# CHAPTER 3
# ORTHOGONAL SETS OF FUNCTIONS

Fourier series are only one of a large class of interesting and useful infinite series expansions for functions that are based on so-called *orthogonal systems* or *orthogonal sets* of functions. This chapter is devoted to explaining the general conceptual framework for understanding such systems, and to showing how they arise from certain kinds of differential equations. Underlying these ideas is a profound analogy between the algebra of Fourier series and the algebra of $n$-dimensional vectors, which we now investigate.

## 3.1 Vectors and inner products

We recall some ideas from elementary 3-dimensional vector algebra and recast them in a more general form. We identify 3-dimensional vectors with ordered triples of real numbers; that is, we write

$$\mathbf{a} = (a_1, a_2, a_3) \qquad \text{rather than} \qquad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

The dot product or inner product of two vectors is then defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

and the norm or length of a vector is defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

We propose to generalize these ideas in two ways: by working in an arbitrary number $k$ of dimensions, and by using complex numbers rather than real ones. This generalization is not just a mathematical fantasy. Although $k$-dimensional vectors do not have an immediate geometrical interpretation in physical space, they are still useful for dealing with problems involving $k$ independent variables. For our purposes, the main motivation for the use of complex numbers is their connection with the exponentials $e^{i\theta}$; but it should be noted that the use of complex vectors is essential in quantum physics. However, in visualizing the ideas we shall be discussing, the reader should just think of real 3-dimensional vectors.

A **(complex) $k$-dimensional vector** is an ordered $k$-tuple of complex numbers:

$$\mathbf{a} = (a_1, a_2, \ldots, a_k).$$

The vector $\mathbf{a}$ is called **real** if its components $a_j$ are all real numbers. Addition and scalar multiplication are defined just as in the 3-dimensional case, but now the scalars are allowed to be complex:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_k + b_k),$$
$$c\mathbf{a} = (ca_1, \ldots, ca_k) \qquad (c \in \mathbf{C}).$$

We denote the zero vector $(0, 0, \ldots, 0)$ by $\mathbf{0}$, and we denote the space of all complex $k$-dimensional vectors by $\mathbf{C}^k$.

The **inner product** of two vectors is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_k\overline{b}_k, \tag{3.1}$$

and the **norm** of a vector is defined by

$$\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} = \left(a_1\overline{a}_1 + \cdots + a_k\overline{a}_k\right)^{1/2} = \left(|a_1|^2 + \cdots + |a_k|^2\right)^{1/2}. \tag{3.2}$$

The reason for the complex conjugates in the definition of the inner product is to make the norm (3.2) positive, for we wish to interpret $\|\mathbf{a}\|$ as the magnitude or length of the vector $\mathbf{a}$. (Recall that the absolute value of a complex number $z = x+iy$ is $(x^2+y^2)^{1/2}$, and this is $(z\overline{z})^{1/2}$ rather than $(z^2)^{1/2}$.) Notice, however, that for real vectors, (3.1) and (3.2) become

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + \cdots + a_k b_k, \qquad \|\mathbf{a}\| = (a_1^2 + \cdots + a_k^2)^{1/2},$$

the obvious generalization of the familiar 3-dimensional case.

A word about the notation: The inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ is often denoted by $\mathbf{a} \cdot \mathbf{b}$ or $(\mathbf{a}, \mathbf{b})$. Also, in the physics literature it is customary to switch the roles of $\mathbf{a}$ and $\mathbf{b}$, that is, to put the complex conjugates on the first variable rather than the second. This discrepancy is regrettable, but by now it is firmly entrenched in common usage.

