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THIN-VERY TALL COMPACT SCATTERED SPACES WHICH ARE HEREDITARILY SEPARABLE

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Abstract. We strengthen the property Δ of a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ considered by Baumgartner and Shelah. This allows us to consider new types of amalgamations in the forcing used by Rabus, Juhász and Soukup to construct thin-very tall compact scattered spaces. We consistently obtain spaces K as above where K^n is hereditarily separable for each $n \in \mathbb{N}$. This serves as a counterexample concerning cardinal functions on compact spaces as well as having some applications in Banach spaces: the Banach space $C(K)$ is an Asplund space of density $\aleph_2$ which has no Fréchet smooth renorming, nor an uncountable biorthogonal system.

1. Introduction

Given a compact scattered space K, we call the derivative of K (denoted by K') the subset of K formed by its accumulation points, and we inductively define $K^{(\alpha)} = (K^{(\beta)})'$ if $\alpha = \beta+1$ and $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$ if α is a limit ordinal. The height of K, $ht(K)$, is the smallest ordinal α such that $K^{(\alpha)}$ is finite and nonempty, and the width of K, $wd(K)$, is the supremum of the cardinalities $|K^{(\alpha)} \setminus K^{(\alpha+1)}|$ for $\alpha < ht(K)$. We call $K = \bigcup_{\alpha<ht(K)} K^{(\alpha)} \setminus K^{(\alpha+1)}$ the Cantor-Bendixson decomposition of K and $K^{(\alpha)} \setminus K^{(\alpha+1)}$ its α^{th} Cantor-Bendixson level.

The purpose of this work is to show that the existence of compact hereditarily separable scattered spaces of height ω_2 is consistent with the usual axioms of set theory. For a given ordinal θ let us consider the following notation:

- A cw(θ) space is a compact scattered space of countable width and height equal to θ.
- An hs(θ) space is a compact scattered space which is hereditarily separable and of height equal to θ.

cw(ω_1) spaces are usually called thin-tall spaces and cw(ω_2) spaces are the thin-very tall spaces. First we remark that any hs(θ) space is a cw(θ) space as the Cantor-Bendixson levels form discrete subspaces. Whether there is or is not in ZFC a

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cw(ω_1) space was a question posed by Telgársky in 1968 (unpublished) and first (consistently) answered by Ostaszewski [21], using $\diamondsuit$. Rajagopalan constructed the first ZFC example of a cw(ω_1) in [23]. Further, Juhász and Weiss generalized these results (and simplified their proofs) in [12] proving in ZFC that for any ordinal $\theta < \omega_2$, there is a cw(θ) space.

For higher θ's the situation changes: in any model of CH there are no cw(ω_2) spaces and Just proved in [13] that neither are there such spaces in the Cohen model (where $\neg$ CH holds). On the other hand, Baumgartner and Shelah [2] constructed by forcing the first consistent example of a cw(ω_2) space. An interesting point of this forcing construction was the use of a new combinatorial device, called a function with the property Δ.

The main purpose of this work is to prove the consistency of the existence of an hs(ω_2) space. In fact, our space has even stronger properties: each of its finite powers is hereditarily separable. Whether consistently there are hs(ω_3) or even cw(ω_3) spaces remains a well-known open question. On the other hand, Martínez in [17] adopted the method of [2] to obtain the consistency of the existence of cw(θ) spaces for each $\theta < \omega_3$.

It follows from an old result of Shapirovskiĭ [25] that for any compact space K, $hd(K) \leq hL(K)^+$. Our construction shows that the dual inequality does not follow from ZFC, since for our compact space K, we have that $hL(K) = \aleph_2 \not\leq \aleph_1 = hd(K)^+$. Nevertheless, the dual inequality holds under GCH for regular spaces: since the weight $w(K)$ of a regular space K is less than or equal to $2^{d(K)}$ (see, for example, [8]), we trivially conclude that $hL(K) \leq w(K) \leq 2^{d(K)} = d(K)^+ \leq hd(K)^+$ under GCH.

Turning to properties of Banach spaces, let us first recall some definitions and results: a Banach space X is an Asplund space if every continuous and convex real-valued function on X is Fréchet smooth at all points of a G_δ dense subset of X. For separable Banach spaces, this is equivalent to admitting a Fréchet smooth renorming (see [4]). Namioka and Phelps proved in [19] that $C(K)$ is Asplund if and only if K is scattered. Thus, our $C(K)$ is an Asplund space.

Haydon constructed in [7] the first nonseparable Asplund space $C(K)$ which does not admit a Fréchet smooth renorming, concluding that the situation changes for nonseparable Asplund spaces. Later, Jiménez Sevilla and Moreno analyzed in [10] the structural properties of the space $C(K)$, where K is the well-known Kunen line constructed under CH (see [20]). They showed, for the Kunen line K, that $C(K)$ is also a nonseparable Asplund space with no Fréchet smooth renorming.

The weight of our space K is $\aleph_2$, so that $C(K)$ is an Asplund space of density $\aleph_2$. The fact that K is compact scattered and every finite power of K is hereditarily separable implies, in the same way as for the Kunen line, that $C(K)$ does not admit any Fréchet smooth renorming, but as in the case of the Kunen line we do not know if it admits a Gâteaux smooth renorming, or a Fréchet smooth bump function.

A biorthogonal system on a Banach space X is a family $(x_\alpha, \varphi_\alpha)_{\alpha<\kappa} \subseteq X \times X^*$ such that $\varphi_\alpha(x_\beta) = \delta_{\alpha,\beta}$, and a semi-biorthogonal system on a Banach space X is a sequence $(x_\alpha, \varphi_\alpha)_{\alpha<\kappa} \subseteq X \times X^*$ such that $\varphi_\alpha(x_\beta) = 1$ if $\alpha = \beta$, $\varphi_\alpha(x_\beta) = 0$ if $\alpha < \beta$ and $\varphi_\alpha(x_\beta) \geq 0$ if $\beta < \alpha$. Todorcevic showed in [29] (Theorem 9 together with the results of [3]) the existence of uncountable semi-biorthogonal systems in Banach spaces $C(K)$ of density strictly greater than $\aleph_1$. On the other hand, the fact that our space K is compact scattered and every finite power of K is hereditarily separable implies, in the same way as for the Kunen line, that $C(K)$ does not admit an uncountable biorthogonal system. It follows that Todorcevic's result cannot be improved in ZFC by replacing the existence of uncountable semi-biorthogonal systems by the existence of uncountable biorthogonal systems in spaces $C(K)$ of large density. On the other hand it is proved in [29] that it is consistent that every nonseparable Banach space has an uncountable biorthogonal system, showing that the existence of a Banach space such as ours or Kunen's cannot be proved in ZFC.

Our construction is based on the Juhász and Soukup [11] interpretation of Rabus' work [22], where he modified the Baumgartner-Shelah forcing from [2] to obtain a countably tight space which is initially ω_1-compact and noncompact, answering a question of Dow and van Douwen.

This paper is organized as follows: we finish this section by reviewing the method of Juhász and Soukup and some related results and definitions which we will need afterwards. In Section 2 we prove the key lemma which enables us to prove our main result in a straightforward way. This lemma introduces a new way of amalgamating conditions in forcings which add thin-very tall spaces. One can apply these amalgamations in the generic construction if one strengthens the property Δ of a function involved in the forcing. In Section 3, we introduce the strong property Δ and, assuming the existence of a function which satisfies it, we prove the main results and analyze their consequences in topological and functional analytic terms. Section 4 is devoted to establishing the consistency of the existence of a function with the strong property Δ. Section 4 is due to the second author and the remaining sections to the first author.

The notation and terminology used are those of [11]. Given a set X, $\wp(X)$ is the power set of X and, given a cardinal κ, $[X]^\kappa$ (resp. $[X]^{\leq\kappa}$ and $[X]^{<\kappa}$) denotes the family of subsets of X of cardinality equal to κ (resp. less than or equal to κ and less than κ).

Let us start by recalling the definition of the property Δ:

Definition 1.1 (Baumgartner, Shelah, [2], p.122)**.** A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if $f(\{\xi,\eta\}) \subseteq \min\{\xi,\eta\}$ for all $\{\xi,\eta\} \in [\omega_2]^2$ and for any uncountable family $\mathcal{A}$ of finite subsets of ω_2, there are distinct $a, b \in \mathcal{A}$ such that for any $\tau \in a \cap b$, any $\xi \in a \setminus b$ and any $\eta \in b \setminus a$ we have:

1) $a \cap b \cap \min\{\xi,\eta\} \subseteq f(\{\xi,\eta\})$;
2) $\tau < \xi \ \Rightarrow f(\{\tau,\eta\}) \subseteq f(\{\xi,\eta\})$;
3) $\tau < \eta \ \Rightarrow f(\{\tau,\xi\}) \subseteq f(\{\xi,\eta\})$.

Now, we fix a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ with the property Δ.

