_{\alpha<\lambda} X_\alpha$ and $X = \overline{span}\bigcup_{\alpha<\omega_1} X_\alpha$. Using Hahn-Banach Theorem, we can obtain a normalized sequence $(h_\alpha)_{\alpha\in\omega_1}$ of linearly independent functionals in X^* such that for every $x \in D$, $h_\alpha(x) = 0$ for all but countably many $\alpha \in \omega_1$.

Let

$$\mathcal{I}_1 = \{I \in [\omega_1]^\omega : (\forall x \in D)(\forall \varepsilon \in (0,1)\cap\mathbb{Q}) \quad \{\alpha \in I : |h_\alpha(x)| \geq \varepsilon\} \text{ is finite}\}$$

and by Lemma 5, $\mathcal{I}_1$ is a P-ideal. Apply the PID and notice that if alternative (i) holds, we are done. If alternative (ii) of PID holds, let $\Gamma \in [\omega_1]^{\omega_1}$ be such that $|I \cap \Gamma| < \omega$ for every $I \in \mathcal{I}_1$. Hence $0 \notin Seq(\{h_\alpha : \alpha \in \Gamma\})$. Indeed, if $0 \in Seq(\{h_\alpha : \alpha \in \Gamma\})$, then by Lemma 8 there is be $(\alpha_n)_{n\in\omega} \subseteq \Gamma$ such that $(h_{\alpha_n})_{n\in\omega}$ is weak* convergent to 0, so that $I = \{h_{\alpha_n} : n \in \omega\} \in \mathcal{I}_1$. This contradicts the fact that $|I \cap \Gamma| < \omega$ for every $I \in \mathcal{I}_1$. On the other hand, $0 \in \overline{\{h_\alpha : \alpha \in \Gamma\}}^{w^*}$.

Consider $K = Seq(\{h_\alpha : \alpha \in \Gamma\})$, $S = \{g - f : f \neq g \text{ and } f, g \in K\}$ and

$$\mathcal{I}_2 = \{I \in [S]^\omega : (\forall x \in D)(\forall \varepsilon \in (0,1) \cap \mathbb{Q}) \quad \{h \in I : |h(x)| \geq \varepsilon\} \text{ is finite}\}.$$

By Lemma 5, $\mathcal{I}_2$ is a P-ideal. We shall prove that alternative (ii) of the PID cannot hold.

Given a weak*-open neighborhood V of 0, since $0 \in \overline{\{h_\alpha : \alpha \in \Gamma\}}^{w^*}$, we know that $V \cap \{h_\alpha : \alpha \in \Gamma\}$ is infinite, so that its sequential closure $Seq(V \cap \{h_\alpha : \alpha \in \Gamma\})$ is a weak* sequentially closed subset of K which doesn't contain 0.

Let $\mathcal{F}$ be a maximal filter of weak* sequentially closed subsets of K containing $Seq(V \cap \{h_\alpha : \alpha \in \Gamma\})$ for all weak*-open neighborhood V of 0. Using the weak* sequential compactness of the dual unit ball and the maximality of $\mathcal{F}$ we get that it is σ-complete in the sense that the intersection of countably many elements of $\mathcal{F}$ belongs to $\mathcal{F}$. Let $\mathcal{F}^+$ be the co-ideal of positive subsets of K, that is, all subsets Z of K which intersect every element of $\mathcal{F}$. Observe that if $Z \in \mathcal{F}^+$, then $Seq(Z) \in \mathcal{F}$. Also, if $\bigcup_{n\in\omega} Z_n \in \mathcal{F}^+$, then there is $n \in \omega$ such that $Z_n \in \mathcal{F}^+$.

