ON UNIVERSAL BANACH SPACES
OF DENSITY CONTINUUM

BY

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ABSTRACT

We consider the question whether there exists a Banach space X of density
continuum such that every Banach space of density at most continuum
isomorphically embeds into X (called a universal Banach space of density
$\mathfrak{c}$). It is well known that ℓ_∞/c_0 is such a space if we assume the continuum
hypothesis. Some additional set-theoretic assumption is indeed needed, as
we prove in the main result of this paper that it is consistent with the usual
axioms of set-theory that there is no universal Banach space of density $\mathfrak{c}$.
Thus, the problem of the existence of a universal Banach space of density
$\mathfrak{c}$ is undecidable using the usual axioms of set-theory.

We also prove that it is consistent that there are universal Banach
spaces of density $\mathfrak{c}$, but ℓ_∞/c_0 is not among them. This relies on the
proof of the consistency of the nonexistence of an isomorphic embedding
of $C([0, \mathfrak{c}])$ into ℓ_∞/c_0.

1. Introduction

This paper concerns the impact of infinitary combinatorics on basic isomorphic
properties of nonseparable Banach spaces. Quite often these properties cannot
be decided based on the usual axioms of set theory. Well known examples of
such phenomena include Haydon's, Levy's and Odell's results on quotients of
Grothendieck spaces [8], Todorcevic's results on uncountable biorthogonal sys-
tems ([21]), Avilés's work on the Radon–Nikodym property ([1]), the second
author's work on support sets ([10]) or Lopez-Abad's and Todorcevic's con-
structions of generic Banach spaces ([15]).

Let $\mathcal{C}$ be a class of Banach spaces. We will use the following standard notions
of universal Banach spaces for $\mathcal{C}$: X is universal (resp. isometrically universal)
for $\mathcal{C}$ if $X \in \mathcal{C}$ and for any $Y \in \mathcal{C}$ there is an isomorphic (resp. isometric)
embedding $T : Y \to X$. If κ is a cardinal, then universal (resp. isometrically
universal) of density κ will mean universal (resp. isometrically universal) for the
class of Banach spaces of density at most κ.

Similarly, one can define a universal Boolean algebra for a class of Boolean
algebras where the embedding is a monomorphism (injective homomorphism)
of Boolean algebras. For topological compact Hausdorff spaces it is natural in
this context to use the dual notion: K is universal for a class $\mathcal{T}$ of compact
Hausdorff spaces if $K \in \mathcal{T}$ and for any $L \in \mathcal{T}$ there is a continuous surjection
$T : K \to L$.

Probably the most known and useful result about the existence of universal Banach spaces is the classical Banach–Mazur theorem (Theorem 8.7.2 of [18]) which says that $C([0,1])$ is an isometrically universal space for the class of all separable Banach spaces. On the other hand, Szlenk's theorem proved in [20] says that there are no universal spaces for the class of separable reflexive Banach spaces.

It is well-known that topological and Boolean algebraic objects translate into Banach-theoretic ones but, in general, not vice-versa (see Chapters 7, 8 and 16 of [18]); in particular we have the following:

FACT 1.1: *If there is a universal Boolean algebra of cardinality κ or a universal totally disconnected compact space K of weight κ or a universal continuum of weight κ, then there is an isometrically universal Banach space of density κ. On the other hand, if there is an (isomorphically) universal Banach space of density κ, then there is one of the form $C(K)$ for K totally disconnected and there is one of the form $C(K)$ for K connected.*

Proof. By the Stone duality, the existence of a universal Boolean algebra of cardinality κ is equivalent to the existence of a universal totally disconnected compact space of weight κ. Also, any compact Hausdorff topological space has a totally disconnected preimage of the same weight (Proposition 8.3.5 of [18]) and so, a universal totally disconnected compact space of weight κ is also universal among all compact spaces of weight κ.

Recall that any Banach space X is isometric to a subspace of $C(B_{X^*})$ (Proposition 6.1.9 of [18]), where B_{X^*} is the dual unit ball of X considered with the weak* topology, which is connected and has weight equal to the density of X.

To prove the first assertion, suppose there is a universal compact space K (either totally disconnected or connected) of weight κ. Given any Banach space of density κ we get, in both cases, a continuous surjection $\phi : K \to B_{X^*}$. The fact that ϕ induces an isometric embedding of $C(B_{X^*})$ into $C(K)$ (Theorem 4.2.2 of [18]) and that X can be isometrically embedded in $C(B_{X^*})$ implies that $C(K)$ is an isometrically universal Banach space of density κ.

Let us now prove the second assertion. If there is a universal Banach space X of density κ, as X can be isometrically embedded in $C(B_{X^*})$, any Banach space of density κ can be embedded as well, so that $C(B_{X^*})$ is a universal Banach space of density κ (and B_{X^*} is connected). But B_{X^*} has a continuous

preimage K which is totally disconnected and of the same weight, so that $C(K)$ is a universal Banach space of density κ as well.   ∎

In this paper we consider the question of the existence of a universal Banach space of density continuum, denoted $\mathfrak{c}$.

As Parovičenko proved in [16] that under CH, $\wp(\mathbb{N})/Fin$ is a universal Boolean algebra of cardinality $\mathfrak{c}$, the same hypothesis implies the existence of an isometrically universal Banach space of density $\mathfrak{c}$, namely $C(K)$ where K is the Stone space of $\wp(\mathbb{N})/Fin$ (homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$). Moreover, $C(K)$ is isometric to ℓ_∞/c_0 (see, for example, 2.11 of [13]). Conversely, using quite general model-theoretic methods, Shelah and Usvyatsov showed in [19] among others that it is consistent that there is no isometrically universal Banach space of density $\mathfrak{c}$. We can summarize these results as:

THEOREM 1.2 ([16], [19]): *Assuming CH, ℓ_∞/c_0 is an isometrically universal Banach space of density $\mathfrak{c}$. On the other hand, it is consistent that there is no isometrically universal Banach space of density $\mathfrak{c}$.*

The main result of our paper, proved in Section 2, shows that for the existence of a universal Banach space of density $\mathfrak{c}$ (where not only isometric isomorphisms are allowed as embeddings, but we allow all isomorphisms) some extra set-theoretic assumption is also necessary:

THEOREM 1.3: *It is consistent that there is no universal Banach space of density $\mathfrak{c}$.*

Our proof is quite inspired by the proof in [3] that it is consistent that there is no universal totally disconnected compact space (nor continuum) of density $\mathfrak{c}$. However, the proof of [3] is allowed to rely on the fact that their embeddings (homeomorphisms) preserve set-theoretic operations as well as the inclusion, which is not true in general for linear operators, i.e., $A \subseteq B$ does not imply that $T(\chi_A) \leq T(\chi_B)$, etc. Actually, by the Kaplansky theorem if there is an order isomorphism between Banach spaces $C(K)$ and $C(K')$ then K and K' are homeomorphic, and so there exists an isometry of the Banach spaces (Theorem 7.8.1 of [18]). Hence preserving the order lies at the heart of the difference between universal and isometrically universal Banach spaces.