Definition 1.2 (Juhász, Soukup [11], Definition 2.1)**.** Let $\mathbb{P}_f$ be the forcing formed by conditions $p = (D_p, h_p, i_p)$, where:

1. $D_p \in [\omega_2]^{<\omega}$,
2. $h_p : D_p \to \wp(D_p)$ and for all $\xi \in D_p$, $\max h_p(\xi) = \xi$,
3. $i_p : [D_p]^2 \to [D_p]^{<\omega}$ and for all $\xi, \eta \in D_p$, $\xi < \eta$, we have that:
   (a) if $\xi \in h_p(\eta)$, then $h_p(\xi) \setminus h_p(\eta) \subseteq \bigcup_{\gamma \in i_p(\{\xi,\eta\})} h_p(\gamma)$,
   (b) if $\xi \notin h_p(\eta)$, then $h_p(\xi) \cap h_p(\eta) \subseteq \bigcup_{\gamma \in i_p(\{\xi,\eta\})} h_p(\gamma)$,
   (c) $i_p(\{\xi,\eta\}) \subseteq f(\{\xi,\eta\})$,

ordered by $p \leq q$ if $D_p \supseteq D_q$, for all $\xi \in D_q$, $h_p(\xi) \cap D_q = h_q(\xi)$ and $i_p|_{[D_q]^2} = i_q$.

To simplify notation, it is convenient to define the following:

Definition 1.3 (Juhász, Soukup [11])**.** Given finite nonempty sets of ordinals x and y such that $\max x < \max y$, we define

$$x * y = \begin{cases} x \setminus y & \text{if } \max x \in y, \\ x \cap y & \text{if } \max x \notin y. \end{cases}$$

We now rewrite conditions 3(a) and 3(b) of the definition of the forcing as

$$h_p(\xi) * h_p(\eta) \subseteq \bigcup_{\gamma \in i_p(\{\xi,\eta\})} h_p(\gamma).$$

To define the space K_f, fix the ground model V and a generic filter G.

Definition 1.4 (Juhász, Soukup [11], Definition 2.3)**.** For each $\xi < \eta < \omega_2$, working in $V^{\mathbb{P}_f}$, let

$$h(\xi) = \bigcup_{p \in G} h_p(\xi) \quad \text{and} \quad i(\{\xi,\eta\}) = \bigcup_{p \in G} i_p(\{\xi,\eta\}),$$

and let L_f be the topological space (ω_2, τ), where τ is the topology on ω_2 which has the family of sets

$$\{h(\xi) : \xi < \omega_2\} \cup \{\omega_2 \setminus h(\xi) : \xi < \omega_2\}$$

as a topological subbasis. We call $h(\xi)$ the generic neighborhood of ξ.

From Theorem 1.5 of [11], it follows that for all $\xi < \omega_2$, $h(\xi)$ is a compact subspace of (ω_2, τ) and it easy to check that

$$(+) \qquad \{h(\xi) \setminus \bigcup_{\eta \in F} h(\eta) : F \in [\xi]^{<\omega}\} \text{ forms a local topological basis at } \xi.$$

Therefore L_f is a locally compact scattered zero-dimensional space.

We are now ready to define K_f:

Definition 1.5. In $V^{\mathbb{P}_f}$, K_f is the one-point compactification of L_f. The point of compactification is denoted $*$; thus $K_f \setminus L_f = \{*\}$.

In particular, we use the following results.

Theorem 1.6 (Rabus [22], Lemma 4.1; Juhász, Soukup [11], Lemma 2.8)**.** $\mathbb{P}_f$ *satisfies c.c.c.*

Proposition 1.7. $V^{\mathbb{P}_f}$ *satisfies "K_f is a compact scattered zero-dimensional space".*

2. Amalgamating conditions

In this section, we present the key lemma needed to prove our main result. Let us start with some preliminaries and auxiliary lemmas.

Definition 2.1. Let $p_1 = (D_1, h_1, i_1)$, $p_2 = (D_2, h_2, i_2) \in \mathbb{P}_f$ be two conditions. We say that p_1 and p_2 are isomorphic conditions if there is an order-preserving bijective function $e : D_1 \to D_2$ satisfying the following conditions:

(a) if $\xi, \eta \in D_1$, then $\xi \in h_1(\eta)$ if and only if $e(\xi) \in h_2(e(\eta))$;
(b) if $\xi \in D_1 \cap D_2$, then $e(\xi) = \xi$.

In this case, if the order-preserving bijection e is such that $\xi \leq e(\xi)$ for every $\xi \in D_1$ we say that p_1 is lower than p_2.

For example we have the following:

Lemma 2.2. *Let $p_1 = (D_1, h_1, i_1)$, $p_2 = (D_2, h_2, i_2) \in \mathbb{P}_f$ be two isomorphic conditions and let $e : D_1 \to D_2$ be the order-preserving bijection. Then for every $\xi \in D_1 \cap D_2$,*

(a) $(h_1(\xi) \cup h_2(\xi)) \cap D_1 = h_1(\xi)$,
(b) $(h_1(\xi) \cup h_2(\xi)) \cap D_2 = h_2(\xi)$,
(c) $h_1(\xi) = e^{-1}[h_2(\xi)]$.

Proof. Directly from Definition 2.1. □

Definition 2.3 (Juhász, Soukup [11])**.** Given $p_1 = (D_1, h_1, i_1), p_2 = (D_2, h_2, i_2) \in \mathbb{P}_f$, define a mapping $\delta_2 : dom(\delta_2) \to D_1 \cap D_2$, where

$$dom(\delta_2) = \{\eta \in D_2 : \text{there is } \delta \in D_1 \cap D_2 \text{ such that } \eta \in h_2(\delta)\}$$

and

$$\delta_2(\eta) = \min\{\delta \in D_1 \cap D_2 : \eta \in h_2(\delta)\}.$$

Lemma 2.4. *Suppose that $p_1 = (D_1, h_1, i_1)$ and $p_2 = (D_2, h_2, i_2) \in \mathbb{P}_f$ are two conditions. Then,*

(a) for all $\eta \in dom(\delta_2) \setminus D_1$, we have that $\eta < \delta_2(\eta)$ and
(b) for all $\eta \in D_1 \cap D_2$ we have $\eta \in dom(\delta_2)$ and $\delta_2(\eta) = \eta$.

Proof. Directly from Definition 2.3. □

We prove the next lemma, for the reader's convenience.

Lemma 2.5 (Juhász, Soukup [11])**.** *Let $p_1 = (D_1, h_1, i_1)$, $p_2 = (D_2, h_2, i_2) \in \mathbb{P}_f$ be two isomorphic conditions. If $\xi \in D_1 \cap D_2$, then*

$$h_2(\xi) = \delta_2^{-1}[h_1(\xi)].$$

Proof. Let $\xi \in D_1 \cap D_2$.

Suppose that $\eta \in dom(\delta_2)$ and $\delta_2(\eta) \in h_1(\xi)$. Since $\delta_2(\eta), \xi \in D_1 \cap D_2$, it follows from Definition 2.1(a) and (b) that $\delta_2(\eta) \in h_2(\xi)$. Suppose that $\eta \notin h_2(\xi)$. Then, $\eta \in h_2(\delta_2(\eta)) * h_2(\xi)$ so that there is $\delta \in i_2(\{\delta_2(\eta), \xi\})$ such that $\eta \in h_2(\delta)$, which contradicts the minimality of $\delta_2(\eta)$ and concludes the proof of the inclusion $\delta_2^{-1}[h_1(\xi)] \subseteq h_2(\xi)$.

Reciprocally, if $\eta \in h_2(\xi)$, then $\eta \in dom(\delta_2)$. Suppose that $\delta_2(\eta) \notin h_2(\xi)$. Then, $\eta \in h_2(\delta_2(\eta)) * h_2(\xi)$ so that there is $\delta \in i_2(\{\delta_2(\eta), \xi\})$ such that $\eta \in h_2(\delta)$, which contradicts the minimality of $\delta_2(\eta)$. So, $\delta_2(\eta) \in h_2(\xi)$ and since $\delta_2(\eta), \xi \in D_1 \cap D_2$, it follows from Definition 2.1(a) and (b) that $\delta_2(\eta) \in h_1(\xi)$, concluding the proof of the lemma. □

In the proof of c.c.c., Rabus, Juhász and Soukup considered the minimal amalgamation which is constructed in a symmetric way with respect to both of the conditions being extended. We will consider an asymmetric amalgamation. The lack of symmetry in our amalgamation is the result of using two functions, δ_2 and e, in the definition of the amalgamation. The final auxiliary lemma below characterizes the sets given by the operation $*$ for elements of the extended condition. The role of the function g will be played by δ_2 or by e.

Lemma 2.6. *Let $p = (D_p, h_p, i_p) \in \mathbb{P}_f$ and let $D_q \in [\omega_2]^{<\omega}$, $h_q : D_q \to \wp(D_q)$ and $g : dom(g) \to D_p$ be such that*

(i) $D_p \subseteq D_q$, $dom(g) \subseteq D_q$,
(ii) for all $\xi \in D_p \cap dom(g)$ we have $g(\xi) = \xi$,
(iii) for all $\xi \in D_p$, $h_q(\xi) = h_p(\xi) \cup g^{-1}[h_p(\xi)]$.