The inner product (3.1) is clearly linear as a function of its first variable but *antilinear* or *conjugate linear* as a function of its second variable; that is, for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and any complex numbers $z$, $w$,

$$\begin{aligned} \langle z\mathbf{a} + w\mathbf{b},\ \mathbf{c} \rangle &= z\langle \mathbf{a}, \mathbf{c} \rangle + w\langle \mathbf{b}, \mathbf{c} \rangle, \\ \langle \mathbf{a},\ z\mathbf{b} + w\mathbf{c} \rangle &= \overline{z}\langle \mathbf{a}, \mathbf{b} \rangle + \overline{w}\langle \mathbf{a}, \mathbf{c} \rangle \end{aligned} \tag{3.3}$$

Also, the inner product is *Hermitian symmetric*, which means that

$$\langle \mathbf{b}, \mathbf{a} \rangle = \overline{\langle \mathbf{a}, \mathbf{b} \rangle}, \tag{3.4}$$

and the norm satisfies the conditions

$$\|c\mathbf{a}\| = |c|\,\|\mathbf{a}\| \qquad (c \in \mathbf{C}), \tag{3.5}$$

$$\|\mathbf{a}\| > 0 \quad \text{for all } \mathbf{a} \neq \mathbf{0}. \tag{3.6}$$

Using these facts, we now derive some fundamental properties of inner products and norms.

**Lemma 3.1.** *For any* **a** *and* **b** *in* $\mathbf{C}^k$,

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\operatorname{Re}\langle\mathbf{a},\mathbf{b}\rangle + \|\mathbf{b}\|^2.$$

*Proof:* By (3.3), (3.4), and the definition of the norm,

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \langle\mathbf{a}+\mathbf{b},\ \mathbf{a}+\mathbf{b}\rangle = \langle\mathbf{a},\mathbf{a}\rangle + \langle\mathbf{a},\mathbf{b}\rangle + \langle\mathbf{b},\mathbf{a}\rangle + \langle\mathbf{b},\mathbf{b}\rangle \\ &= \langle\mathbf{a},\mathbf{a}\rangle + \langle\mathbf{a},\mathbf{b}\rangle + \overline{\langle\mathbf{a},\mathbf{b}\rangle} + \langle\mathbf{b},\mathbf{b}\rangle = \|\mathbf{a}\|^2 + 2\operatorname{Re}\langle\mathbf{a},\mathbf{b}\rangle + \|\mathbf{b}\|^2.\end{aligned}$$ ∎

**The Cauchy-Schwarz Inequality.** *For any* **a** *and* **b** *in* $\mathbf{C}^k$,

$$\left|\langle\mathbf{a},\mathbf{b}\rangle\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\|. \tag{3.7}$$

*Proof:* We may assume that $\mathbf{b} \neq \mathbf{0}$, since otherwise both sides of (3.7) are 0. Also, neither $\left|\langle\mathbf{a},\mathbf{b}\rangle\right|$ nor $\|\mathbf{a}\|\,\|\mathbf{b}\|$ is affected if we multiply **a** by a scalar of absolute value one, so we may replace **a** by $c\mathbf{a}$, with $|c| = 1$, so as to make $\langle\mathbf{a},\mathbf{b}\rangle$ real. (That is, if $\langle\mathbf{a},\mathbf{b}\rangle = re^{i\theta}$, we take $c = e^{-i\theta}$.) Assuming then that $\langle\mathbf{a},\mathbf{b}\rangle$ is real, by Lemma 3.1 we see that for any real number $t$,

$$0 \le \|\mathbf{a}+t\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2t\langle\mathbf{a},\mathbf{b}\rangle + t^2\|\mathbf{b}\|^2.$$

This last expression is a quadratic function of $t$, since $\|\mathbf{b}\| \neq 0$, and (by elementary calculus) it achieves its minimum value at $t = -\langle\mathbf{a},\mathbf{b}\rangle/\|\mathbf{b}\|^2$. If we substitute this value for $t$, we obtain

$$0 \le \|\mathbf{a}\|^2 - 2\frac{\langle\mathbf{a},\mathbf{b}\rangle^2}{\|\mathbf{b}\|^2} + \frac{\langle\mathbf{a},\mathbf{b}\rangle^2}{\|\mathbf{b}\|^4}\|\mathbf{b}\|^2 = \|\mathbf{a}\|^2 - \frac{\langle\mathbf{a},\mathbf{b}\rangle^2}{\|\mathbf{b}\|^2},$$

or

$$0 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \langle\mathbf{a},\mathbf{b}\rangle^2,$$

which, since $\langle\mathbf{a},\mathbf{b}\rangle$ is assumed real, is equivalent to (3.7). ∎