To overcome these difficulties we use a strong almost disjoint family of subsets of ω_1 (first constructed in [2]), i.e., a family $(X_\xi : \xi < \omega_2)$ of subsets of ω_1 such

that $X_\xi \cap X_\eta$ is finite for distinct $\xi, \eta < \omega_2$. Already the existence of such a family cannot be proved without extra set-theoretic assumptions. Similar uses of almost disjoint families of $\mathbb{N}$ instead of ω_1 were apparently initiated by Whitley in his proof of the fact that c_0 is not complemented in ℓ_∞ ([22]).

In Section 3 we prove that even if there are universal Banach spaces of density continuum, ℓ_∞/c_0 does not have to be one of them.

THEOREM 1.4: *It is consistent that there are universal Banach spaces of density $\mathfrak{c}$, but ℓ_∞/c_0 is not among them.*

The model where this takes place is the standard Cohen model. This time we follow the main idea of the proof by Kunen in [11] of the fact that in this model the algebra $\wp(\mathbb{N})/Fin$ does not contain well-ordered chains of length $\mathfrak{c}$. The main trick is to use the richness of the group of automorphisms of Cohen's forcing which are induced by permutations of ω_2. This allows us to prove that ℓ_∞/c_0 does not contain an isomorphic copy of $C(K)$, where K is the Stone space of a well-ordered chain of length $\mathfrak{c}$ (such a K is simply homeomorphic to $[0, \mathfrak{c}] = [0, \omega_2]$ with the order topology). However, not all permutations which can be used in Kunen's proof would work in our argument.

The terminology concerning forcing is based on [12] and the one concerning $C(K)$ spaces is based on [18]. The results of this paper answer questions 5 and 6 from [9].

2. Nonexistence of a universal space of density $\mathfrak{c}$

The model in which there will be no universal Banach spaces of density $\mathfrak{c}$ is the model obtained by a product of two forcings, $\mathbb{P}_1$ and $\mathbb{P}_2$. $\mathbb{P}_1$ is the c.c.c. forcing of Section 6 of [2] which adds a strong almost disjoint family $(X_\xi : \xi < \omega_2)$ of uncountable subsets of ω_1, that is, $X_\xi \cap X_\eta$ is finite whenever $\xi \neq \eta$ and $\mathbb{P}_2$ is the standard σ-closed and ω_2-c.c. forcing for adding ω_3 subsets of ω_1 with countable conditions ($Fn(\omega_3 \times \omega_1, 2, \omega_1)$ of Definition 6.1 of [12]). The ground model V is a model of GCH.

Definition 2.1 ([2] Section 6): Fix a family $(Y_\xi)_{\xi<\omega_2}$ of uncountable subsets of ω_1 such that $Y_\xi \cap Y_\eta$ is countable for different $\xi, \eta \in \omega_2$ and let $\mathbb{P}_1$ be the partial order of functions f whose domain $\mathrm{dom} f$ is a finite subset of ω_2, $f(\xi) \in [Y_\xi]^{<\omega}$ for every $\xi \in \mathrm{dom} f$ and, given $f, g \in \mathbb{P}_1$, put $f \leq g$ if

- $\mathrm{dom}g \subseteq \mathrm{dom}f$;
- $g(\xi) \subseteq f(\xi)$ for every $\xi \in \mathrm{dom}g$;
- $f(\xi) \cap f(\eta) = g(\xi) \cap g(\eta)$ for different $\xi, \eta \in \mathrm{dom}g$.

Let us recall some properties of $\mathbb{P}_1$ which we will need.

LEMMA 2.2: *The following assertions hold:*

(a) $\mathbb{P}_1$ *is c.c.c. of cardinality* ω_2.

(b) $\mathbb{P}_1$ *has precaliber* ω_2*, that is, every set of cardinality* ω_2 *has a centered subset of cardinality* ω_2.

(c) $\mathbb{P}_1$ *forces that there is a strong almost disjoint family* $(X_\xi : \xi < \omega_2)$ *of uncountable subsets of* ω_1 *(this is denoted* $A(\aleph_1, \aleph_2, \aleph_1, \aleph_0)$ *in* [2]*).*

(d) *If the ground model is a model of GCH, then* $\mathbb{P}_1$ *forces* $\mathfrak{c} = \omega_2$ *and that GCH holds at other cardinals.*

Proof. For (a) and (c) see Section 6 of [2]. To prove (b), fix $(f_\alpha)_{\alpha \in \omega_2} \subseteq \mathbb{P}_1$. By the Δ-system lemma, there is a subset $S \subseteq \omega_2$ of cardinality ω_2 such that $(\mathrm{dom}f_\alpha)_{\alpha \in S}$ is a Δ-system of root Δ. Since for each $\xi \in \Delta$ there are at most ω_1 possibilities for $f_\alpha(\xi)$ (because $f_\alpha(\xi) \in [Y_\xi]^{<\omega}$ and $|Y_\xi| = \omega_1$), by thinning out the family a finite number ($|\Delta|$) of times, we can assume without loss of generality that $f_\alpha|_\Delta = f_\beta|_\Delta$ for every $\alpha, \beta \in S$, which makes $(f_\alpha)_{\alpha \in S}$ a centered family. For $\mathfrak{c} \leq \omega_2$ and GCH at other cardinals in (d), use the standard argument with nice-names (Lemma VII 5.13 of [12]). To obtain that $\mathfrak{c} \geq \omega_2$ in $V^{\mathbb{P}_1}$, use Theorem 3.4 (a) of [2], where it is proved that under CH there is no strong almost disjoint family of size ω_2. ∎

Definition 2.3: Let $\mathbb{P}_2$ be the forcing formed by partial functions f whose domain $\mathrm{dom}f$ is a countable subset of $\omega_3 \times \omega_1$ and whose range is included in $2 = \{0, 1\}$, ordered by extension of functions. Given a subset $A \subseteq \omega_3$, we denote by $\mathbb{P}_2(A)$ the forcing formed by the elements of $\mathbb{P}_2$ whose domain is included in $A \times \omega_1$.

We summarize in the next lemma the properties of $\mathbb{P}_2$ which we will use.