Then, for all $\xi, \eta \in D_p$, $\xi < \eta$, we have that

$$(h_q(\xi) * h_q(\eta)) \cap D_p = h_p(\xi) * h_p(\eta)$$

and

$$(h_q(\xi) * h_q(\eta)) \cap (D_q \setminus D_p) = g^{-1}[h_p(\xi) * h_p(\eta)] \cap (D_q \setminus D_p).$$

Proof. Since $\xi, \eta \in D_p$, $\xi < \eta$, by (ii) we have that $\xi \in h_p(\eta)$ if and only if $\xi \in h_q(\eta)$, so (ii) obviously gives $(h_q(\xi) * h_q(\eta)) \cap D_p = h_p(\xi) * h_p(\eta)$.

Now suppose $\xi \in h_q(\eta)$, so $\xi \in h_p(\eta)$ and so by (iii),

$$(h_q(\xi) \setminus h_q(\eta)) \cap (D_q \setminus D_p) = (h_q(\xi) \cap (D_q \setminus D_p)) \setminus (h_q(\eta) \cap (D_q \setminus D_p))$$

$$= (g^{-1}[h_p(\xi)] \cap (D_q \setminus D_p)) \setminus (g^{-1}[h_p(\eta)] \cap (D_q \setminus D_p)) = g^{-1}[h_p(\xi) \setminus h_p(\eta)] \cap (D_q \setminus D_p).$$

On the other hand, if $\xi \notin h_q(\eta)$, then $\xi \notin h_p(\eta)$ and so by (iii),

$$(h_q(\xi) \cap h_q(\eta)) \cap (D_q \setminus D_p) = (h_q(\xi) \cap (D_q \setminus D_p)) \cap (h_q(\eta) \cap (D_q \setminus D_p))$$

$$= (g^{-1}[h_p(\xi)] \cap (D_q \setminus D_p)) \cap (g^{-1}[h_p(\eta)] \cap (D_q \setminus D_p)) = g^{-1}[h_p(\xi) \cap h_p(\eta)] \cap (D_q \setminus D_p),$$

concluding the proof of the lemma. □

Now we go to our key lemma: a strong hypothesis about the behaviour of the function f allows us to amalgamate two isomorphic conditions, one lower than the other, into a common extension q in such a way that $h(\xi) \cap D_q \subseteq h[e(\xi)] \cap D_q$ for ξ in the domain of the lower of the two conditions.

Lemma 2.7. *Let $p_1 = (D_1, h_1, i_1)$, $p_2 = (D_2, h_2, i_2) \in \mathbb{P}_f$ be two isomorphic conditions and suppose p_1 is lower than p_2. Let $e : D_1 \to D_2$ be the order-preserving bijective function and assume that*

(A) if $\xi, \eta \in D_1 \cap D_2$ and $\xi \neq \eta$, then $i_1(\{\xi, \eta\}) = i_2(\{\xi, \eta\})$;
(B) for all $\zeta \in D_1 \cap D_2$, all $\xi \in D_1 \setminus D_2$ and all $\eta \in D_2 \setminus D_1$:
(i) if $\zeta < \xi$, then $f(\{\zeta, \eta\}) \subseteq f(\{\xi, \eta\})$;
(ii) $D_1 \cap \xi \cap \eta \subseteq f(\{\xi, \eta\})$.

Then there is $q \in \mathbb{P}_f$, $q \leq p_1, p_2$, such that for all $\xi \in D_1$ and all $\eta \in D_2$:

$$\xi \in h_q(\eta) \textit{ if and only if } e(\xi) \in h_2(\eta).$$

Proof. We define $q = (D_q, h_q, i_q)$ by: $D_q = D_1 \cup D_2$;

$$h_q(\xi) = \begin{cases} h_1(\xi) \cup \delta_2^{-1}[h_1(\xi)] & \text{if } \xi \in D_1, \\ h_2(\xi) \cup e^{-1}[h_2(\xi)] & \text{if } \xi \in D_2, \end{cases}$$

and

$$i_q(\{\xi, \eta\}) = \begin{cases} i_1(\{\xi, \eta\}) & \text{if } \xi, \eta \in D_1, \\ i_2(\{\xi, \eta\}) & \text{if } \xi, \eta \in D_2, \\ f(\{\xi, \eta\}) \cap D_q & \text{otherwise.} \end{cases}$$

Note that (A) implies that the set $i_q(\{\xi, \eta\})$ is well-defined for any $\xi, \eta \in D_1 \cap D_2$, $\xi \neq \eta$; clearly i_q is well-defined for the other pairs. Also, if $\xi \in D_1 \cap D_2$, then the set $h_q(\xi)$ is well-defined because both of the conditions reduce to $h_q(\xi) = h_1(\xi) \cup h_2(\xi)$ by Lemmas 2.5 and 2.2(c).

We have to show that $q \in \mathbb{P}_f$, i.e., that q satisfies conditions 1, 2 and 3 from Definition 1.2. The fact that q satisfies conditions 1.2.1 and 1.2.3(c) follows directly from the definition of q and from the fact that $p_1, p_2 \in \mathbb{P}_f$. Condition 1.2.2 is satisfied because $p_1, p_2 \in \mathbb{P}_f$ and the functions e and δ_2 are nondecreasing. In what follows we will be using Lemma 2.6 for $p = p_1, p_2$ and $g = \delta_2, e$, respectively. The hypothesis of the lemma is satisfied for these objects by 2.4(b) and 2.1(b).

Now we check conditions 1.2.3(a) and (b). Let $\xi, \eta \in D_q$, $\xi < \eta$, and consider the following cases:

Case 1. $\xi, \eta \in D_1$.

It follows from the definition of q and from Lemma 2.6 that

$$(h_q(\xi) * h_q(\eta)) \cap D_1 = h_1(\xi) * h_1(\eta)$$

and

$$(h_q(\xi) * h_q(\eta)) \cap (D_2 \setminus D_1) = \delta_2^{-1}[h_1(\xi) * h_1(\eta)] \cap (D_2 \setminus D_1).$$

Now let $\zeta \in h_q(\xi) * h_q(\eta)$.

Subcase 1.1. $\zeta \in D_1$.

In this subcase, $\zeta \in h_1(\xi) * h_1(\eta)$ and there is $\gamma \in i_1(\{\xi, \eta\}) = i_q(\{\xi, \eta\})$ such that $\zeta \in h_1(\gamma) \subseteq h_q(\gamma)$, as we wanted.

Subcase 1.2. $\zeta \in D_2 \setminus D_1$.

In this subcase, $\delta_2(\zeta) \in h_1(\xi) * h_1(\eta)$ and, since $\delta_2(\zeta), \xi, \eta \in D_1$ and $p_1 \in \mathbb{P}_f$, there is $\gamma \in i_1(\{\xi, \eta\}) = i_q(\{\xi, \eta\})$ such that $\delta_2(\zeta) \in h_1(\gamma)$. Since $\gamma \in D_1$, it follows by the definition of q that $\zeta \in h_q(\gamma)$, as we wanted.

Case 2. $\xi, \eta \in D_2$.

It follows from the definition of q and from Lemma 2.6 that

$$(h_q(\xi) * h_q(\eta)) \cap D_1 = h_1(\xi) * h_1(\eta)$$

and

$$(h_q(\xi) * h_q(\eta)) \cap (D_2 \setminus D_1) = e^{-1}[h_1(\xi) * h_1(\eta)] \cap (D_2 \setminus D_1).$$

Now let $\zeta \in h_q(\xi) * h_q(\eta)$.

Subcase 2.1. $\zeta \in D_1 \setminus D_2$.

In this subcase, $e(\zeta) \in h_2(\xi) * h_2(\eta)$ and, since $e(\zeta), \xi, \eta \in D_2$ and $p_2 \in \mathbb{P}_f$, there is $\gamma \in i_2(\{\xi, \eta\}) = i_q(\{\xi, \eta\})$ such that $e(\zeta) \in h_2(\gamma)$. Since $\gamma \in D_2$, it follows by the definition of q that $\zeta \in h_q(\gamma)$, as we wanted.

Subcase 2.2. $\zeta \in D_2$.

In this subcase, $\zeta \in h_2(\xi) * h_2(\eta)$ and there is $\gamma \in i_2(\{\xi, \eta\}) = i_q(\{\xi, \eta\})$ such that $\zeta \in h_2(\gamma) \subseteq h_q(\gamma)$, as we wanted.

Case 3. $\xi \in D_1 \setminus D_2$ and $\eta \in D_2 \setminus D_1$.

Here we fix $\zeta \in h_q(\xi) * h_q(\eta)$ and we consider the following subcases:

Subcase 3.1. $\zeta \in D_1$.

In this subcase, $\zeta \in D_1 \cap \xi \cap \eta$ and it follows from (B)(ii) that $\zeta \in f(\{\xi, \eta\})$. Hence, $\zeta \in D_1 \cap f(\{\xi, \eta\}) \subseteq D_q \cap f(\{\xi, \eta\}) = i_q(\{\xi, \eta\})$. Taking $\gamma = \zeta$, we conclude that $\zeta \in h_q(\gamma)$ and $\gamma \in i_q(\{\xi, \eta\})$, as we wanted.