**The Triangle Inequality.** *For any* **a** *and* **b** *in* $\mathbf{C}^k$,

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|. \tag{3.8}$$

*Proof:* By Lemma 3.1, the Cauchy-Schwarz inequality, and the fact that $\operatorname{Re} z \le |z|$, we have

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\operatorname{Re}\langle\mathbf{a},\mathbf{b}\rangle + \|\mathbf{b}\|^2 \\ &\le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 \\ &= \left(\|\mathbf{a}\| + \|\mathbf{b}\|\right)^2.\end{aligned}$$ ∎

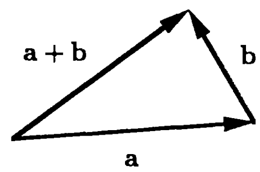



FIGURE 3.1. The sum of two vectors.

Geometrically, the triangle inequality just says that one side of a triangle can be no longer than the sum of the other two sides; see Figure 3.1. This picture is perfectly accurate, for the vectors **a**, **b**, and $\mathbf{a}+\mathbf{b}$ always lie in the same plane no matter how many dimensions they live in.

We recall that two real 3-dimensional vectors are orthogonal or perpendicular to each other precisely when their inner product is zero. We shall take this as a definition in the general case: two complex $k$-dimensional vectors **a** and **b** are **orthogonal** if $\langle \mathbf{a}, \mathbf{b}\rangle = 0$. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are called **mutually orthogonal** if $\langle \mathbf{a}_i, \mathbf{a}_j\rangle = 0$ for all $i \neq j$. With this terminology, we have a generalization of the classic theorem about the lengths of the sides of a right triangle:

**The Pythagorean Theorem.** *If* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *are mutually orthogonal, then*

$$\|\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n\|^2 = \|\mathbf{a}_1\|^2 + \|\mathbf{a}_2\|^2 + \cdots + |\mathbf{a}_n\|^2. \tag{3.9}$$

*Proof:* We have

$$\|\mathbf{a}_1 + \cdots + \mathbf{a}_n\|^2 = \langle \mathbf{a}_1 + \cdots + \mathbf{a}_n,\ \mathbf{a}_1 + \cdots + \mathbf{a}_n\rangle.$$

If we multiply out the right side by (3.3), all the cross terms vanish because of the orthogonality condition, and we are left with

$$\langle \mathbf{a}_1, \mathbf{a}_1\rangle + \cdots + \langle \mathbf{a}_n, \mathbf{a}_n\rangle = \|\mathbf{a}_1\|^2 + \cdots + \|\mathbf{a}_n\|^2.$$ ∎

*Important Remark.* The proofs of the Cauchy-Schwarz and triangle inequalities and the Pythagorean theorem depend only on the properties (3.3) and (3.4) of the inner product and the definition $\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a}\rangle^{1/2}$, not on the specific formula (3.1). They therefore remain valid for any other "inner product" that satisfies (3.3) and (3.4) and the "norm" associated to it.

Some more terminology: We say that a vector **u** is **normalized**, or is a **unit vector**, if $\|\mathbf{u}\| = 1$. Any nonzero vector **a** can be normalized by multiplying it by the reciprocal of its norm: If $\mathbf{u} = \|\mathbf{a}\|^{-1}\mathbf{a}$, then $\|\mathbf{u}\| = \|\mathbf{a}\|^{-1}\|\mathbf{a}\| = 1$. We shall call a collection $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ of vectors an **orthogonal set** if its elements are mutually orthogonal and nonzero, and an **orthonormal set** if its elements are mutually orthogonal and normalized. (See Figure 3.2.) Of course, any orthogonal set can be made into an orthonormal set by normalizing each of its elements. Thus, a set $\{\mathbf{a}_1, \mathbf{a}_2, \dots\}$ is orthonormal if and only if

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \delta_{ij}, \tag{3.10}$$

where $\delta_{ij}$ is the **Kronecker $\delta$-symbol:**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.11}$$

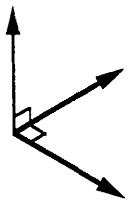


FIGURE 3.2. An orthonormal set of vectors.