LEMMA 2.4: *Assume GCH. The following assertions hold:*

(a) $\mathbb{P}_2$ *is isomorphic to* $\mathbb{P}_2(A) \times \mathbb{P}_2(\omega_3 \setminus A)$*, for any* $A \subseteq \omega_3$.

(b) $\mathbb{P}_2$ *is* σ*-closed and* ω_2*-c.c.*

Proof. See Section VII 6 of [12]. ∎

And finally, we conclude some properties of the product $\mathbb{P}_1 \times \mathbb{P}_2$.

LEMMA 2.5: *Assume GCH. The following assertions hold:*

(i) $\mathbb{P}_2$ *forces that* $\check{\mathbb{P}}_1$ *is c.c.c.*

(ii) $\mathbb{P}_1 \times \mathbb{P}_2$ *is* ω_2-*c.c.*

(iii) $\mathbb{P}_1 \times \mathbb{P}_2$ *preserves cardinals and in* $V^{\mathbb{P}_1 \times \mathbb{P}_2}$ *we have that* $\mathfrak{c} = \omega_2$.

(iv) *Let* $\kappa = \omega_1$ *or* $\kappa = \omega_2$ *and let* $A \subseteq \omega_3$. *If* X *is in the model* $V^{\mathbb{P}_1 \times \mathbb{P}_2(A)}$ *and* $(Y_\xi : \xi < \kappa) \in V^{\mathbb{P}_1 \times \mathbb{P}_2}$ *is a sequence of subsets of* X *all of cardinality* $\leq \kappa$, *then there is in* V *a subset* A' *of* ω_3 *of cardinality* $\leq \kappa$ *such that* $(Y_\xi : \xi < \kappa) \in V^{\mathbb{P}_1 \times \mathbb{P}_2(A \cup A')}$.

(v) *Suppose* $A \subseteq \omega_3$ *and* $\beta \in \omega_3 \setminus A$. *If* $X \in V^{\mathbb{P}_1}$ *is an uncountable subset of* ω_1, *then* $X \cap G_\beta \notin V^{\mathbb{P}_1 \times \mathbb{P}_2(A)}$ *where* $G_\beta = G \cap \mathbb{P}_2(\{\beta\})$ *and* G *is* $\mathbb{P}_1 \times \mathbb{P}_2$-*generic over* V.

Proof. In this proof we will be often using the product lemma (Theorem VIII 1.4. of [12]). It implies that $\mathbb{P}_1 \times \mathbb{P}_2$ can be viewed as the forcing iterations $\mathbb{P}_1 * \check{\mathbb{P}}_2$ or $\mathbb{P}_2 * \check{\mathbb{P}}_1$.

Note that in $V^{\mathbb{P}_2}$ we have that $\check{\mathbb{P}}_1 = \mathbb{P}_1$, and so Lemma 2.2 (a) implies (i). For (ii) note that any product of an ω_2-c.c. forcing and a forcing which has precaliber ω_2 is ω_2-c.c., so Lemma 2.2 (b) and Lemma 2.4 (b) imply (ii).

For (iii) note that ω_1 is preserved by $\mathbb{P}_2 * \check{\mathbb{P}}_1$, since it is preserved by $\mathbb{P}_2$ by Lemma 2.4 (b) and later by $\check{\mathbb{P}}_1$ by (i). Other cardinals are preserved by (ii). In $V^{\mathbb{P}_1}$ we have $\mathfrak{c} = \omega_2$ by Lemma 2.2 (d). It is also true in $V^{\mathbb{P}_1 \times \mathbb{P}_2}$, since in $V^{\mathbb{P}_1}$ the forcing $\check{\mathbb{P}}_2$ is ω_1-Baire and hence it does not add reals.

(iv) is a consequence of the standard factorization, as for example in Lemma VIII 2.2. of [12], which can be applied by (ii) and Lemma 2.4 (a).

For (v) consider in $V^{\mathbb{P}_1 \times \mathbb{P}_2(A)}$

$$D_Y = \{p \in \check{\mathbb{P}}_2(\{\beta\}) : p^{-1}(\{1\}) \cap X \cap \mathrm{dom}(p) \neq Y \cap X \cap \mathrm{dom}(p)\}$$

where Y is any subset of ω_1 in $V^{\mathbb{P}_1 \times \mathbb{P}_2(A)}$. Since for any $q \in \check{\mathbb{P}}_2(\{\beta\})$ one can find a finite extension p satisfying $p^{-1}(\{1\}) \cap X \cap \mathrm{dom}(p) \neq Y \cap X \cap \mathrm{dom}(p)\}$, we may conclude that $p \in \check{\mathbb{P}}_2(\{\beta\})$, and hence D_Y is a dense subset of $\check{\mathbb{P}}_2(\{\beta\})$ which belongs to $V^{\mathbb{P}_1 \times \mathbb{P}_2(A)}$. Now, by the product lemma, G_β as in (v) is a $\check{\mathbb{P}}_2(\{\beta\})$-generic over $V^{\mathbb{P}_1 \times \mathbb{P}_2(A)}$ and so we may conclude (v). ■

To prove the main result of this paper, we need a combinatorial lemma concerning measures over a Boolean subalgebra of $\wp(\omega_1)$.

LEMMA 2.6: *Let $\mathcal{B}$ be a Boolean subalgebra of $\wp(\omega_1)$ which contains all finite sets of ω_1, L its Stone space and let $(X_\gamma)_{\gamma \in \omega_2} \subseteq \mathcal{B}$ be a strong almost disjoint family. Given a family $(\mu_\xi)_{\xi \in \omega_1}$ in $C(L)^*$, there is $\gamma_0 \in \omega_2$ such that*

$$\forall \gamma \in (\gamma_0, \omega_2) \quad \forall \xi \in \omega_1 \quad \forall X \subseteq X_\gamma \quad \text{if } X \in \mathcal{B}, \text{ then } \mu_\xi([X]) = \sum_{\lambda \in X} \mu_\xi([\{\lambda\}]),$$

where $[X]$ denotes the clopen subset of L corresponding to X by the Stone duality.