Claim. *If $S = \bigcup_{n\in\omega} S_n$, then there is $n_0 \in \omega$ such that $0 \in Seq(S_{n_0})$.*

Proof of Claim 1. If $S = \bigcup_{n\in\omega} S_n$, a Fubini-type argument gives us an $n_0 \in \omega$ and a set $Z \in \mathcal{F}^+$ such that

$$(\forall f \in Z) \quad Z_f = \{g \in \Gamma : g - f \in S_{n_0}\} \in \mathcal{F}^+.$$

Fix $f_0 \in Seq(Z)$. Inductively, given

$$f_n \in Seq(Z) \cap (\bigcap\{Seq(Z_{f_k^i}) : k \in \omega, i < n\}),$$

use Lemma 8 to get $(f_k^n)_{k\in\omega} \subseteq Z$ such that $f_n = \lim_k f_k^n$. Since by our hypothesis, for each $k \in \omega$, $Z_{f_k^n} \in \mathcal{F}^+$, we get that

$$Seq(Z) \cap (\bigcap\{Seq(Z_{f_k^i}) : k \in \omega, i < n+1\}) \in \mathcal{F},$$

so that we can fix

$$f_{n+1} \in Seq(Z) \cap (\bigcap\{Seq(Z_{f_k^i}) : k \in \omega, i < n+1\}).$$

Now we have a sequence $(f_n)_{n\in\omega} \subseteq Seq(Z)$ and by the hypothesis that B_{X^*} is weak* sequentially compact, we may assume without loss of generality (passing to a subsequence) that $(f_n)_{n\in\omega}$ converges to some f, so that $(f_{2n} - f_{2n+1})_{n\in\omega}$ converges to 0.

For each $n \in \omega$, since $f_{2n+1} \in \bigcap\{Seq(Z_{f_k^{2n}}) : k \in \omega\}$, let for each $k \in \omega$ $(g_i^{n,k})_{i\in\omega} \subseteq Z_{f_k^{2n}}$ be such that $f_{2n+1} = \lim_i g_i^{n,k}$ for every $k \in \omega$, again by Lemma 8, and for each $n \in \omega$, there is an increasing sequence $(i_k)_{k\in\omega}$ such that $f_{2n+1} = \lim_k g_{i_k}^{n,k}$, so that $(f_k^{2n} - g_{i_k}^{n,k})_{k\in\omega}$ converges to $f_{2n} - f_{2n-1}$. Applying Lemma 8 once again, there is an increasing sequence $(k_n)_{n\in\omega}$ such that $(f_{k_n}^{2n} - g_{i_{k_n}}^{n,k_n})_{n\in\omega}$ converges to 0. On the other hand, let us go back to the choice of these sequences to see that $(f^{2n}_{k_n} - g^{n,k_n}_{i_{k_n}})_{n\in\omega} \subseteq S_{n_0}$ since $g^{n,k_n}_{i_{k_n}} \in Z_{f^{2n}_{k_n}}$. This concludes the proof of the claim. □

The previous claim guarantees that alternative (ii) of PID cannot hold for $\mathcal{I}_2$. Hence, alternative (i) holds. Thus we get an uncountable family of functionals $(f_\alpha)_{\alpha\in\omega_1}$ such that for every $x \in D$, hence every $x \in X$, $(f_\alpha(x))_{\alpha\in\omega_1} \in c_0(\omega_1)$. This concludes the proof of the theorem. □

We are now ready to prove the next step in the main result of this section.

Proposition 10. *Assume the PID and $\mathfrak{b} > \omega_1$. If X is a Banach space of density ω_1 and D is a $\mathbb{Q}$-linear dense subspace of X of cardinality ω_1 and $(f_\alpha : \alpha < \omega_1)$ is an uncountable bounded sequence of (distinct) functionals in X^* such that*

$$\forall x \in X \quad (f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1) \tag{1}$$

Then there is $\Gamma \in [\omega_1]^{\omega_1}$ such that

$$\forall x \in D \quad (f_\alpha(x) : \alpha < \omega_1) \in \ell_1(\Gamma). \tag{2}$$

Proof. Let

$$\mathcal{I} = \{I \in [\omega_1]^\omega : (\forall x \in D) \quad \sum_{\alpha\in I} |f_\alpha(x)| < +\infty\}.$$

Let us prove that $\mathcal{I}$ is a P-ideal.