Subcase 3.2. $\zeta \in D_2 \setminus D_1$.

First note that, regardless of the fact whether $h_q(\xi) * h_q(\eta) = h_q(\xi) \cap h_q(\eta)$ or $h_q(\xi) * h_q(\eta) = h_q(\xi) \setminus h_q(\eta)$, the assumption $\zeta \in h_q(\xi) * h_q(\eta)$ implies that $\zeta \in h_q(\xi)$.

In this subcase, it follows from the definition of $h_q(\xi)$ that $\delta_2(\zeta) \in h_1(\xi)$, so that $\delta_2(\zeta) \in D_1 \cap \xi \cap \eta$, and it follows from (B)(ii) that $\delta_2(\zeta) \in f(\{\xi, \eta\}) \cap D_q = i_q(\{\xi, \eta\})$. By the definition of $\delta_2(\zeta)$, $\zeta \in h_2(\delta_2(\zeta)) \subseteq h_q(\delta_2(\zeta))$. Taking $\gamma = \delta_2(\zeta)$, we have that $\zeta \in h_q(\gamma)$ and $\gamma \in i_q(\{\xi, \eta\})$, concluding the proof of this subcase.

Case 4. $\xi \in D_2 \setminus D_1$ and $\eta \in D_1 \setminus D_2$.

Again we fix $\zeta \in h_q(\xi) * h_q(\eta)$ and we consider the following subcases:

Subcase 4.1. $\zeta \in D_1$.

The proof in this subcase follows identically to the proof of Subcase 3.1.

Subcase 4.2.[1] $\zeta \in D_2 \setminus D_1$. We start this last subcase by proving the following:

Fact 1. $\{\zeta, \xi\} \cap dom(\delta_2)$ is a nonempty set such that $\min \delta_2[\{\zeta, \xi\}] < \eta$ and if both ζ and ξ are in $dom(\delta_2)$, then $\delta_2(\zeta) \neq \delta_2(\xi)$.

Proof of Fact 1. First remark that, from the definition of $*$, it follows that if $\xi \notin h_q(\eta)$, then $\zeta \in h_q(\xi) * h_q(\eta) = h_q(\xi) \cap h_q(\eta)$ and therefore $\zeta \in h_q(\eta)$. Analogously, if $\xi \in h_q(\eta)$, then $\zeta \in h_q(\xi) * h_q(\eta) = h_q(\xi) \setminus h_q(\eta)$ and therefore $\zeta \notin h_q(\eta)$. So,

$$|\{\zeta, \xi\} \cap h_q(\eta)| = 1.$$

From the definition of q we have that, since $\xi, \zeta \notin D_1$ and $\eta \in D_1$ in this subcase, the above means that

$$|\{\zeta, \xi\} \cap \delta_2^{-1}[h_1(\eta)]| = 1,$$

so that $\{\zeta, \xi\} \cap dom(\delta_2)$ is a nonempty set. The above observation also implies that $\delta_2[\{\zeta, \xi\}] \cap h_1(\eta) \neq \emptyset$ and so, $\min \delta_2[\{\zeta, \xi\}] \leq \eta$. Now since $\eta \notin D_2$ and the range of δ_2 is included in $D_1 \cap D_2$, the inequality must be strict.

Finally, we have seen that $\zeta \in h_q(\eta)$ if and only if $\xi \notin h_q(\eta)$ and so, if both ζ and ξ are in the domain of δ_2, it follows that $\delta_2(\zeta) \in h_1(\eta)$ if and only if $\delta_2(\xi) \notin h_1(\eta)$, so that $\delta_2(\zeta) \neq \delta_2(\xi)$, concluding the proof of Fact 1.

Take $\theta = \min\{\delta_2(\xi), \delta_2(\zeta)\}$ and note that $\theta \neq \xi$ since $\xi \in D_2 \setminus D_1$ and the range of δ_2 is included in $D_1 \cap D_2$. We go now to the following subcases:

Subcase 4.2.1. $\theta < \xi$.

Here, $\theta \neq \delta_2(\xi)$ and therefore $\delta_2(\zeta) = \theta \in D_1 \cap \xi \cap \eta$. From condition (B)(ii), it follows that $\delta_2(\zeta) \in f(\{\xi, \eta\}) \cap D_q = i_q(\{\xi, \eta\})$. Since $\zeta \in h_2(\delta_2(\zeta)) \subseteq h_q(\delta_2(\zeta))$, taking $\gamma = \delta_2(\zeta)$, we have that $\zeta \in h_q(\gamma)$ and $\gamma \in i_q(\{\xi, \eta\})$, as we wanted.

Subcase 4.2.2. $\theta > \xi$.

Note that $\zeta \in h_q(\xi) * h_q(\eta) \subseteq h_q(\xi)$. Since ζ and ξ satisfying the hypothesis of Case 4.2 are in $D_2 \setminus D_1$, it follows from the definition of h_q that $\zeta \in h_2(\xi)$. To finish, let us show the following:

Fact 2. $\zeta \in h_2(\xi) * h_2(\theta)$.

[1] This case is similar to Subcase 2.2 in the proof of Claim 2.7.2 of [11].

Proof of Fact 2. First suppose $\theta = \delta_2(\xi)$. If $\zeta \notin dom(\delta_2)$, then $\zeta \notin h_2(\delta_2(\xi))$. If $\zeta \in dom(\delta_2)$, from Fact 1 and the minimality of $\delta_2(\zeta)$, it follows that $\zeta \notin h_2(\delta_2(\xi))$. Since $\xi \in h_2(\delta_2(\xi))$, we have that $\zeta \in h_2(\xi) \setminus h_2(\delta_2(\xi)) = h_2(\xi) * h_2(\theta)$.

Now suppose $\theta = \delta_2(\zeta)$. Analogously we prove that $\xi \notin h_2(\delta_2(\zeta))$ and $\zeta \in h_2(\xi) \cap h_2(\delta_2(\zeta)) = h_2(\xi) * h_2(\theta)$, concluding the proof of Fact 2.

Finally, since $p_2 \in \mathbb{P}_f$, there is $\gamma \in i_2(\{\xi, \theta\})$ such that $\zeta \in h_2(\gamma) \subseteq h_q(\gamma)$. By condition (B)(i), which can be used by Fact 1, we have that

$$i_2(\{\xi,\theta\}) \subseteq f(\{\xi,\theta\}) \cap D_2 \subseteq f(\{\xi,\eta\}) \cap D_q = i_q(\{\xi,\eta\}).$$

Hence, $\gamma \in i_q(\{\xi,\eta\})$ and $\zeta \in h_q(\gamma)$, concluding the proof of Subcase 4.2, Case 4 and thus concluding the proof of Claim 2.

Now that we know that $q \in \mathbb{P}_f$, let us check the other conclusions: it follows easily from the definition of q and Lemma 2.5 that $q \le p_1$ and analogously it follows from the definition of q and Lemma 2.4 that $q \le p_2$.

Finally, we verify the condition we want q to satisfy, that is, $\xi \in h_2(\eta) \cup e^{-1}[h_2(\eta)]$ if and only if $e(\xi) \in h_2(\eta)$: let $\xi \in D_1$ and $\eta \in D_2$ and consider again the following cases:

Case 1. $\xi \in D_1 \cap D_2$.

It follows from the fact that in this case $e(\xi) = \xi$.

Case 2. $\xi \in D_1 \setminus D_2$.

In this case, $\xi \in h_2(\eta) \cup e^{-1}[h_2(\eta)]$ if and only if $\xi \in e^{-1}[h_2(\eta)]$ if and only if $e(\xi) \in h_2(\eta)$, concluding the proof of the lemma. $\square$

3. The main results

To apply the key lemma proved in the previous section, the function f on which the forcing $\mathbb{P}_f$ depends must satisfy a stronger version of the property Δ:

Definition 3.1. A function $f : [\omega_2]^2 \to [\omega_2]^{\le\omega}$ has the strong property Δ if $f(\{\xi,\eta\}) \subseteq \min\{\xi,\eta\}$ for all $\{\xi,\eta\} \in [\omega_2]^2$ and for any uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2, there are distinct $a, b \in \mathcal{A}$ and an order-preserving bijection $e : a \to b$ which is the identity on $a \cap b$ and such that $\xi \le e(\xi)$ for all $\xi \in a$ and for any $\tau \in a \cap b$, any $\xi \in a \setminus b$ and any $\eta \in b \setminus a$ we have:

1) $a \cap \min\{\xi,\eta\} \subseteq f(\{\xi,\eta\})$,
2) $\tau < \xi \;\Rightarrow f(\{\tau,\eta\}) \subseteq f(\{\xi,\eta\})$,
3) $\tau < \eta \;\Rightarrow f(\{\tau,\xi\}) \subseteq f(\{\xi,\eta\})$.

Finally we arrive at the main result of this paper.