Proof. Let us first prove the following:

CLAIM: *There is $\gamma' \in \omega_2$ such that*

$$\forall \gamma \in (\gamma', \omega_2) \quad \forall \xi \in \omega_1 \quad \forall X \subseteq X_\gamma, \quad \text{if } X \in \mathcal{B}, \text{ then } \mu_\xi([X]) = \mu_\xi\left(\bigcup_{\lambda \in X} [\{\lambda\}]\right).$$

Proof of the Claim. Suppose by contradiction that the claim does not hold. Then, there is $A \subseteq \omega_2$ of cardinality ω_2 such that for every $\gamma \in A$ there are $\xi_\gamma \in \omega_1$ and $Y_\gamma \subseteq X_\gamma$ such that

$$\mu_{\xi_\gamma}([Y_\gamma]) \neq \mu_{\xi_\gamma}\left(\bigcup_{\lambda \in Y_\gamma} [\{\lambda\}]\right).$$

Since there are at most ω_1 possibilities for ξ_γ, we may assume without loss of generality that $\xi_\gamma = \xi$ for a fixed $\xi \in \omega_1$. Also, we can assume without loss of generality that there is a natural number m such that for all $\gamma \in A$,

$$\left| \mu_\xi([Y_\gamma]) - \mu_\xi\left(\bigcup_{\lambda \in Y_\gamma} [\{\lambda\}]\right) \right| > \frac{1}{m}.$$

Let n be a natural number greater than $m \cdot \|\mu_\xi\|$ and let $\gamma_1, \ldots, \gamma_n$ be different ordinals in A such that

$$\mu_\xi([Y_{\gamma_i}]) - \mu_\xi\left(\bigcup_{\lambda \in Y_{\gamma_i}} [\{\lambda\}]\right)$$

are either all positive or all negative.

Since $(X_\gamma)_{\gamma \in \omega_2}$ is a strong almost disjoint family in $\mathcal{B}$, it follows that $(Y_\gamma)_{\gamma \in A}$ is also a strong almost disjoint family in $\mathcal{B}$ and so, putting

$$E_\gamma = [Y_\gamma] \setminus \bigcup_{\lambda \in Y_\gamma} [\{\lambda\}],$$

we have that $(E_\gamma)_{\gamma \in A}$ is a pairwise disjoint family of Borel subsets of L. Then,

$$\left| \mu_\xi \left(\bigcup_{i=1}^{n} E_{\gamma_i} \right) \right| = \sum_{i=1}^{n} \left| \mu_\xi([Y_{\gamma_i}]) - \mu_\xi \left(\bigcup_{\lambda \in Y_{\gamma_i}} [\{\lambda\}] \right) \right| \geq n \cdot \frac{1}{m} > \|\mu_\xi\|,$$

a contradiction, which concludes the proof of the claim.   ■

Let us now prove the lemma. Suppose by contradiction that the lemma does not hold. Then, there is $A \subseteq (\gamma', \omega_2)$ of cardinality ω_2 such that for every $\gamma \in A$ there are $\xi_\gamma \in \omega_1$ and $Y_\gamma \subseteq X_\gamma$ such that

$$\mu_{\xi_\gamma}([Y_\gamma]) \neq \sum_{\lambda \in Y_\gamma} \mu_{\xi_\gamma}([\{\lambda\}]).$$

By the previous claim, we conclude that

$$\mu_{\xi_\gamma} \left(\bigcup_{\lambda \in Y_\gamma} [\{\lambda\}] \right) \neq \sum_{\lambda \in Y_\gamma} \mu_{\xi_\gamma}([\{\lambda\}]).$$

Since there are at most ω_1 possibilities for ξ_γ, we may assume without loss of generality that $\xi_\gamma = \xi$ for a fixed $\xi \in \omega_1$. Also, we can assume without loss of generality that there is a natural number m such that for all $\gamma \in A$,

$$\left| \mu_\xi \left(\bigcup_{\lambda \in Y_\gamma} [\{\lambda\}] \right) - \sum_{\lambda \in Y_\gamma} \mu_\xi([\{\lambda\}]) \right| > \frac{1}{m}.$$

Fix $\gamma \in A$. Let $Z_\gamma = \{\lambda \in Y_\gamma : \mu_\xi([\{\lambda\}]) \neq 0\}$ and $W_\gamma = Y_\gamma \setminus Z_\gamma$. Now Z_γ is a countable set and, since μ_ξ is σ-additive,

$$\mu_\xi \left(\bigcup_{\lambda \in Z_\gamma} [\{\lambda\}] \right) = \sum_{\lambda \in Z_\gamma} \mu_\xi([\{\lambda\}]).$$

Then, putting

$$\delta_\gamma = \mu_\xi \left(\bigcup_{\lambda \in Y_\gamma} [\{\lambda\}] \right) - \sum_{\lambda \in Y_\gamma} \mu_\xi([\{\lambda\}]),$$

using the fact that $([\{\lambda\}])_{\lambda \in \omega_1}$ is a pairwise disjoint family and that $\mu_\xi([\{\lambda\}]) = 0$ for any $\lambda \in W_\gamma$, we get that

$$\delta_\gamma = \mu_\xi \left(\bigcup_{\lambda \in Z_\gamma} [\{\lambda\}] \right) + \mu_\xi \left(\bigcup_{\lambda \in W_\gamma} [\{\lambda\}] \right) - \sum_{\lambda \in Z_\gamma} \mu_\xi([\{\lambda\}]) - \sum_{\lambda \in W_\gamma} \mu_\xi([\{\lambda\}])$$

$$= \mu_\xi \left(\bigcup_{\lambda \in Z_\gamma} [\{\lambda\}] \right) - \sum_{\lambda \in Z_\gamma} \mu_\xi([\{\lambda\}]) + \mu_\xi \left(\bigcup_{\lambda \in W_\gamma} [\{\lambda\}] \right) = \mu_\xi \left(\bigcup_{\lambda \in W_\gamma} [\{\lambda\}] \right).$$

Let n be a natural number greater than $m \cdot \|\mu_\xi\|$ and let $\gamma_1, \ldots, \gamma_n$ be different ordinals in A such that $(\delta_{\gamma_j})_{1 \leq j \leq n}$ are either all positive or all negative. Since $W_\gamma \subseteq Y_\gamma \subseteq X_\gamma$ and $(X_\gamma)_{\gamma \in \omega_2}$ is strong almost disjoint, then for each $1 \leq j \leq n$, let F_j be a finite subset of W_{γ_j} such that $(W_{\gamma_j} \setminus F_j)_{1 \leq j \leq n}$ are pairwise disjoint. Note that by the additivity of μ_ξ,

$$\mu_\xi \Big(\bigcup_{\lambda \in W_{\gamma_j} \setminus F_j} [\{\lambda\}] \Big) = \mu_\xi \Big(\bigcup_{\lambda \in W_{\gamma_j}} [\{\lambda\}] \Big) - \sum_{\lambda \in F_j} \mu_\xi([\{\lambda\}]) = \mu_\xi \Big(\bigcup_{\lambda \in W_{\gamma_j}} [\{\lambda\}] \Big) = \delta_{\gamma_j},$$