The vectors in any orthogonal set $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ are linearly independent; that is, the equation

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

can hold only when all the scalars $c_j$ are zero. To see this, take the inner product of both sides with $\mathbf{a}_j$ $(1 \leq j \leq n)$; because of the orthogonality and the fact that $\mathbf{a}_j \neq \mathbf{0}$, the result is

$$c_j\langle \mathbf{a}_j, \mathbf{a}_j \rangle = c_j\|\mathbf{a}_j\|^2 = 0, \quad \text{hence} \quad c_j = 0.$$

It follows that *the number of vectors in any orthogonal set in $\mathbf{C}^k$ is at most $k$*, since $\mathbf{C}^k$ is $k$-dimensional.

An example of an orthonormal set of $k$ vectors is given by the standard basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$, where

$$\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0) \qquad (1 \text{ in the } j\text{th position, 0 elsewhere}).$$

For any $\mathbf{a} = (a_1, \dots, a_k) \in \mathbf{C}^k$, we clearly have

$$\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k,$$

so $\mathbf{a}$ is expressed in a simple way as a linear combination of the $\mathbf{e}_j$'s. But sometimes it is more convenient to use other orthonormal sets that are adapted to a particular problem, and here too there is a simple way of expressing arbitrary vectors as linear combinations of the orthonormal vectors.

Indeed, suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set in $\mathbf{C}^k$. If a vector $\mathbf{a} \in \mathbf{C}^k$ is expressed as a linear combination of the $\mathbf{u}_j$'s,

$$\mathbf{a} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k,$$

by taking the inner product of both sides with $\mathbf{u}_j$ and using (3.10) we find that the coefficients $c_j$ are given by

$$c_j = \langle \mathbf{a}, \mathbf{u}_j \rangle \qquad (1 \le j \le k). \tag{3.12}$$

Conversely, if $\mathbf{a}$ is any vector in $\mathbf{C}^n$, we may define the constants $c_1, \ldots, c_k$ by (3.12) and form the linear combination

$$\tilde{\mathbf{a}} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k.$$

Then the difference $\mathbf{b} = \mathbf{a} - \tilde{\mathbf{a}}$ is orthogonal to all the $u_j$'s:

$$\langle \mathbf{b}, \mathbf{u}_j \rangle = \langle \mathbf{a}, \mathbf{u}_j \rangle - \langle \tilde{\mathbf{a}}, \mathbf{u}_j \rangle = c_j - c_j = 0.$$

But this means that $\mathbf{b} = \mathbf{0}$, for otherwise $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{b}\}$ would be an orthogonal set with $k + 1$ elements, which is impossible. In other words, $\tilde{\mathbf{a}} = \mathbf{a}$, and we have the following result.

**Theorem 3.1.** *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set of $k$ vectors in $\mathbf{C}^k$. For any $\mathbf{a} \in \mathbf{C}^k$ we have*

$$\mathbf{a} = \langle \mathbf{a}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{a}, \mathbf{u}_k \rangle \mathbf{u}_k.$$

*Moreover,*

$$\|\mathbf{a}\|^2 = |\langle \mathbf{a}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{a}, \mathbf{u}_k \rangle|^2.$$

*Proof:* The first assertion has just been proved, and the second one follows from it by the Pythagorean theorem. ∎

***EXERCISES***

1. Show that $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right)$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
2. Suppose $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ is an orthogonal set in $\mathbf{C}^k$, not necessarily normalized. Use Theorem 3.1 to show that for any $\mathbf{a} \in \mathbf{C}^k$,
$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{y}_1 \rangle \mathbf{y}_1}{\|\mathbf{y}_1\|^2} + \cdots + \frac{\langle \mathbf{a}, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{y}_k\|^2}.$$
3. Let $\mathbf{y}_1 = (2, 3i, 5)$ and $\mathbf{y}_2 = (3i, 2, 0)$.
   a. Show that $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = 0$ and find a nonzero $\mathbf{y}_3$ that is orthogonal to both $\mathbf{y}_1$ and $\mathbf{y}_2$.
   b. What are the norms of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$?
   c. Use Theorem 3.1 or Exercise 2 to express the vectors $(1, 2, 3i)$ and $(0, 1, 0)$ as linear combinations of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$.