Theorem 3.2. *If $f : [\omega_2]^2 \to [\omega_2]^{\le\omega}$ has the strong property Δ, then $V^{\mathbb{P}_f}$ satisfies "for all $n \in \mathbb{N}$, K_f^n is hereditarily separable".*

Proof. We prove this by induction on $n \in \mathbb{N}$: in $V^{\mathbb{P}_f}$, fix $n \in \mathbb{N}$ and suppose that for all $0 \le i < n$, K_f^i is hereditarily separable (take $K_f^0 = \{*\}$) and let us show that K_f^n is hereditarily separable. We will be using a well-known fact that a regular space is hereditarily separable if and only if it has no uncountable left-separated sequence (see Theorem 3.1 of [24]).

In V, suppose $(\dot{x}_\alpha)_{\alpha<\omega_1}$ is a sequence of names such that $\mathbb{P}_f$ forces that $(\dot{x}_\alpha)_{\alpha<\omega_1}$ is a left-separated sequence in K_f^n of cardinality $\aleph_1$ and for each $\alpha < \omega_1$, we have that $\dot{x}_\alpha = (\dot{x}_1^\alpha, \dots, \dot{x}_n^\alpha)$, where each $\dot{x}_i^\alpha$ is a name for an element of K_f.

Notice that if

$$\mathbb{P}_f \Vdash \exists 1 \leq i \leq n,\ \exists X \subseteq \omega_1,\ |X| = \aleph_1 \text{ such that } \forall \alpha, \beta \in X,\ \dot{x}_i^\alpha = \dot{x}_i^\beta,$$

then

$$\mathbb{P}_f \Vdash \exists 1 \leq i \leq n,\ \exists X \subseteq \omega_1,\ |X| = \aleph_1 \text{ such that } ((\dot{x}_1^\alpha, \ldots, \dot{x}_{i-1}^\alpha, \dot{x}_{i+1}^\alpha, \ldots, \dot{x}_n^\alpha))_{\alpha \in X}$$
$$\text{is a left-separated sequence in } K_f^{n-1},$$

contradicting the inductive hypothesis. Therefore, we can assume without loss of generality that $\mathbb{P}_f$ forces that for all $1 \leq i \leq n$ and all $\alpha < \beta < \omega_1$, $\dot{x}_i^\alpha \neq \dot{x}_i{}^\beta$ and $\dot{x}_i^\alpha \in L_f = K_f \setminus \{*\}$.

By assertion $(+)$ following Definition 1.4, for each $\alpha < \omega_1$, there are names $\dot{F}_1^\alpha, \ldots, \dot{F}_n^\alpha$ for finite subsets of ω_2 such that $\mathbb{P}_f$ forces that

$$\forall \alpha < \omega_1 \quad \forall 1 \leq i \leq n \quad \dot{x}_i^\alpha \in h(\dot{x}_i^\alpha) \setminus \bigcup_{\xi \in \dot{F}_i^\alpha} h(\xi)$$

and

$$\forall \alpha < \beta < \omega_1 \quad \exists 1 \leq i \leq n \quad \dot{x}_i^\alpha \notin h(\dot{x}_i^\beta) \setminus \bigcup_{\xi \in \dot{F}_i^\beta} h(\xi).$$

For each $\alpha < \omega_1$, let $p_\alpha = (D_\alpha, h_\alpha, i_\alpha) \in \mathbb{P}_f$, $x_1^\alpha, \ldots, x_n^\alpha \in \omega_2$ and $F_1^\alpha, \ldots, F_n^\alpha \subseteq \omega_2$ be finite such that

$$p_\alpha \Vdash \forall 1 \leq i \leq n \quad \dot{x}_i^\alpha = \check{x}_i^\alpha \text{ and } \dot{F}_i^\alpha = \check{F}_i^\alpha.$$

By Lemma 2.2 of [11], we can assume without loss of generality that for all $\alpha < \omega_1$ and all $1 \leq i \leq n$, $F_i^\alpha \subseteq D_\alpha$ and $x_i^\alpha \in D_\alpha$.

By the Δ-system Lemma, we can assume as well that $(D_\alpha)_{\alpha<\omega_1}$ forms a Δ-system with root D. Since for each pair $\{\xi, \eta\} \subseteq D$ and each $\alpha < \omega_1$, we have that $i_\alpha(\{\xi, \eta\}) \in [f(\{\xi, \eta\})]^{<\omega}$, we may assume that for all $\alpha < \beta < \omega_1$, if $\xi, \eta \in D$, $\xi \neq \eta$, then $i_\alpha(\{\xi, \eta\}) = i_\beta(\{\xi, \eta\})$.

By thinning out, we can assume without loss of generality that $(D_\alpha)_{\alpha<\omega_1}$ forms a Δ-system with root D such that for every $\alpha < \beta < \omega_1$:

- p_α is isomorphic to p_β;
- p_α is lower than p_β;
- if $e_{\alpha\beta} : D_\alpha \to D_\beta$ is the order-preserving bijective function, then $e_{\alpha\beta}(x_i^\alpha) = x_i^\beta$, for all $1 \leq i \leq n$.

Finally, we may assume that for all $1 \leq i \leq n$ we have: either $x_i^\alpha = x_i^\beta$ for all $\alpha < \beta < \omega_1$, or $x_i^\alpha \notin D$ for all $\alpha < \omega_1$, and actually the second case holds by our initial assumption about the sequence.

Since f has the strong property Δ, there are $\alpha < \beta < \omega_1$ such that for all $\zeta \in D$, all $\xi \in D_\alpha \setminus D$ and all $\eta \in D_\beta \setminus D$:

(i) $D_\alpha \cap \xi \cap \eta \subseteq f(\{\xi, \eta\})$;
(ii) if $\zeta < \xi$, then $f(\{\zeta, \eta\}) \subseteq f(\{\xi, \eta\})$;
(iii) if $\zeta < \eta$, then $f(\{\zeta, \xi\}) \subseteq f(\{\xi, \eta\})$.

Note that p_α and p_β satisfy the hypothesis of Lemma 2.7. Hence, there is $q \leq p_\alpha, p_\beta$ in $\mathbb{P}_f$ such that for all $\xi \in D_\alpha$ and all $\eta \in D_\beta$,

$$\xi \in h_q(\eta) \text{ if and only if } e_{\alpha\beta}(\xi) \in h_{p_\beta}(\eta).$$

Then, for all $1 \leq i \leq n$ and all $\xi \in D_\beta$, we have that

$$x_i^\alpha \in h_q(\xi) \text{ if and only if } x_i^\beta \in h_{p_\beta}(\xi).$$

So we have that
$$x_i^\alpha \in h_q(x_i^\beta) \setminus \bigcup_{\xi \in F_i^\beta} h_q(\xi).$$
But $q \leq p_\alpha, p_\beta$ and then
$$q \Vdash \forall 1 \leq i \leq n,\ \dot{x}_i^\alpha = \check{x}_i^\alpha,\ \dot{x}_i^\beta = \check{x}_i^\beta \text{ and } \dot{F}_i^\beta = \check{F}_i^\beta.$$
Therefore,
$$q \Vdash \forall 1 \leq i \leq n,\ \dot{x}_i^\alpha = \check{x}_i^\alpha \in h(\check{x}_i^\beta) \setminus \bigcup_{\xi \in \check{F}_i^\beta} h(\xi) = h(\dot{x}_i^\beta) \setminus \bigcup_{\xi \in \dot{F}_i^\beta} h(\xi),$$
contradicting the hypothesis about $\dot{x}_i^\alpha, \dot{x}_i^\beta$ and $\dot{F}_i^\beta$. □

Corollary 3.3. *It is relatively consistent with ZFC that there is a hereditarily separable compact scattered space of height ω_2.*

Proof. Since each level of the Cantor-Bendixson decomposition of K_f is a discrete subset of K_f, it follows that every level of it is countable. But $|K_f| = \aleph_2$ and $K_f = \bigcup_{\alpha < ht(K_f)} K_f^{(\alpha)} \setminus K_f^{(\alpha+1)}$, so that $ht(K_f) \geq \omega_2$. It is easy to see that $\bigcap_{\alpha<\omega_2} K_f^{(\alpha)} = \{*\}$ concluding that $ht(K_f) = \omega_2$. □

Corollary 3.4. *It is relatively consistent with ZFC that there is a hereditarily separable compact space with hereditary Lindelöf degree equal to $\aleph_2$. In particular, it is relatively consistent with ZFC that there is a compact space K such that $hL(K) \not\leq hd(K)^+$.*

Proof. It follows from the fact that $hL(K_f) \leq |K_f| = \aleph_2$ and that $\{K_f \setminus K_f^{(\alpha)} : \alpha < \omega_2\}$ is an open covering of $K_f \setminus \{*\}$ which does not admit a subcovering of strictly smaller cardinality. □

Corollary 3.5. *It is relatively consistent with ZFC that there is an Asplund space $C(K)$ of density $\aleph_2$ which does not admit any Fréchet smooth renorming and which does not contain an uncountable biorthogonal system.*

Proof. Since every finite power of K_f is hereditarily separable, Lemma 4.37 and Theorem 4.38 of [6] imply that $C(K_f)$ is hereditarily Lindelöf relative to its pointwise convergence topology. But for compact scattered spaces K, the pointwise convergence topology and the weak topology of $C(K)$ coincide (see Theorem 7.4 of [20]), so that $C(K_f)$ is hereditarily Lindelöf relative to its weak topology.