since $\mu_\xi([\{\lambda\}]) = 0$ for every $\lambda \in W_\gamma$. Then,

$$\Big| \mu_\xi \Big(\bigcup_{j=1}^n \bigcup_{\lambda \in W_{\gamma_j} \setminus F_j} [\{\lambda\}] \Big) \Big| = \Big| \sum_{j=1}^n \mu_\xi \Big(\bigcup_{\lambda \in W_{\gamma_j} \setminus F_j} [\{\lambda\}] \Big) \Big| = \Big| \sum_{j=1}^n \delta_{\gamma_j} \Big| \geq n \cdot \frac{1}{m} > \|\mu_\xi\|,$$

which is a contradiction and completes the proof of the lemma. ∎

For the sake of the proof of the main result, let us adopt the following notation: if $\mathcal{A}$ is a Boolean algebra, then $C_\mathbb{Q}(\mathcal{A})$ is the set of all formal linear combinations of elements of $\mathcal{A}$ with rational coefficients. If $\mathcal{A} \subseteq \mathcal{B}$ are Boolean algebras, then $C_\mathbb{Q}(\mathcal{A})$ can be identified with a (nonclosed) linear subspace of $C(K)$, where K is the Stone space of $\mathcal{B}$. If $\mathcal{A} = \mathcal{B}$, then the subspace is norm-dense by the Stone–Weierstrass theorem. The closure $\overline{C_\mathbb{Q}(\mathcal{A})}$ will mean the norm closure. We can talk about linear bounded functionals ν defined on the spaces $C_\mathbb{Q}(\mathcal{A})$, which correspond to finitely additive bounded measures on $\mathcal{A}$ (see Section 18.7 of [18]). If $\mathcal{A} = \mathcal{B}$, they have unique extensions to continuous linear functionals on $C(K)$, where K is the Stone space of $\mathcal{A}$, which can be interpreted as Radon measures on K. It turns out that families of finitely additive measures viewed as functionals on $C_\mathbb{Q}(\mathcal{A})$ can code all the information about operators between Banach spaces $C(K)$. This is useful in the forcing context, since if $\mathcal{A}_0 \subseteq \wp(\omega_2)$ is in an intermediate model, then such a measure could be interpreted as a subset of $\wp(\omega_2) \times \wp(\omega)$ which belongs to the intermediate model, so that the factorization of Lemma 2.5 (iv) can be applied, while the corresponding Radon measure is at least as big as the Stone space of the Boolean algebra. On the other hand, representing operators T into a $C(K)$ space as functions sending $x \in K$ to $T^*(\delta_x)$ in the dual space is quite classical (see Theorem VI 7.1. of [4]). Here $T^* : C^*(K) \to C^*(K)$ is the adjoint operator of T given by $T^*(\mu)(f) = \mu(T(f))$, where by the Riesz representation theorem the elements of $C^*(K)$ are identified with the Radon measures on K.

Proof of Theorem 1.3. Let V be a model of GCH. By Lemma 2.5, we have that $\mathbb{P}_1 \times \mathbb{P}_2$ preserves cardinals and in the extension $V^{\mathbb{P}_1 \times \mathbb{P}_2}$ we have $\mathfrak{c} = \omega_2$.

By Fact 1.1, it is enough to prove that there is no universal Banach space of density $\mathfrak{c}$ which is of the form $C(K)$ where K is totally disconnected, i.e., where K is the Stone space of a Boolean algebra. In the extension $V^{\mathbb{P}_1 \times \mathbb{P}_2}$, take any Boolean algebra $\mathcal{A}$ of cardinality $\omega_2 = \mathfrak{c}$. We will prove that $C(K)$ is not a universal Banach space of density $\mathfrak{c}$, where K is the Stone space of $\mathcal{A}$.

Since $\mathcal{A}$ has cardinality ω_2, we may assume that $\mathcal{A}$ is a subalgebra of $\wp(\omega_2)$ and so Lemma 2.5 (iv) applies to the sequence of its elements, and hence there is $\alpha < \omega_3$ such that $\mathcal{A} \in V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha)}$.

Let $\mathcal{B}$ be the Boolean subalgebra of $\wp(\omega_1)$ of all subsets of ω_1 which are in $V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha + \omega_2)}$ and let L be its Stone space in $V^{\mathbb{P}_1 \times \mathbb{P}_2}$.

We will prove that in $V^{\mathbb{P}_1 \times \mathbb{P}_2}$ the space $C(L)$ cannot be isomorphically embedded in $C(K)$, which will give that $C(K)$ is not a universal Banach space of density $\mathfrak{c}$, concluding the proof. Suppose it can be isomorphically embedded and let us derive a contradiction.

Work in $V^{\mathbb{P}_1 \times \mathbb{P}_2}$.

Let $T : C(L) \to C(K)$ be an isomorphic embedding and $T^{-1} : T[C(L)] \to C(L)$ its inverse. Let $\mathcal{B}_0 \subseteq \mathcal{B} \subseteq \wp(\omega_1)$ be the Boolean algebra of all finite and cofinite subsets of ω_1.

CLAIM 1: *There is a Boolean algebra $\mathcal{A}_0 \subseteq \mathcal{A}$ and a bounded sequence of finitely additive bounded measures $(\nu_\xi : \xi \in \omega_1)$ on $\mathcal{A}_0$ such that:*

(1) $|\mathcal{A}_0| \leq \omega_1$.
(2) *If $\xi \in \omega_1$ and $\overline{\rho}_\xi \in C(K)^*$ is such that $\overline{\rho}_\xi(\chi_{[a]}) = \nu_\xi(a)$ for each $a \in \mathcal{A}_0$, then for each $\lambda \in \omega_1$ we have $\overline{\rho}_\xi(T(\chi_{[\{\lambda\}]})) = \delta_\xi(\{\lambda\})$.*

Proof of Claim 1. For each $f \in C(K)$ there is a countable subalgebra $\mathcal{A}_f \subseteq \mathcal{A}$ such that $f \in \overline{C_{\mathbb{Q}}(\mathcal{A}_f)}$, because we can approximate f by finite linear combinations with rational coefficients of characteristic functions of clopen sets. So, take any $\mathcal{A}_0$ such that $\mathcal{A}_{T(\chi_{[\{\lambda\}]})} \subseteq \mathcal{A}_0$ for every $\lambda \in \omega_1$.