4. Let $\mathbf{u}_1 = \frac{1}{3}(1, 2i, -2i, 0)$, $\mathbf{u}_2 = \frac{1}{5}(2-4i, -2, i, 0)$, $\mathbf{u}_3 = \frac{1}{15}(4+2i, 5+8i, 4+10i, 0)$, and $\mathbf{u}_4 = (0, 0, 0, i)$.
   a. Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_4\}$ is an orthonormal set in $\mathbf{C}^4$.
   b. Express the vectors $(1, 0, 0, 0)$ and $(2, 10-i, 10-9i, -3)$ as linear combinations of $\mathbf{u}_1, \dots, \mathbf{u}_4$ by using Theorem 3.1.
5. Suppose $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is an orthonormal set in $\mathbf{C}^k$ with $m < k$. Show that for any $\mathbf{a} \in \mathbf{C}^k$ there is a unique set of constants $\{c_1, \dots, c_m\}$ such that $\mathbf{a} - \sum_1^m c_j \mathbf{u}_j$ is orthogonal to all the $\mathbf{u}_j$'s, and determine these constants explicitly. (Hint: Consider the proof of Theorem 3.1.)

The following problems deal with $k \times k$ complex matrices $T = (T_{ij})$. We recall that if $T = (T_{ij})$ and $S = (S_{ij})$ are $k \times k$ matrices, $TS$ is the matrix whose $(ij)$th component is $\sum_l T_{il} S_{lj}$, and if $\mathbf{a} \in \mathbf{C}^k$, $T\mathbf{a}$ is the vector whose $i$th component is $\sum_j T_{ij} a_j$. The (Hermitian) **adjoint** of the matrix $T$ is the matrix $T^*$ obtained by interchanging rows and columns and taking complex conjugates, that is, $(T^*)_{ij} = \overline{T_{ji}}$.

6. Show that $\langle T\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, T^*\mathbf{b}\rangle$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
7. Show that if $T = T^*$, the "product" defined by $\langle \mathbf{a}, \mathbf{b}\rangle_T = \langle T\mathbf{a}, \mathbf{b}\rangle$ satisfies properties (3.3) and (3.4).
8. Let $\mathbf{t}_j = (T_{1j}, \dots, T_{kj})$ be the vector that makes up the $j$th row of $T$. Show that the following properties of the matrix $T$ are equivalent. (Hint: Show that the $(ij)$th component of $T^*T$ is $\langle \mathbf{t}_j, \mathbf{t}_i\rangle$.)
   (i) $\{\mathbf{t}_1, \dots, \mathbf{t}_k\}$ is an orthonormal basis for $\mathbf{C}^k$.
   (ii) $T^*T$ is the identity matrix, i.e., $(T^*T)_{ij} = \delta_{ij}$.
   (iii) $\|T\mathbf{a}\| = \|\mathbf{a}\|$ for all $\mathbf{a} \in \mathbf{C}^k$.
9. Show that $|\langle \mathbf{a}, \mathbf{b}\rangle| = \|\mathbf{a}\|\,\|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are complex scalar multiples of one another, and that $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are positive scalar multiples of one another. (Examine the proofs of the Cauchy-Schwarz and triangle inequalities to see when equality holds.)

## 3.2 Functions and inner products

A vector $\mathbf{a} = (a_1, \dots, a_k)$ in $\mathbf{C}^k$ can be regarded as a function on the set $\{1, \dots, k\}$ that assigns to the integer $j$ the $j$th component $\mathbf{a}(j) = a_j$, and with this notation we can write the inner product and norm as follows:

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_1^k \mathbf{a}(j)\overline{\mathbf{b}(j)}, \qquad \|\mathbf{a}\| = \left(\sum_1^k |\mathbf{a}(j)|^2\right)^{1/2}. \tag{3.13}$$

We now make a leap of imagination: Consider the space $PC(a, b)$ of piecewise continuous functions on the interval $[a, b]$, and think of functions $f \in PC(a, b)$ as infinite-dimensional vectors whose "components" are the values $f(x)$ as $x$ ranges over the interval $[a, b]$. The operations of vector addition and scalar multiplication are just the usual addition of functions and multiplication of functions by constants. To define the inner product and the norm, we simply replace the sums in (3.13) by their continuous versions, i.e., integrals:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx, \qquad \|f\| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}. \tag{3.14}$$