Given a pairwise disjoint sequence $(I_n)_{n\in\omega}$ of elements of $\mathcal{I}$, fix e an enumeration of $\bigcup_{n\in\omega} I_n$ and for each $x \in D$, let $g_x \in \omega^\omega$ be such that

$$\sum_{\alpha\in I_n\setminus e[g_x(n)]} |f_\alpha(x)| < \frac{1}{2^n}.$$

Since $\mathfrak{b} > \omega_1$, there is $g \in \omega^\omega$ such that $g_x \leq^* g$ for every $x \in D$. Let

$$I = \bigcup_{n\in\omega} I_n \setminus e[g(n)].$$

Clearly $I_n \subseteq^* I$ for every $n \in \omega$ and let us prove that $I \in \mathcal{I}$. Given $x \in D$, let $n_0 \in \omega$ be such that $n \geq n_0$ implies that $g_x(n) \leq g(n)$. Then, for each $n < n_0$,

$$\sum_{\alpha\in I_n\setminus e[g(n)]} |f_\alpha(x)| \leq \sum_{\alpha\in I_n} |f_\alpha(x)| < +\infty$$

and for $n \geq n_0$,

$$\sum_{\alpha\in I_n\setminus e[g(n)]} |f_\alpha(x)| \leq \sum_{\alpha\in I_n\setminus e[g_x(n)]} |f_\alpha(x)| < \frac{1}{2^n}.$$

Hence,

$$\sum_{\alpha\in I} |f_\alpha(x)| = \sum_{n\in\omega} \sum_{\alpha\in I_n\setminus e[g(n)]} |f_\alpha(x)|$$

$$= \sum_{n<n_0} \sum_{\alpha\in I_n\setminus e[g(n)]} |f_\alpha(x)| + \sum_{n\geq n_0} \sum_{\alpha\in I_n\setminus e[g(n)]} |f_\alpha(x)| < +\infty,$$

so that $I \in \mathcal{I}$, which concludes the proof that $\mathcal{I}$ is a P-ideal.

Apply the PID and notice that if alternative (i) holds, we are done. Let us see that alternative (ii) of PID cannot hold, because any infinite subset of ω_1 contains an infinite subset which is in $\mathcal{I}$. Let $I \in [\omega_1]^\omega$ and fix e an enumeration of I. For each $x \in D$, let $h_x \in \omega^\omega$ be such that for every $n \geq h_x(k)$, $|f_{e(n)}(x)| < \frac{1}{2^k}$.

Since $\mathfrak{b} > \omega_1$, there is $h \in \omega^\omega$ such that $h_x \leq^* h$ for every $x \in D$. Without loss of generality we may assume that h is strictly increasing. Let $B = \{e(h(n)) : n \in \omega\} \in [I]^\omega$ and let us see that $B \in \mathcal{I}$: given $x \in D$, let $n_0 \in \omega$ be such that $n \geq n_0$ implies $h_x(n) \leq h(n)$, so that $|f_{e(h(n))}(x)| < \frac{1}{2^n}$. Hence,

$$\sum_{\alpha \in B} |f_\alpha(x)| = \sum_{n \in \omega} |f_{e(h(n))}(x)| = \sum_{n < n_0} |f_{e(h(n))}(x)| + \sum_{n \geq n_0} |f_{e(h(n))}(x)|$$

$$\leq \sum_{n < n_0} |f_{e(h(n))}(x)| + \sum_{n \geq n_0} \frac{1}{2^n} < +\infty.$$

This guarantees that B belongs to the ideal $\mathcal{I}$, so that (ii) of PID cannot happen and this concludes the proof. □

Proof of Theorem 6. Since Asplund spaces have weak* sequentially compact dual unit ball (see [5]), we can directly apply Propositions 9 and 10. □

3. Nonseparable quotient

Our main purpose in this section is to prove the following result, which corresponds to (a) implies (d') in the Main Theorem.