Let $\phi_\xi = (T^{-1})^*(\delta_\xi)$ be a bounded linear functional on $T[C(L)]$ which corresponds to δ_ξ on $C(L)$. In particular, we have that

$$\phi_\xi(T(\chi_{[\{\lambda\}]})) = \delta_\xi(\{\lambda\}).$$

Of course, $\|\phi_\xi\| \leq \|(T^{-1})^*\|$, so the sequence of ϕ_ξ's is bounded. By the Hahn–Banach theorem, for any $\xi \in \omega_1$, ϕ_ξ has a norm-preserving extension ψ_ξ

which is defined on $C(K)$. Finally, for $\xi \in \omega_1$, let ν_ξ be the finitely additive bounded measure (see Section 18.7 of [18]) on $\mathcal{A}_0$ defined by

$$\nu_\xi(a) = \psi_\xi(\chi_{[a]}).$$

Now suppose that $\overline{\rho}_\xi \in C(K)^*$ is such that $\overline{\rho}_\xi(\chi_{[a]}) = \nu_\xi(a)$ for each $a \in \mathcal{A}_0$. Then, $\overline{\rho}_\xi(\chi_{[a]}) = \psi_\xi(\chi_{[a]})$ for each $a \in \mathcal{A}_0$, and so, by the linearity, the Stone–Weierstrass theorem and the continuity, we may conclude that $\overline{\rho}_\xi|\overline{C_{\mathbb{Q}}(\mathcal{A}_0)} = \psi_\xi|\overline{C_{\mathbb{Q}}(\mathcal{A}_0)}$. But by the choice of $\mathcal{A}_0$, we have that $T(\chi_{[\{\lambda\}]}) \in \overline{C_{\mathbb{Q}}(\mathcal{A}_0)}$ for each $\lambda \in \omega_1$ and so

$$\overline{\rho}_\xi(T(\chi_{[\{\lambda\}]}) = \psi_\xi(T(\chi_{[\{\lambda\}]})) = \phi_\xi(T(\chi_{[\{\lambda\}]})) = \delta_\xi(\{\lambda\}),$$

which concludes the proof of Claim 1. ∎

By Lemma 2.5 (iv), we can find $B \subseteq \omega_3 \setminus \alpha$ of cardinality ω_1 such that $B \in V$ and the sequence $(\nu_\xi : \xi < \omega_1)$ and Boolean algebra $\mathcal{A}_0$ are in $V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha \cup B)}$. Now working in $V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha \cup B)}$, apply Tarski's theorem (Proposition 17.2.9 of [18]) to extend ν_ξ's to norm-preserving finitely additive bounded measures ρ_ξ on $\mathcal{A}$.

CLAIM 2: *For every $g \in C_{\mathbb{Q}}(\mathcal{A})$, the sequence $(\int g d\rho_\xi : \xi < \omega_1)$ belongs to $V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha \cup B)}$.*

Proof of Claim 2. Both the algebra $\mathcal{A}$ and the sequence $(\rho_\xi : \xi < \omega_1)$ belong to $V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha \cup B)}$. Hence $C_{\mathbb{Q}}(\mathcal{A})$ is in this model and the evaluation of the integrals follows their linearity and depends only on the values of the measures on $\mathcal{A}$. This completes the proof of Claim 2. ∎

Now work again in $V^{\mathbb{P}_1 \times \mathbb{P}_2}$. Consider the strong almost disjoint family $(X_\gamma : \gamma < \omega_2)$ added by $\mathbb{P}_1$ (by Lemma 2.2) and the adjoint operator $T^* : C(K)^* \to C(L)^*$. For $\xi \in \omega_1$, let $\overline{\rho}_\xi$ be the unique functional on $C(K)$ extending ρ_ξ (see Section 18.7 of [18]), so for each $a \in \mathcal{A}_0$ we have

$$\overline{\rho}_\xi(\chi_{[a]}) = \rho_\xi(a) = \nu_\xi(a).$$

By Claim 1, we have that $\overline{\rho}_\xi(T(\chi_{[\{\lambda\}]})) = \delta_\xi(\{\lambda\})$ for each $\xi, \lambda \in \omega_1$.

Now, let μ_ξ be the Radon measure on L corresponding to the functional $T^*(\overline{\rho}_\xi)$ on $C(L)$. In particular, for each $\xi, \lambda \in \omega_1$ we have

$$(2.1) \qquad \mu_\xi([\{\lambda\}]) = \overline{\rho}_\xi(T(\chi_{[\{\lambda\}]})) = \delta_\xi(\{\lambda\}).$$

By Lemma 2.6, there is $\gamma \in \omega_2$ such that for each $\xi \in \omega_1$, we have

$$\mu_\xi([X]) = \sum_{\lambda \in X} \mu_\xi([\{\lambda\}])$$

for every $X \subseteq X_\gamma$, $X \in \mathcal{B}$. Then, if $\beta < \alpha + \omega_2$ is such that $\sup(\alpha + \omega_2) \cap B < \beta$ (which certainly exists because B has cardinality ω_1) and G_β is the projection of the $\mathbb{P}_1 \times \mathbb{P}_2$-generic G over V on the β-th coordinate in $\mathbb{P}_2$, we have that

$$(2.2) \qquad \mu_\xi([X_\gamma \cap G_\beta]) = \sum_{\lambda \in X_\gamma \cap G_\beta} \mu_\xi([\{\lambda\}]).$$

Taking $f = T(\chi_{[X_\gamma \cap G_\beta]})$ and combining equalities (2.1) and (2.2), it follows that

$$\overline{\rho}_\xi(f) = \mu_\xi([X_\gamma \cap G_\beta]) = \sum_{\lambda \in X_\gamma \cap G_\beta} \delta_\xi(\{\lambda\}) = \begin{cases} 1 & \text{if } \xi \in X_\gamma \cap G_\beta, \\ 0 & \text{if } \xi \notin X_\gamma \cap G_\beta. \end{cases}$$

Now, take $g \in C_{\mathbb{Q}}(\mathcal{A})$ such that $||g - f|| < 1/3||(T^*)^{-1}||$.