This inner product on functions evidently satisfies the linearity and symmetry properties (3.3) and (3.4), and it is related to the norm by the equation $\|f\| = \langle f, f\rangle^{1/2}$. *Hence the Cauchy-Schwarz inequality, the triangle inequality, and the Pythagorean theorem remain valid in this context, with the same proofs.* Explicitly, in terms of integrals, they say the following:

$$\left|\int_a^b f(x)\overline{g(x)}\,dx\right| \le \sqrt{\int_a^b |f(x)|^2\,dx}\sqrt{\int_a^b |g(x)|^2\,dx}, \tag{3.15}$$

$$\sqrt{\int_a^b |f(x)+g(x)|^2\,dx} \le \sqrt{\int_a^b |f(x)|^2\,dx} + \sqrt{\int_a^b |g(x)|^2\,dx}, \tag{3.16}$$

and

$$\int_a^b \left|\sum_1^n f_j(x)\right|^2 dx = \sum_1^n \int_a^b |f_j(x)|^2\,dx \quad \text{when} \quad \int_a^b f_i(x)\overline{f_j(x)}\,dx = 0 \quad \text{for } i \ne j. \tag{3.17}$$

The homogeneity property (3.5) of the norm, i.e., $\|cf\| = |c|\,\|f\|$, is clearly valid in the present situation, but there is a slight problem with the positivity property (3.6). The integral of a function is not affected by altering the value of the function at a finite number of points, so if $f$ is a function on $[a, b]$ that is zero except at a finite number of points, then $\|f\| = 0$ although $f$ is not the zero function. For the class $PC(a, b)$ with which we are working, there are two ways out of this difficulty. One is to use the convention suggested by the Fourier convergence theorem, that is, to consider only functions $f \in PC(a, b)$ with the property that

$$f(x) = \tfrac{1}{2}\left[f(x-) + f(x+)\right] \quad \text{for all } x \in (a, b), \quad f(a) = f(a+), \quad f(b) = f(b-).$$

If $f \in PC(a, b)$ satisfies this condition and $f(x_0) \ne 0$, then $|f(x)| > 0$ on some interval containing $x_0$, and hence $\|f\| > 0$. (See Exercises 6 and 7.) The other is simply to agree to consider two functions as equal if they agree except at finitely many points. The reader can use whichever of these devices seems most comfortable; at any rate, we shall not worry any more about this matter.

The concepts of orthogonal and orthonormal sets of functions are defined just as for vectors in $\mathbf{C}^k$, and we can ask whether there is an analogue of Theorem 3.1. That is, given an orthonormal set $\{\phi_n\}$ in $PC(a, b)$, can we express an arbitrary $f \in PC(a,b)$ as $\sum\langle f,\phi_n\rangle\phi_n$? Here, for the first time, we have to confront the fact that the space $PC(a,b)$, unlike $\mathbf{C}^k$, is infinite-dimensional. This means, in particular, that we cannot tell whether the set $\{\phi_n\}$ contains "enough" functions to span the whole space just by counting how many functions are in it; after all, if one removes finitely many elements from an infinite set, there are still infinitely many left. It also means that the sum $\sum\langle f,\phi_n\rangle\phi_n$ will be an infinite series, so we have to worry about convergence. Hence there is some work to be done; but we can see that we are on the track of something very interesting by reconsidering the results of the previous chapter in the light of the ideas we have just developed.

Consider the functions

$$\phi_n(x) = (2\pi)^{-1/2}e^{inx}, \qquad n = 0, \pm1, \pm2, \dots$$

We regard these functions as elements of the space $PC(-\pi,\pi)$; we then have

$$\langle\phi_m,\phi_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}\overline{e^{inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Thus $\{\phi_n\}_{-\infty}^{\infty}$ is an orthonormal set. Moreover, if the Fourier coefficients $c_n$ of $f \in PC(-\pi,\pi)$ are defined as in Chapter 2, we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}\,dx = (2\pi)^{-1/2}\langle f,\phi_n\rangle,$$

and hence

$$\sum_{-\infty}^{\infty} c_n e^{inx} = \sum_{-\infty}^{\infty}\Big[(2\pi)^{-1/2}\langle f,\phi_n\rangle\Big]\Big[(2\pi)^{1/2}\phi_n(x)\Big] = \sum_{-\infty}^{\infty}\langle f,\phi_n\rangle\phi_n(x).$$

Thus, the Fourier series of $f$ is just its expansion with respect to the orthonormal set $\{\phi_n\}$, as one would expect from the discussion in §3.1!