Theorem 11. *Assume PID and $\mathfrak{b} > \omega_1$. Every Asplund space X of density ω_1 has a fundamental biorthogonal system.*

The proof of this result is a simplified version of [20, Corollary 26.5] or its original version [19, Theorem 1]. It is simplified since, as opposed to the original paper, we start from a sequence of functionals given by Theorem 6 satisfying properties (1) and (2), which already defines the desired bounded linear operator. All we have to do is to refine this sequence to guarantee that the image of the quotient operator has a monotone long Schauder basis. This refinement is obtained with no use of the PID or $\mathfrak{b} > \omega_1$.

Proposition 12. *Let X be a Banach space with weak* sequentially compact dual unit ball which contains a $\mathbb{Q}$-linear dense subspace D of cardinality ω_1 and let $(f_\alpha : \alpha < \omega_1)$ be an uncountable bounded sequence $(f_\alpha)_{\alpha \in \omega_1}$ of (distinct) functionals in X^* such that*

$$(1) \qquad (\forall x \in X) \quad (f_\alpha(x) : \alpha < \omega_1) \in c_0(\omega_1)$$

and

$$(2) \qquad (\forall x \in D) \quad (f_\alpha(x) : \alpha < \omega_1) \in \ell_1(\omega_1).$$

Then there is $\Gamma \in [\omega_1]^{\omega_1}$ such that $(f_\alpha)_{\alpha \in \Gamma}$ is a monotone long Schauder basic sequence in X^* and $\overline{span}\{f_\alpha^* : \alpha \in \Gamma\}$ is a (nonseparable) quotient of X with (with monotone Schauder basis $(f_\alpha^*)_{\alpha \in \Gamma}$).

Proof. From Corollary 7, we can find $\Gamma \in [\omega_1]^{\omega_1}$ and $(x_\alpha)_{\alpha \in \Gamma}$ in X such that $(x_\alpha, f_\alpha)_{\alpha \in \Gamma}$ is a biorthogonal system. For each $\Gamma_0 \in [\Gamma]^{\omega_1}$, we can consider the map $Q_{\Gamma_0} : X \to (\overline{span}\{f_\alpha : \alpha \in \Gamma_0\})^*$, defined by

$$Q_{\Gamma_0}(x) = \sum_{\alpha \in \Gamma_0} f_\alpha(x) f_\alpha^*.$$

For $x \in D$, $\sum_{\alpha\in\Gamma_0} |f_\alpha(x)| < \infty$ and therefore Q_{Γ_0} is well defined. Moreover, $Q_{\Gamma_0}(x_\beta) = \sum_{\alpha\in\Gamma_0} f_\alpha(x_\beta) f^*_\alpha = f^*_\beta$ for all $\beta \in \Gamma_0$, so that

$$\overline{span}\{f^*_\alpha : \alpha \in \Gamma_0\}) \subseteq ImQ_{\Gamma_0} \subseteq (\overline{span}\{f_\alpha : \alpha \in \Gamma_0\})^*.$$

If we find Γ_0 such that $(f_\alpha)_{\alpha\in\Gamma_0}$ is a long monotone basic sequence, then

$$\overline{span}\{f^*_\alpha : \alpha \in \Gamma_0\}) = ImQ_{\Gamma_0} = (\overline{span}\{f_\alpha : \alpha \in \Gamma_0\})^*,$$

which guarantees that Q_{Γ_0} is the desired quotient mapping. It remains to find Γ_0.