Using the fact that

$$||\overline{\rho}_\xi|| = ||\phi_\xi|| = ||(T^*)^{-1}(\delta_\xi)|| \leq ||(T^*)^{-1}|| \cdot ||\delta_\xi|| \leq ||(T^*)^{-1}||,$$

we get that $|\overline{\rho}_\xi(g - f)| < 1/3$, hence

$$\int g d\rho_\xi = \int g d\overline{\rho}_\xi \begin{cases} > 2/3 & \text{if } \xi \in X_\gamma \cap G_\beta, \\ < 1/3 & \text{if } \xi \notin X_\gamma \cap G_\beta. \end{cases}$$

Since the formulas $\int g\rho_\xi > 2/3$ and $\int g\rho_\xi < 1/3$ are absolute, by Claim 2 we conclude that $X_\gamma \cap G_\beta$ belongs to $V^{\mathbb{P}_1 \times \mathbb{P}_2(\alpha \cup B)}$, which contradicts Lemma 2.5 (v) and concludes the proof. ∎

3. ℓ_∞/c_0 may fail to be among existing universal Banach spaces

Our main purpose in this section is to prove that the Banach space ℓ_∞/c_0 may fail to be a universal space of density $\mathfrak{c}$ and at the same time there may exist universal Banach spaces of density $\mathfrak{c}$. Actually, we prove that this situation takes place in the model obtained by adding ω_2 Cohen reals to a model of GCH. Moreover, the reason why ℓ_∞/c_0 is not universal can be seen quite explicitly, namely in that model it contains no isomorphic copy of $C([0, \mathfrak{c}])$.

Definition 3.1: Let $\mathbb{P}$ be the forcing formed by partial functions f whose domains $\mathrm{dom} f$ are finite subsets of $\omega_2 \times \omega$ and whose ranges are included in $2 = \{0,1\}$, ordered by extension of functions.

THEOREM 3.2: *Assume GCH; $\mathbb{P}$ forces that there is a universal Banach space of density $\mathfrak{c}$.*

Proof. This is just the conjunction of the results of [3] and [7]. As noted in [3] at the beginning of Section 6, in [7] it is proved that the existence of a $\mathfrak{c}$-saturated ultrafilter on $\mathbb{N}$ is equivalent to the conjunction of Martin's Axiom for countable partial orders and $2^{<\mathfrak{c}} = \mathfrak{c}$. The former holds in any model obtained by adding at least $\mathfrak{c}$ Cohen reals and the latter holds in any model obtained by a c.c.c. forcing of size ω_2 which adds ω_2 reals over a model of CH+$2^{\omega_1} = \omega_2$. Hence, if we assume GCH, $\mathbb{P}$ forces that there is a $\mathfrak{c}$-saturated ultrafilter on $\mathbb{N}$. Now we use the observation included at the end of Section 5 of [3] that the existence of a $\mathfrak{c}$-saturated ultrafilter on $\mathbb{N}$ implies that there is a universal continuum of weight $\mathfrak{c}$ and apply Fact 1.1. ∎

Now we proceed to the proof of the fact that $\mathbb{P}$ forces that ℓ_∞/c_0 contains no isomorphic copy of $C([0, \omega_2])$. This is motivated by a result and the proof of Kunen in [11] saying that $\mathbb{P}$ forces that $\wp(\mathbb{N})/Fin$ has no well-ordered chains of length $\mathfrak{c}$.

Definition 3.3: A nice-name for an element of ℓ_∞ is a name of the form

$$\dot{f} = \bigcup_{n,m \in \mathbb{N} \times \mathbb{N}} \{\langle [[\check{n}, \check{m}], \check{q}_{n,m}(p)], p \rangle : p \in A_{n,m}\},$$

where $[[\check{n}, \check{m}], \check{q}]$ stands for the canonical name for an ordered pair whose first element is the ordered pair $\langle \check{n}, \check{m} \rangle$ and whose second element is q; $A_{n,m}$'s are maximal antichains in $\mathbb{P}$; and $q_{n,m} : A_{n,m} \to \mathbb{Q}$ are functions.

Given a nice-name $\dot{f}$ for an element of ℓ_∞ as above, we define the support of $\dot{f}$ by

$$\mathrm{supp}(\dot{f}) = \bigcup \{\mathrm{dom}(p) : p \in A_{n,m}\}.$$

Thus, formally the value of a nice-name for an element of ℓ_∞ is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$. This can be treated as a code for an element of ℓ_∞, for example if we associate with such an f the element of ℓ_∞ (formally a subset of $\mathbb{R}^{\mathbb{N}}$) equal to $\lim_{m \to \infty} f(n, m)$ at n.

THEOREM 3.4: *Assume CH; $\mathbb{P}$ forces that there is no isomorphism of $C([0, \omega_2])$ into ℓ_∞/c_0.*

Proof. Assume CH and suppose that there was in $V^{\mathbb{P}}$ an isomorphism of $C([0, \omega_2])$ into ℓ_∞/c_0: let $\dot{T}$ be a name for it. Fix $k \in \mathbb{N}$ and $q \in \mathbb{Q}$ such that $\mathbb{P} \Vdash \|\dot{T}\| \leq \check{q}$ and $\mathbb{P} \Vdash \check{q} \cdot \|\dot{T}^{-1}\| < k - 1$.

For each $\alpha < \omega_2$, let $\dot{f}_\alpha$ be a nice-name for an element of ℓ_∞ such that $\mathbb{P} \Vdash [\dot{f}_\alpha]_{c_0} = \dot{T}(\check{\chi}_{[0,\alpha]})$, where $[\dot{f}_\alpha]_{c_0}$ denotes the equivalence class of $\dot{f}_\alpha$ in ℓ_∞/c_0. Let $A_\alpha = \text{supp}(\dot{f}_\alpha) \subseteq \omega_2$.

By CH and the Δ-system lemma, there is $X \subseteq \omega_2$ of cardinality ω_2 such that $(A_\alpha)_{\alpha \in X}$ form a Δ-system and $\alpha \in A_\alpha$.

Then, by thinning out using standard counting arguments, we may assume w.l.o.g. that whenever $\alpha < \beta$ and $\alpha, \beta \in X$, there is an order-preserving function $\sigma_{\alpha,\beta} : A_\alpha \to A_\beta$ such that $\sigma_{\alpha,\beta}(\alpha) = \beta$, which is constant on $A_\alpha \cap A_\beta$ and such that it lifts up to an isomorphism $\pi_{\alpha,\beta} : \mathbb{P}(A_\alpha \cup A_\beta) \to \mathbb{P}(A_\alpha \cup A_\beta)$ such that $\pi_{\alpha,\beta}^2 = Id_{\mathbb{P}(A_\alpha \cup A_\beta)}$ and $\pi_{\alpha,\beta}^*(\dot{f}_\alpha) = \dot{f}_\beta$, where π^* is the lifting of π to the $\mathbb{P}$-names as in Definition VII 7.12. of [12]. As any finite permutation is a composition of cycles, for any finite $F \subseteq X$ and any permutation $\sigma : F \to F$ there is a permutation $\pi_\sigma : \omega_2 \to \omega_2$ such that

$$(3.1) \qquad \pi_\sigma^*(\dot{f}_\alpha) = \dot{f}_{\sigma(\alpha)}.$$

Now, let σ be a permutation of ω_2 with the following property: there are $\alpha_1 < \cdots < \alpha_{2k} < \omega_2$ all in X such that

$$(3.2) \qquad \sigma(\alpha_{2i-1}) = \alpha_i \quad \text{and} \quad \sigma(\alpha_{2i}) = \alpha_{2k-(i-1)}, \quad \text{for all } 1 \leq i \leq k.$$