Now, if $C(K_f)$ admits a Fréchet smooth renorming, by Corollaries 8.34 (due to Mazur [18]) and 8.36 of [6] (due to Jiménez Sevilla and Moreno [10]) it contains an uncountable bounded subset A such that for every $x_0 \in A$, x_0 is not in the (norm-)closed convex hull of $A \setminus \{x_0\}$, that is, $x_0 \notin \overline{conv}(A \setminus \{x_0\})$. Since the weak and norm convex closures coincide in Banach spaces, A turns out to be an uncountable discrete family of $C(K_f)$ relative to its weak topology, which contradicts the fact that $C(K_f)$ is hereditarily Lindelöf relative to its weak topology.

Now, if $C(K_f)$ admits an uncountable biorthogonal system $(x_\alpha, \varphi_\alpha)_{\alpha<\omega_1} \subseteq C(K_f) \times C(K_f)^*$, then $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable discrete family of $C(K_f)$ relative to its weak topology, contradicting the fact that $C(K_f)$ is hereditarily Lindelöf relative to its weak topology. □

One should compare the above corollary to Theorem 4.41 of [6] (due to Ostaszewski [21]) and to Corollary 8.37 of [6] (due to Jiménez Sevilla and Moreno [10]).

4. The existence of the required function f

In this section we prove the consistency of the existence of a function with the strong property Δ. It turns out that we are even able to prove the consistency of the existence of such a function with its range included in the family of finite (rather than countable) subsets of ω_2. The method is quite involved but, as shown at the end of this section, forcings preserving CH (as in [2]) cannot serve for this purpose even if we were interested in a function with its range included in countable subsets of ω_2.

4.1. **Forcing with side conditions in Velleman's simplified morasses.** To construct a forcing which adds the required auxiliary function on pairs of ω_2 we will need a family of countable subsets of ω_2 with some strong properties. The following proposition establishes a list of the most useful properties:

Proposition 4.1. *It is relatively consistent with ZFC+CH that there exists a family $\mathcal{F} \subseteq [\omega_2]^\omega$ which satisfies the following properties:*

1) $(\mathcal{F}, \subseteq)$ is well-founded (thus, one can talk about $rank(X)$ for $X \in \mathcal{F}$).
2) $\mathcal{F}$ is stationary in $[\omega_2]^\omega$ (see [1]).
3) If $\alpha \in X, Y \in \mathcal{F}$ and $rank(X) \leq rank(Y)$, then $X \cap \alpha \subseteq Y \cap \alpha$.

If M is a countable elementary submodel of $H(\omega_3)$ containing $\omega_1, \omega_2, \mathcal{F}$ and $X = M \cap \omega_2 \in \mathcal{F}$, then

4) $M \cap \omega_1 = rank(X)$.
5) $Y \subset X$, $Y \in \mathcal{F}$ implies $Y \in M$.
6) $X_1, ..., X_n \in \mathcal{F}$ for $n \in N$ and $rank(X_i) < rank(X)$ for $1 \leq i \leq n$ implies that there is $Z \in \mathcal{F}$ such that $Z \in M$ and $X \cap (X_1 \cup ... \cup X_n) \subseteq Z$.

Proof. We will prove that a simplified Velleman's $(\omega_1, 1)$-morass (see [30]) which is a stationary coding set (see [31]) satisfies the above properties. The proof relies heavily on the properties of Velleman's morasses obtained in [15]. We will often refer to this paper; in particular, we adopt definitions of simplified morass and stationary coding set from this paper (Section 2). The consistency of the existence of such morasses can be immediately obtained from the corresponding proof for semimorasses in [14], Theorem 3, Section 2.

1) follows from Definition 2.1 of [15] and 2) from the fact that $\mathcal{F}$ is assumed to be a stationary coding set. To prove 3) apply 2.5 of [15]. Now 4) is Fact 2.7 of [15] and 5) is Fact 2.6 of [15]. To obtain 6) apply Fact 2.8 of [15] to each X_i obtaining $Z(X_i)$ such that $Z(X_i) \in M \cap \mathcal{F}$ and $X_i \cap X \subseteq Z(X_i)$. Now use the elementarity of M and the directedness of $\mathcal{F}$ (see Definition 2.1 of [15]) to obtain Z as in 6). $\square$

Now we will adopt a few facts from [16] and [15] concerning forcing with side conditions in $\mathcal{F}$. As explained in these papers, to use elements of $\mathcal{F}$ as side conditions means to use forcings P whose conditions are of the form (p, A) where p is a finite condition of a natural forcing adding the structure in question and A is a finite subset of $\mathcal{F}$. This is like using models as side conditions in the method of forcing with models as side conditions developed by Todorcevic (see [27]). The order is given by the forcing order on the first coordinate and inverse inclusion on the second coordinate. In addition we require the existence of some natural projections of p onto the elements of A as a part of the definition of the forcing notion. The properties 1) - 6) above allow us to perform many maneuvers with ease; also the definitions are simpler. This method appears to be equivalent to the variant of Todorcevic's method where one employs matrices of models (see [28], Section 4, for an example with detailed definitions). The price we need to pay for this convenience is that P is not proper (unlike Todorcevic's forcings,) but only $\mathcal{F}$-proper, i.e., there is a club $\mathcal{C} \subseteq [\omega_2]^\omega$ such that for models $M \prec H(\omega_3)$ such that $M \in \mathcal{F} \cap \mathcal{C}$ and $p \in P \cap M$, there are (P, M)-generic conditions stronger than p. As $\mathcal{F}$ may be assumed to be stationary, $\mathcal{F}$-properness implies the preservation of ω_1 (for a proof as for proper forcings, see [1]). The preservation of bigger cardinals follows from the ω_2-chain condition. Note that the fact that the forcing is not proper but preserves cardinals is no limitation in the applications that one seeks here, i.e., consistent existence of structures of sizes bigger than ω_1. Let us describe basic notions related to forcing with side conditions in $\mathcal{F}$ that we will use.

Definition 4.2. Suppose $\mathcal{F} \subseteq [\omega_2]^\omega$. We say that a forcing notion P is $\mathcal{F}$-proper if there is $\theta > (2^{|P|})^+$ and a club set $\mathcal{C} \subseteq [H(\theta)]^\omega$ such that whenever $p \in M \in \mathcal{C}$ and $M \cap \omega_2 \in \mathcal{F}$, then there is a (P, M)-generic $p_0 \leq p$; i.e., $D \cap M$ is predense below p_0 for every $D \in M$ which is dense in P.

Fact 4.3. Suppose $\mathcal{F} \subseteq [\omega_2]^\omega$ is a stationary set and P is an $\mathcal{F}$-proper forcing notion. Then P preserves ω_1.

Proof. The proof is a straightforward version of Shelah's paradigmatic proof of the preservation of ω_1 by proper forcings (see [26] or [1]). □

The following definition and lemmas are formulations of well-known techniques (originated in Shelah's use of elementary submodels in forcing) and will simplify our further arguments.

Definition 4.4. Let P be a notion of forcing, $q \in P$ and let $\theta > (2^{|P|})^+$. Suppose $M \prec H(\theta)$ and $P, \pi_1, ..., \pi_k \in M$. We say that a formula $\phi(x_0, x_1, ..., x_k)$ well reflects q in $(M; \pi_1, ..., \pi_k)$ whenever the following are satisfied:

i) $\phi(q, \pi_1, ..., \pi_k)$ holds in $H(\theta)$;
ii) whenever $s \in M$ is such that $\phi(s, \pi_1, ..., \pi_k)$ holds in M, then q and s are compatible.

Definition 4.5. Suppose $\mathcal{F} \subseteq [\omega_2]^\omega$ and suppose P is a notion of forcing. We say that P is simply $\mathcal{F}$-proper if there is θ such that whenever

a) $p \in P$,
b) $M \prec H(\theta)$, M countable,
c) $p,\ P,\ \mathcal{F} \in M$,
d) $M \cap \omega_2 \in \mathcal{F}$,

then there is $p_0 \leq p$ such that if $q \geq p_0$, then there are $\pi_1, ..., \pi_k \in M$ and a formula $\phi(x_0, x_1, ..., x_k)$ which well reflects q in $(M, \pi_1, ..., \pi_k)$.