Let us try this again for Fourier cosine series on the interval $[0,\pi]$. From the trigonometric identity

$$\cos a\cos b = \tfrac{1}{2}\Big[\cos(a+b) + \cos(a-b)\Big]$$

and the fact that

$$\int_0^{\pi}\cos kx\,dx = \begin{cases} k^{-1}\sin kx|_0^{\pi} = 0 & \text{for } k \neq 0, \\ x|_0^{\pi} = \pi & \text{for } k = 0, \end{cases}$$

we see that for $m, n \geq 0$,

$$\begin{aligned}\int_0^{\pi}\cos mx\cos nx\,dx &= \frac{1}{2}\int_0^{\pi}\Big[\cos(m+n)x + \cos(m-n)x\Big]dx \\ &= \begin{cases} \pi & \text{if } m = n = 0, \\ \frac{1}{2}\pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n. \end{cases}\end{aligned}$$

That is, if we define

$$\psi_0(x) = (1/\pi)^{1/2}, \qquad \psi_n(x) = (2/\pi)^{1/2}\cos nx \quad \text{for } n > 0,$$

then $\{\psi_n\}_0^\infty$ is an orthonormal set in $PC(0,\pi)$. Moreover, if the Fourier cosine coefficients $a_n$ of $f \in PC(0,\pi)$ are defined as before,

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \begin{cases} 2(1/\pi)^{1/2}\langle f, \psi_0\rangle & \text{for } n = 0, \\ (2/\pi)^{1/2}\langle f, \psi_n\rangle & \text{for } n > 0, \end{cases}$$

we have

$$\tfrac{1}{2}a_0 + \sum_0^\infty a_n \cos nx = \sum_0^\infty \langle f, \psi_n\rangle \psi_n(x).$$

The reader may verify that the trigonometric form of the Fourier series on $[-\pi,\pi]$ and the Fourier sine series on $[0,\pi]$ are also instances of expansions with respect to orthonormal sets.

Now, we have been a bit cavalier in this discussion. The reader will recall that we proved the validity of Fourier expansions only for piecewise smooth functions; for functions that are merely piecewise continuous there is no guarantee that the Fourier series will converge at any given point. What this means is that we need to take a closer look at questions of convergence in the context of the ideas from vector geometry that we are now using.

### *EXERCISES*

1. Show that $\left\{(2/l)^{1/2}\sin(n-\tfrac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0,l)$.
2. Show that $\left\{(2/l)^{1/2}\cos(n-\tfrac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal set in $PC(0,l)$.
3. Show that $f_0(x) = 1$ and $f_1(x) = x$ are orthogonal on $[-1,1]$, and find constants $a$ and $b$ so that $f_2(x) = x^2 + ax + b$ is orthogonal to both $f_0$ and $f_1$ on $[-1,1]$. What are the normalizations of $f_0$, $f_1$, and $f_2$?
4. Suppose $\{\phi_n\}$ is an orthonormal set in $PC(0,l)$, and let $\phi_n^+$ and $\phi_n^-$ be the even and odd extensions of $\phi_n$ to $[-l,l]$. Show that $\left\{2^{-1/2}\phi_n^+\right\} \cup \left\{2^{-1/2}\phi_n^-\right\}$ is an orthonormal set in $PC(-l,l)$. (Hint: First show that $\left\{2^{-1/2}\phi_n^+\right\}$ and $\left\{2^{-1/2}\phi_n^-\right\}$ are orthonormal, and then that $\langle\phi_n^+, \phi_m^-\rangle = 0$ for all $m, n$.)
5. Let $\{\phi_n : n \geq 0\}$ be an orthonormal set in $PC(-l,l)$ such that $\phi_n$ is even when $n$ is even and $\phi_n$ is odd when $n$ is odd. Show that $\{\sqrt{2}\,\phi_n : n \text{ even}\}$ and $\{\sqrt{2}\,\phi_n : n \text{ odd}\}$ are orthonormal sets in $PC(0,l)$.
6. Suppose $f \in PC(a,b)$ and $f(x) = \frac{1}{2}\left[f(x-) + f(x+)\right]$ for all $x \in (a,b)$. Show that if $f(x_0) \neq 0$ for some $x_0 \in (a,b)$, then $f(x) \neq 0$ for all $x$ in some interval containing $x_0$. ($x_0$ may be an endpoint of the interval.)
7. Show that if $f \in PC(a,b)$, $f \geq 0$, and $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ except perhaps at finitely many points. (Hint: By redefining $f$ at its discontinuities, you can make $f$ satisfy the conditions of Exercise 6.)