For each $x \in D$ and each $\varepsilon > 0$, fix $\Gamma_{x,\varepsilon} \in [\Gamma]^{<\omega}$ such that

$$\sum_{\alpha\in\omega_1\setminus\Gamma_{x,\varepsilon}} |f_\alpha(x)| < \varepsilon.$$

Claim. *For every $\delta, \varepsilon \in (0, 1)$ and every $\Delta \in [\Gamma\cap\lambda]^{<\omega}$ there is a finite subset $F \subseteq \Gamma$ such that for every norm-one $f^* \in (span\{f_\alpha : \alpha \in \Delta\})^*$ there is $x \in D$ such that $\|x\| \in (1-\delta, 1+\delta)$, $\Gamma_{x,\varepsilon} \subseteq F$ and $|f^*(e)-e(x)| \leq \delta\cdot\|e\|$ for all $e \in span\{f_\alpha : \alpha \in \Delta\}$.*

Proof of the Claim. Let D^* be a finite subset of $(span\{f_\alpha : \alpha \in \Delta\})^*$ which is $\frac{\delta}{3}$-dense. For each $g^* \in D^*$, by the principle of local reflexivity there is $y_{g^*} \in X$ with $\|y_{g^*}\| \in (1-\frac{\delta}{3}, 1+\frac{\delta}{3})$ such that $|g^*(e)-e(y_{g^*})| \leq \frac{\delta}{3}\cdot\|e\|$ for all $e \in span\{f_\alpha : \alpha \in \Delta\}$. Let $x_{g^*} \in D$ be such that $\|y_{g^*} - x_{g^*}\| < \frac{\delta}{3}$. Notice that $\|x_{g^*}\| \in (1 - \delta, 1 + \delta)$. Let $F = \bigcup\{\Gamma_{x_{g^*},\varepsilon} : g^* \in D^*\}$, which is clearly finite.

Given a norm-one $f^* \in (span\{f_\alpha : \alpha \in \Delta\})^*$, let $g^* \in D^*$ be such that $\|f^* - g^*\| < \frac{\delta}{3}$. We know that $\|x_{g^*}\| \in (1 - \delta, 1 + \delta)$ and $\Gamma_{x_{g^*},\varepsilon} \subseteq F$. Moreover, for every $e \in span\{f_\alpha : \alpha \in \Delta\}$ we have that

$$\begin{aligned} |f^*(e) - e(x)| &\leq |f^*(e) - g^*(e)| + |g^*(e) - e(y_{g^*})| + |e(y_{g^*}) - e(x_{g^*})| \\ &\leq \|f^* - g^*\| \cdot \|e\| + |g^*(e) - e(y_{g^*})| + \|y_{g^*} - x_{g^*}\| \cdot \|e\| < \delta \cdot \|e\|, \end{aligned}$$

which concludes the proof of Claim 1. □

Let Ω be the set of limit ordinals $\lambda < \omega_1$ with the following property: for every $\delta, \varepsilon \in (0, 1)$, every $\Delta \in [\omega_1\cap\lambda]^{<\omega}$ and for every norm-one $f^* \in (span\{f_\alpha : \alpha \in \Delta\})^*$ there is $x \in X$ such that $\|x\| \in (1 - \delta, 1 + \delta)$ and $|f^*(e) - e(x)| \leq \delta \cdot \|e\|$ for all $e \in span\{f_\alpha : \alpha \in \Delta\}$ and $\sum_{\alpha\in\omega_1\setminus\lambda} |f_\alpha(x)| < \varepsilon$.

Claim. *Ω is closed and unbounded in ω_1.*

Proof of the Claim. It is easy to see that it is closed. Given any subset $S \subseteq \Omega$, let $\lambda = \sup S$. For every $\Delta \in [\omega_1\cap\lambda]^{<\omega}$ and every norm-one $f^* \in (span\{f_\alpha : \alpha \in \Delta\})^*$, let $\lambda' \leq \lambda$, $\lambda' \in S$ such that $\Delta \subseteq \lambda'$. Then, given $\delta, \varepsilon \in (0, 1)$, there is $x \in X$ such that $\|x\| \in (1 - \delta, 1 + \delta)$ and $|f^*(e) - e(x)| \leq \delta \cdot \|e\|$ for all $e \in span\{f_\alpha : \alpha \in \Delta\}$ and $\sum_{\alpha\in\omega_1\setminus\lambda} |f_\alpha(x)| \leq \sum_{\alpha\in\omega_1\setminus\lambda'} |f_\alpha(x)| < \varepsilon$. Hence, $\lambda \in \Omega$.