By Lemma VII 7.13 (c) of [12] and (3.1), for any formula ϕ and any permutation σ of X we have

$$(3.3) \qquad P \Vdash \phi(\dot{f}_{\alpha_1}, \ldots, \dot{f}_{\alpha_{2k}}) \quad \text{iff} \quad P \Vdash \phi(\dot{f}_{\sigma(\alpha_1)}, \ldots, \dot{f}_{\sigma(\alpha_{2k})}).$$

Notice that $((\alpha_{2i-1}, \alpha_{2i}])_{1 \leq i \leq k}$ are pairwise disjoint clopen intervals, and so it follows that

$$\left\| \sum_{i=1}^k \chi_{[0,\alpha_{2i}]} - \chi_{[0,\alpha_{2i-1}]} \right\| = \left\| \sum_{i=1}^k \chi_{(\alpha_{2i-1},\alpha_{2i}]} \right\| = \left\| \sum_{i=1}^k \chi_{(\alpha_1,\alpha_{2k}]} \right\| = 1,$$

which implies that

$$\mathbb{P} \Vdash \left\| \sum_{i=1}^k \chi_{[0,\alpha_{2i}]} - \chi_{[0,\alpha_{2i-1}]} \right\| = 1,$$

and consequently

$$\mathbb{P} \Vdash \Big\| \sum_{i=1}^{k} [\dot{f}_{\alpha_{2i}}]_{c_0} - [\dot{f}_{\alpha_{2i-1}}]_{c_0} \Big\| = \Big\| \dot{T}\Big(\sum_{i=1}^{k} \chi_{[0,\alpha_{2i}]} - \chi_{[0,\alpha_{2i-1}]} \Big) \Big\| \leq \check{q}.$$

Then, by (3.3),

$$(3.4) \qquad \mathbb{P} \Vdash \Big\| \sum_{i=1}^{k} [\dot{f}_{\sigma(\alpha_{2i})}]_{c_0} - [\dot{f}_{\sigma(\alpha_{2i-1})}]_{c_0} \Big\| \leq \check{q}.$$

But (3.2) yields that $\alpha_k \in (\alpha_i, \alpha_{2k-(i-1)}]$ for every $1 \leq i < k$, so that

$$\Big\| \sum_{i=1}^{k} \chi_{[0,\sigma(\alpha_{2i})]} - \chi_{[0,\sigma(\alpha_{2i-1})]} \Big\| = \Big\| \sum_{i=1}^{k} \chi_{[0,\alpha_{2k-(i-1)}]} - \chi_{[0,\alpha_i]} \Big\|$$

$$= \Big\| \sum_{i=1}^{k} \chi_{(\alpha_i, \alpha_{2k-(i-1)}]} \Big\| \geq k - 1,$$

and then, $\mathbb{P}$ also forces that

$$k - 1 \leq \Big\| \sum_{i=1}^{k} \chi_{[0,\sigma(\alpha_{2i})]} - \chi_{[0,\sigma(\alpha_{2i-1})]} \Big\| = \Big\| \dot{T}^{-1}\Big(\sum_{i=1}^{k} [\dot{f}_{\sigma(\alpha_{2i})}]_{c_0} - [\dot{f}_{\sigma(\alpha_{2i-1})}]_{c_0} \Big) \Big\|$$

$$\leq \|\dot{T}^{-1}\| \cdot \Big\| \sum_{i=1}^{k} [\dot{f}_{\sigma(\alpha_{2i})}]_{c_0} - [\dot{f}_{\sigma(\alpha_{2i-1})}]_{c_0} \Big\|,$$

which implies that

$$\mathbb{P} \Vdash \Big\| \sum_{i=1}^{k} [\dot{f}_{\sigma(\alpha_{2i})}]_{c_0} - [\dot{f}_{\sigma(\alpha_{2i-1})}]_{c_0} \Big\| \geq \frac{k-1}{\|\dot{T}^{-1}\|} > \check{q},$$

contradicting equation (3.4) and concluding the proof. ∎

It is natural to ask if we can directly conclude the nonexistence of an embedding of $C([0, \mathfrak{c}])$ into ℓ_∞/c_0 from the fact that $\wp(\mathbb{N})/Fin$ does not have well-ordered chains of length $\mathfrak{c}$ in this model. This could be done, for example, if we could prove that $Clop(K)$ has a well-ordered chain of length κ whenever $C([0, \kappa])$ embeds isomorphically into $C(K)$, for K totally disconnected.

However, this is not the case even if the embedding is isometric. It is clear that $C([0, \kappa])$ isometrically embeds into $C(K)$, where K is the dual ball of $C([0, \kappa])$ with the weak* topology. Using Kaplansky's theorem (Theorem 4.49 of [6]) saying that any Banach space has countable tightness in the weak topology and comparing the weak and the weak* topology in K, we can prove that in

K there are no sequences $(U_\xi)_{\xi<\omega_1}$ of weak*-open sets such that $\overline{U_\xi} \subseteq U_\eta$ for $\xi < \eta < \omega_1$. Moreover, if L is the standard totally disconnected preimage of K, we can prove (using quite tedious and technical arguments which we do not include here) that L has no uncountable well-ordered chains of clopen sets, but as L is a continuous preimage of K we have an isometric copy of $C([0,\kappa])$ inside $C(L)$.

Note, for example, that the situation with antichains instead of well-ordered chains is quite different. For a totally disconnected K, $C(K)$ contains a copy of $C(L)$ where L is the Stone space of the Boolean algebra generated by an uncountable pairwise disjoint family of elements (i.e., $C(L)$ is isomorphic to $c_0(\omega_1)$) if and only if the Boolean algebra $Clop(K)$ contains a pairwise disjoint family of cardinality ω_1 ([17], Theorem 12.30 (ii) of [6]).

However, we still do not know if in the concrete case of $K = \beta\mathbb{N} \setminus \mathbb{N}$ it is possible not to have well-ordered chains of length ω_2 of clopen sets and at the same time have an isomorphic (isometric) copy of $C([0,\omega_2])$ inside $C(\beta\mathbb{N}\setminus\mathbb{N}) \equiv l_\infty/c_0$.

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