Lemma 4.6. *If P is simply $\mathcal{F}$-proper, then P is $\mathcal{F}$-proper.*

Proof. We will prove that whenever $M,\ p$ are as in a) - d) of Definition 4.5, then p_0 is a (P, M)-generic condition. Letting $D \in M$ be dense, we will show that $D \cap M$

is predense below p_0. Letting $q \leq p_0$, we may w.l.o.g. assume that $q \in D$. Let $\pi_1, ..., \pi_k \in M$ and $\phi(x_0, x_1, ..., x_k)$ be such that $\phi(x_0, x_1, ..., x_k)$ well reflects q in $(M, \pi_1, ..., \pi_k)$. By i) of Definition 4.4, we have that $\phi(q, \pi_1, ...\pi_k)$ in $H(\theta)$. By its elementarity, M satisfies the formula "$\exists x \in P\ \phi(x, \pi_1, ...\pi_k)$ & $x \in D$". So let $s \in M$ witness this fact. Now by Definition 4.4ii), s and q are compatible, so $D \cap M$ contains a condition compatible with q, which proves that $D \cap M$ is predense below q, which completes the proof. □

4.2. **Adding a function with the strong property Δ.** Fix a family $\mathcal{F} \subseteq [\omega_2]^\omega$ satisfying 1) - 6) of Proposition 4.1. We will assume familiarity of the reader with elementary submodels of structures $H(\theta)$. In particular we will make use of facts such as that countable elements of such models are their subsets or that such models contain ω. See [5] for more on this subject. We consider the following forcing P whose conditions p are of the form: $p = (a_p, f_p, A_p)$, where

a) $a_p \in [\omega_2]^{<\omega}$;
b) $f_p : [a_p]^2 \to [\omega_2]^{<\omega}$;
c) $A_p \in [\mathcal{F}]^{<\omega}$;
d) $f_p(\alpha, \beta) \subseteq \bigcap\{X : X \in A_p,\ \alpha, \beta \in X\} \cap \min\{\alpha, \beta\}$ for any distinct $\alpha, \beta \in a_p$.

The order is just the inverse inclusion, i.e., $p \leq q$ if and only if $a_p \supseteq a_q$, $f_p \supseteq f_q$, $A_p \supseteq A_q$.

Fact 4.7. P is simply $\mathcal{F}$-proper.

Proof. Let $\theta = \omega_3$ and let M and p be as in a) - d) of Definition 4.5. The existence of such an M follows from the stationarity of $\mathcal{F}$. Let $X_0 = M \cap \omega_2$. Let $p_0 = (a_p, f_p, A_p \cup \{X_0\})$. Finally let $q \leq p_0$. The proof consists of using Lemma 4.6 and finding the parameters $\pi_1, ..., \pi_k \in M$ and a formula $\phi(x_0, x_1, ..., x_k)$ which well reflects q in $(M, \pi_1, ..., \pi_k)$.

Define $q|M = (a_q \cap M, f_q|M, A_q \cap M)$. Introduce the notation $\delta = M \cap \omega_1 = rank(M)$, where the second equality follows from 4) of Proposition 4.1. Note that $A_q \cap M = A_{q|M} = \{X \in A_q : X \subset X_0\}$. This follows from 5) of Proposition 4.1. The fact that $[M]^{<\omega} \subseteq M$ implies that $a_{q|M}, A_{q|M} \in M$. Also, as d) of the definition of the forcing is satisfied for q and $\alpha, \beta \in a_q$, we have that $f_q(\alpha, \beta) \subseteq X_0 = M \cap \omega_2$ for $\alpha, \beta \in a_q \cap X_0$. So, we may conclude that $f_{q|M} \in M$; in other words, we have $q|M \in M \cap P$. It is clear that $q|M \leq p$. By 6) of Proposition 4.1 and the fact that $[M]^{<\omega} \subseteq M$, in M there is a $Z \in \mathcal{F}$ such that $\bigcup\{X \cap M :\ rank(X) < \delta,\ X \in A_q\} \subseteq Z$. Let $\phi(x_0, x_1, x_2, x_3, x_4)$ be the formula which says that x_0 is a condition of the partial order x_4 which extends in x_4 the condition x_3 and such that the difference between the first coordinate of x_0 and x_2 is disjoint from x_1.

Claim. $\phi(x_0, x_1, x_2, x_3, x_4)$ well reflects q in $(M, Z, a_{q|M}, q|M, P)$.

Proof of the Claim. It is clear that $\phi(q, Z, a_{q|M}, q|M, P)$ holds in $H(\omega_3)$. Now let $s \in M$ be a condition satisfying $\phi(s, Z, a_{q|M}, q|M, P)$; i.e., s extends in P the condition $q|M$ and $a_s \setminus a_{q|M}$ is disjoint from Z. Define the common extension r of q and s as follows: $a_r = a_s \cup a_q$, $f_r = f_s \cup f_q \cup h$, $A_r = A_s \cup A_q$, where $h(\{\alpha, \beta\}) = \emptyset$ for $\{\alpha, \beta\} \in [a_s \cup a_q]^2 - ([a_s]^2 \cup [a_q]^2)$. Such an f_r is a function on $[a_r]^2$ since $q|M \geq q, s$. Clearly all clauses of the definition of the forcing P but d) are trivially satisfied by r. So let us prove d). Letting $\alpha, \beta \in a_r$ and $X \in A_r$, we will consider a few cases.

Case 1. $\alpha, \beta \in a_s$, $X \in A_s$.
This is trivial because $s \in P$.

Case 2. $\alpha, \beta \in a_q$, $X \in A_q$.
This is trivial because $q \in P$.

Case 3. $\alpha, \beta \in a_s$, $X \in A_q$.
Since $\phi(s, Z, a_{q|M}, q|M, P)$ holds in M we have that either $rank(X) \geq \delta = rank(M \cap \omega_2) = rank(X_0)$, in which case d) is satisfied because $f_r(\{\alpha, \beta\}) = f_s(\{\alpha, \beta\}) \subseteq X_0 \cap \min\{\alpha, \beta\} \subseteq X \cap \min\{\alpha, \beta\}$ by d) for s and 3) of Proposition 4.1 or $rank(X) < \delta$ and then by the definition of ϕ and Z we get that $\alpha, \beta \in a_s \cap a_q$, so we are again in Case 2.

Case 4. $\alpha, \beta \in a_q$, $X \in A_s$.
This means that $\alpha, \beta \in M$, because $s \in M$, i.e., $\alpha, \beta \in a_s \cap a_q$, so we are again in Case 1.

Case 5. $\alpha \in a_s \setminus a_q$ and $\beta \in a_q \setminus a_s$.
Then $h(\{\alpha, \beta\}) = \emptyset$.

The proof of the claim completes the proof of Fact 4.7. □

Definition 4.8. For $p \in P$, call the set $a_p \cup f[[a_p]^2] \cup \bigcup A_p$ the support of p and denote it by $supp(p)$.

Definition 4.9. We say that two conditions p, q of P are isomorphic (via $\pi : supp(p) \to supp(q)$) if $\pi : supp(p) \to supp(q)$ is an order-preserving bijection constant on $supp(p) \cap supp(q)$ and

i) $\pi[a_p] = a_q$;
ii) $\{\pi[X] : X \in A_p\} = A_q$;
iii) $f_q(\{\pi(\alpha), \pi(\beta)\}) = \pi[f_p(\{\alpha, \beta\})]$ for all $\alpha, \beta \in a_p$.

Lemma 4.10. *Suppose $p, q \in P$ are isomorphic via $\pi : supp(p) \to supp(q)$. Then they are compatible.*

Proof. Define the common extension r of p and q as follows: $a_r = a_p \cup a_q$, $f_r = f_p \cup f_q \cup h$, $A_r = A_p \cup A_q$, where $h(\{\alpha, \beta\}) = \emptyset$ for $\{\alpha, \beta\} \in [a_p \cup a_q]^2 - ([a_p]^2 \cup [a_q]^2)$. The only nonautomatic condition of the definition of P which needs to be checked is d).

Case 1. $\alpha, \beta \in a_r$.
If $X \in A_r$, we are trivially done. If $X \in A_q$ and $\alpha, \beta \in X$, then $\alpha, \beta \in supp(p) \cap supp(q)$; hence $\alpha, \beta \in a_p \cap a_q$ and hence again use d) for q.

Case 2. $\alpha, \beta \in a_q$.
This is similar to the previous case.

Case 3. $\alpha \in a_r \setminus a_q$, $\beta \in a_q \setminus a_r$.
In this case h is empty. □

Fact 4.11. Assuming CH, the forcing P is ω_2-c.c. Thus by Fact 4.7, Lemma 4.6 and Fact 4.3, P preserves cardinals.

Proof. By the previous lemma the proof is a standard application of the Δ-system lemma to the sequence of supports $\{supp(p_\xi) : \xi < \omega_2\}$ of some conditions $p_\xi \in P$ under our cardinal arithmetic assumption. □

Theorem 4.12. *In V^P there is a function $f : [\omega_2]^2 \to [\omega_2]^{<\omega}$ with the strong property Δ.*

Proof. Clearly, we claim that $f = \bigcup\{f_p : p \in G\}$ defines such a function, where G is a P-generic over V. Let $\dot{f}$ be a name for it.

Fix a set $A = \{\dot{a}^\alpha : \alpha < \omega_1\}$ of P-names for elements of an uncountable Δ-system of n-tuples $\dot{a} = \{\dot{a}_i : i < n\}$ of elements of ω_2 for which there are bijections e as in Definition 3.1 (any uncountable Δ-system has such an uncountable subsystem). Fix a condition $p \in P$.

Take a model $M \prec H(\omega_3)$ such that $M \cap \omega_2 = X_0 \in \mathcal{F}$ and $p \in P \cap M$; $\mathcal{F} \in M$ and $\{\dot{a}^\alpha : \alpha < \omega_1\} \in M$. We will show that there are $\alpha_1 < \alpha_2 < \omega_1$ and $r \leq p$ such that r forces 1), 2), 3) of Definition 3.1 for $\dot{a}^{\alpha_1}$ and $\dot{a}^{\alpha_2}$.