To prove that Ω is unbounded, given $\mu < \omega_1$, let $\lambda_0 = \mu$ and recursively construct an increasing sequence $(\lambda_n)_{n\in\omega}$ in ω_1 with the following property:

$$\forall\delta, \varepsilon \in (0, 1)\ \forall\Delta \in [\omega_1 \cap \lambda_n]^{<\omega}\ \forall f^* \in (span\{f_\alpha : \alpha \in \Delta\})^* \text{ such that } \|f^*\| = 1$$
$$\exists x \in X \text{ such that } \|x\| \in (1 - \delta, 1 + \delta),\ \Gamma_{x,\varepsilon} \subseteq \lambda_{n+1} \text{ and}$$
$$\forall e \in span\{f_\alpha : \alpha \in \Delta\} \quad |f^*(e) - e(x)| \leq \delta \cdot \|e\|.$$

Then, taking $\lambda = \sup_{n\in\omega} \lambda_n$, we get that $\lambda > \mu$ and $\lambda \in \Omega$.

For the recursive construction, for each $\delta, \varepsilon \in (0, 1)\cap\mathbb{Q}$ and each $\Delta \in [\omega_1\cap\lambda_n]^{<\omega}$, let $F_{\Delta,\delta,\varepsilon}$ be a finite subset of ω_1 as in Claim 1 and let $\lambda_{n+1} < \omega_1$ be such that

$\max F_{\Delta,\delta,\varepsilon} < \lambda_{n+1}$ for all $\delta, \varepsilon \in \mathbb{Q} \cap (0,1)$ and all $\Delta \in [\lambda_n]^{<\omega}$. It is easy to see that λ_{n+1} has the required properties. □

Let $\Gamma_0 \in [\Gamma]^{\omega_1}$ be such that for every $\mu < \lambda$ consecutive elements of Ω there is at most one element of Γ. Let us see that $(f_\alpha)_{\alpha\in\Gamma_0}$ is a long monotone Schauder basic sequence. Given $g = \sum_{i=1}^n a_i f_{\alpha_i}$ and $f = \sum_{i=1}^m a_i f_{\alpha_i}$ for some $\alpha_1 < \cdots < \alpha_n$ in Γ_0, $a_1, \ldots, a_n \in \mathbb{R}$ and $1 \leq m < n$, we prove that $\|f\| \leq \|g\|$.

Suppose by contradiction that there is $\theta > 1$ such that $\|f\| > \theta \cdot \|g\|$. We may assume WLOG that $\|f\| = 1$. Let $\lambda \in \Omega$ be such that $\alpha_m < \lambda \leq \alpha_{m+1}$, $\Delta = \{\alpha_1, \ldots, \alpha_m\}$ and a norm-one $f^* \in (span\{f_\alpha : \alpha \in \Delta\})^*$ such that $|f^*(f)| = \|f\| = 1$. Given $\delta \in (0,1)$, let $\varepsilon \in (0,1)$ be such that $\max\{|a_i| : 1 \leq i \leq n\} \cdot \varepsilon < \delta$. Since $\Delta \in [\omega_1 \cap \lambda]^{<\omega}$ and $\lambda \in \Omega$, there is $x \in X$ such that $\|x\| \in (1-\delta, 1+\delta)$ and $|f^*(e) - e(x)| \leq \delta \cdot \|e\|$ for all $e \in span\{f_\alpha : \alpha \in \Delta\}$ and $\sum_{\alpha\in\omega_1\setminus\lambda} |f_\alpha(x)| < \varepsilon$. In particular, $|1 - f(x)| = |f^*(f) - f(x)| \leq \delta$ and $\sum_{i=m+1}^n |f_{\alpha_i}(x)| < \varepsilon$, so that $\sum_{i=m+1}^n |a_i f_{\alpha_i}(x)| \leq \max\{|a_i| : m+1 \leq i \leq n\} \cdot \varepsilon < \frac{\delta}{3}$. Then we get that:

$$1 = \|f\| > \theta\|g\| \geq \theta(1-\delta)|g(x)|$$

$$\geq \theta(1-\delta)(|f(x)| - \sum_{i=m+1}^n |a_i f_{\alpha_i}(x)|) \geq \theta(1-\delta)(1-\delta-\delta).$$

For a sufficiently small δ, we get a contradiction. □

Combining previous results we can prove the main result of this section.

Proof of Theorem 11. Apply Theorem 6 and Proposition 12. □

4. Biorthogonal systems in the dual space

The main purpose of this section is to prove the following result:

Theorem 13. *Assume PID and $\mathfrak{b} > \omega_1$. Then the dual of every Asplund space of density ω_1 has an fundamental biorthogonal system.*

Theorem 13 follows directly from the next two results. The following theorem is a ZFC result implicit in the proof of [2, Theorem 4].

Theorem 14. *If X is an Asplund space of density ω_1 and $(x_\alpha, f_\alpha)_{\alpha<\omega_1}$ is a fundamental biorthogonal system, then there is $\Gamma \in [\omega_1]^{\omega_1}$ such that $T : (X^*, w^*) \to (c_0(\Gamma), \tau_p)$ given by $T(f) = (f(x_\alpha))_{\alpha\in\Gamma}$ is a well-defined linear continuous bounded mapping with nonseparable range.*

We finish with the following result:

Theorem 15. *Assume PID and $\mathfrak{b} > \omega_1$. If X is an Asplund space of density ω_1 and $(x_\alpha)_{\alpha<\omega_1}$ is a sequence in X such that $T : (X^*, w^*) \to (c_0(\omega_1), \tau_p)$ given by $T(f) = (f(x_\alpha))_{\alpha\in\omega_1}$ is a well-defined linear continuous bounded mapping with nonseparable range, then X^* has a fundamental biorthogonal system.*

Proof. Since X is Asplund and has density ω_1, X^* has also density ω_1 - see the proof of [2, Theorem 4]. Also, X does not contain any isomorphic copy of ℓ_1. Therefore, thanks to a characterization in [12], we get that for every countable $Z \subseteq B_{X^{**}}$, $\overline{Z}^{w^*}$ is a Rosenthal compact and, therefore, $B_{X^{**}}$ is weak* sequentially compact. With these properties in hand, we can apply several of the previous results to X^*, starting from the sequence $(x_\alpha)_{\alpha<\omega_1}$ seen as a sequence in X^{**}.

Fix D^* a $\mathbb{Q}$-linear dense subset of X^* and apply Proposition 10 to get $\Gamma \in [\omega_1]^{\omega_1}$ such that

$$\forall f \in D^* \quad (f(x_\alpha))_{\alpha\in\Gamma} \in \ell_1(\Gamma).$$

We can now use Proposition 12 to get that X^* has a nonseparable quotient with Schauder basis, which in turn implies that X^* has a fundamental biorthogonal system. □

Remark 16. *The statement of Theorem 15 is inspired by the main result in* [2]*. Namely, we expect that with some extra work it could be guaranteed that the functionals of the fundamental biorthogonal system in the dual space obtained in Theorem 15 are indeed in the space X. If this is so, then by the results of* [8, Theorem 2.4]*, we can add yet another statement to our list of equivalences of $\mathfrak{b} = \omega_2$ under PID: Every Asplund space X of density ω_1 has a renorming with the Mazur intersection property, that is, a renorming such that every closed convex subset of X is the intersection of closed balls of X.*

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