First take a condition $p_0 \leq p$ as in Fact 4.7, i.e., $a_{p_0} = a_p$, $f_{p_0} = f_p$, $A_{p_0} = A_p \cup X_0$. Take $q \leq p_0$ and $\alpha_1 \in \omega_1$ such that there is b such that $b \setminus M \neq \emptyset$, $q \Vdash \dot{a}^{\alpha_1} = \check{b}$ and $b \subseteq a_q$. This can be done as $\{\dot{a}^\alpha : \alpha < \omega_1\}$ is a sequence of names for an uncountable Δ-system of sets and $|M| = \omega$. Proceed as in the proof of Fact 4.7; i.e., choose Z and ϕ as in Fact 4.7. So, we have $\phi(q, Z, a_{q|M}, q|M, P)$ in $H(\omega_3)$ and so by the elementarity of M, we can find an s and α_2 such that $\phi(s, Z, a_{q|M}, q|M, P)$ holds in M and moreover there is a such that $a \setminus (b \cap M) \in [M \setminus Z]^{<\omega}$ and such that $s \Vdash \dot{a}^{\alpha_2} = \check{a}$ and $a \subseteq a_s$. Now we will obtain another amalgamation r of s and q which will force 1), 2) and 3) of Definition 3.1. Let $a_r = a_s \cup a_q$, $f_r = f_s \cup f_q \cup h$. For $\xi \in a_s \setminus a_q$ and $\eta \in a_q \setminus a_s$:

$$h(\{\xi, \eta\}) = [A \cup B \cup C] \cap D, \tag{**}$$

where

$$A = a \cap \min\{\xi, \eta\},$$
$$B = \bigcup\{f_s(\{\tau, \xi\}) : \tau \in a \cap b, \tau < \eta\},$$
$$C = \bigcup\{f_q(\{\tau, \eta\}) : \tau \in a \cap b, \tau < \xi\},$$
$$D = \min\{\xi, \eta\} \cap \bigcap\{X \in A_q : \xi, \eta \in X, rank(X) \geq \delta\}.$$

First let us check that r is a common extension of q and s. The proof also follows the cases as in the Claim in the proof of Fact 4.7. All are checked in the same manner except for Case 5 where one may assume that $X \in A_q$ as $\beta \notin M$. This time the inclusion in the set D guarantees that d) holds in Case 5.

Now we will check 1), 2) and 3) of Definition 3.1 for a, b as above and f_r. This will be enough since $r \Vdash \dot{a}^{\alpha_1} = \check{b}, \dot{a}^{\alpha_2} = \check{a}$ and $r \Vdash f_r \subseteq \dot{f}$. Suppose $\xi \in a \setminus b$ and $\eta \in b \setminus a$. By the form of the definition of $f_r(\{\xi, \eta\}) = h(\{\xi, \eta\})$ it will be enough to prove that the sets A, B and C are actually included in $\min\{\xi, \eta\} \cap X$ for any $X \in A_q$ such that $rank(X) \geq \delta$ and $\xi, \eta \in X$. So, let $X \in A_q$ be any element such that $rank(X) \geq \delta$ and $\xi, \eta \in X$.

For 1) of Definition 3.1, note that since $X_0 = M \cap \omega_2$ and $rank(X_0) = \delta$ we have that $M \cap \min\{\xi, \eta\}$ is included in X by 3) of Proposition 4.1. Hence, as $a \subseteq M$, we have $a \cap \min\{\xi, \eta\} \subseteq \min\{\xi, \eta\} \cap X$; that is, we obtain 1).

To get 2) of Definition 3.1 assume that $\tau \in a \cap b$ and $\tau < \xi$; hence $\min\{\tau, \eta\} \leq \min\{\xi, \eta\}$. Note again, by 3) of Proposition 4.1, that $M \cap \xi \subseteq X \cap \xi$, which implies in this case that $\tau \in X$. Hence, since $\tau, \eta \in a_q$, by d) of the definition of the forcing, we have that $f_q(\{\tau, \eta\}) \subseteq X \cap \min\{\tau, \eta\} \subseteq X \cap \min\{\xi, \eta\}$, so we obtain 2).

To get 3) of Definition 3.1 assume that $\tau \in a \cap b$ and $\tau < \eta$; hence $\min\{\tau, \xi\} \leq \min\{\xi, \eta\}$. We have that $\tau, \xi \in M \cap \xi$, and again $M \cap \xi \subseteq X$. Hence, since $\xi, \tau \in a_s$, by d) of the definition of the forcing, and the fact that $s \in M$, we have that $f_s(\{\tau, \xi\}) \cap \min\{\tau, \xi\} \subseteq X \cap \min\{\tau, \xi\} \subseteq X \cap \min\{\xi, \eta\}$, so we obtain 3). □

Remark 4.13. For any $k \leq \omega$ and any uncountable Δ-system one can have k sets satisfying 1), 2) and 3) of Definition 3.1. This follows from the Dushnik-Miller theorem; see [9], Theorem 9.7.

4.3. **CH and the strong property Δ.** In this section we prove that CH implies that there is no function f such as in the previous section, even if we allow f to take countable sets as values. This also proves that the strong property Δ cannot be obtained as in Baumgartner and Shelah [2], that is, by a forcing which preserves CH.

Proposition 4.14. *(CH) There is no $f : [\omega_2]^2 \to [\omega_2]^\omega$ such that for every Δ-system $\mathcal{A}$ of finite subsets of ω_2 of cardinality $\aleph_1$, there exist distinct $a, b \in \mathcal{A}$ such that*

$$*) \qquad \forall \xi \in a \setminus b \quad \forall \eta \in b \setminus a \quad a \cap \xi \cap \eta \subseteq f(\{\xi, \eta\}).$$

Proof. Suppose that $f : [\omega_2]^2 \to [\omega_2]^\omega$. For an $A \subseteq \omega_2$ and $\xi \in \omega_2 \setminus A$ define

$$f_{A,\xi} : A \to [A]^\omega, \quad f_{A,\xi}(\eta) = f(\{\eta, \xi\}) \cap A \quad \forall \eta \in A.$$

Let $M \prec H(\omega_3)$ be closed under its countable subsets (here we use CH) $|M| = \omega_1$, $\omega_1 \subseteq M$; $\omega_1, \omega_2, f \in M$ and such that $\sup(M \cap \omega_2) = \gamma$ has an uncountable cofinality.

By recursion construct a sequence $(\alpha_\xi, \beta_\xi)_{\xi<\omega_1}$ which satisfies:

1) $\omega_1 < \alpha_\xi, \beta_\xi \in M \cap \gamma$;
2) $\alpha_\xi < \beta_\xi < \alpha_\eta$ for all $\xi < \eta$;
3) $f_{A_\eta, \beta_\eta} = f_{A_\eta, \gamma}$, where $A_\eta = \{\alpha_\xi, \beta_\xi : \xi < \eta\}$;
4) $\alpha_\eta \notin f(\{\beta_\eta, \gamma\})$.

To justify that this construction can be carried out assume that we have A_η satisfying 1)-4) and let us show how to obtain α_η, β_η. As $A_\eta \subseteq M$ and M is closed under its countable sets we have $A_\eta \in M$. Also $f_{A_\eta,\gamma} \in M$ as M is closed under countable sets. Hence, by the elementarity there is $\beta_\eta \in M \setminus \sup(A_\eta)$ such that $f_{A_\eta,\beta_\eta} = f_{A_\eta,\gamma}$ and $cf(\beta_\eta) = \omega_1$. Now $f(\{\beta_\eta, \gamma\}) \cap M$ is in M again, so using the fact that $cf(\beta_\eta) = \omega_1$ we can find $\alpha_\eta \in M$ satisfying $A_\eta < \alpha_\eta < \beta_\eta$ and $\alpha_\eta \notin f(\{\beta_\eta, \gamma\})$, which completes the construction.

Now define $\gamma_\eta < \omega_1$ such that for $\xi < \eta < \omega_1$ we have

$$\gamma_\xi < \gamma_\eta \notin \bigcup\{f(\{\beta_\xi, \beta_\eta\}) : \xi < \eta\}.$$

Finally define $\mathcal{A} = \{\{\gamma_\xi, \alpha_\xi, \beta_\xi\} : \xi < \omega_1\}$. Suppose $\xi < \eta$. Note that, as by 4), $\alpha_\xi \notin f(\{\beta_\xi, \gamma\})$ and by 3), $f(\{\beta_\xi, \gamma\}) = f(\{\beta_\xi, \beta_\eta\})$, we have

$$\alpha_\xi \in (\beta_\xi \cap \beta_\eta) \setminus f(\{\beta_\xi, \beta_\eta\}).$$

But on the other hand, by the definition of γ_η, we have

$$\gamma_\eta \in (\beta_\xi \cap \beta_\eta) \setminus f(\{\beta_\xi, \beta_\eta\}),$$

which shows that the inclusion *) of the proposition holds for no $a, b \in \mathcal{A}$. □

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