## 3.3 Convergence and completeness

If we visualize a $k$-dimensional vector $\mathbf{a}$ as the point in $k$-space with coordinates $(a_1, \ldots, a_k)$ rather than as an arrow, then $\|\mathbf{a} - \mathbf{b}\|$ is just the distance between the points $\mathbf{a}$ and $\mathbf{b}$ as defined by Euclidean geometry. Accordingly, the natural notion of convergence for vectors is that $\mathbf{a}_n \to \mathbf{a}$ if and only if $\|\mathbf{a}_n - \mathbf{a}\| \to 0$. This suggests a new definition of convergence for functions. Namely, if $\{f_n\}$ is a sequence of functions in $PC(a, b)$, we say that $f_n \to f$ **in norm** if $\|f_n - f\| \to 0$, that is,

$$f_n \to f \text{ in norm} \quad \Longleftrightarrow \quad \int_a^b |f_n(x) - f(x)|^2 dx \to 0.$$

Convergence of $f_n$ to $f$ in norm thus means that the difference $f_n - f$ tends to zero in a suitable averaged sense over the interval $[a, b]$. It does not guarantee pointwise convergence, nor does pointwise convergence imply convergence in norm. For example, let $[a, b] = [0, 1]$. If we define

$$f_n(x) = 1 \quad \text{for } 0 \le x \le 1/n, \quad f_n(x) = 0 \quad \text{elsewhere,}$$

then

$$\|f_n\|^2 = \int_0^1 |f_n(x)|^2 \, dx = \int_0^{1/n} dx = 1/n,$$

so $f_n \to 0$ in norm, but $f_n(0) = 1$ for all $n$, so $f_n$ does not converge to zero pointwise. On the other hand, if

$$g_n(x) = n \quad \text{for } 0 < x < 1/n, \quad g_n(x) = 0 \quad \text{elsewhere,}$$

then $g_n \to 0$ pointwise (in fact, $g_n(0) = 0$ for all $n$, and for any $x > 0$, $g_n(x) = 0$ for $n > |x|^{-1}$), but

$$\|g_n\|^2 = \int_0^1 |g_n(x)|^2 \, dx = \int_0^{1/n} n^2 \, dx = n,$$

so $g_n \not\to 0$ in norm. However, we have the following simple and useful result.

**Theorem 3.2.** *If $f_n \to f$ uniformly on $[a, b]$ $(-\infty < a < b < \infty)$, then $f_n \to f$ in norm.*

*Proof:* Uniform convergence means that there is a sequence $\{M_n\}$ of constants such that $|f_n(x) - f(x)| \le M_n$ for all $x \in [a, b]$ and $M_n \to 0$. But then

$$\|f_n - f\|^2 = \int_a^b |f_n(x) - f(x)|^2 \, dx \le \int_a^b M_n^2 \, dx = (b - a) M_n^2,$$

so $\|f_n - f\|$ tends to zero along with $M_n$. ∎

It should be mentioned that the norm and inner product are themselves continuous with respect to convergence in norm; that is, if $f_n \to f$ in norm, then

$$\|f_n\| \to \|f\|, \qquad \langle f_n, g\rangle \to \langle f, g\rangle \quad \text{and} \quad \langle g, f_n\rangle \to \langle g, f\rangle \quad \text{for all } g.$$

The verification is left to the reader (Exercises 1 and 2).

$PC(a,b)$ fails in one crucial respect to be a good infinite-dimensional analogue of Euclidean space, namely, it is not *complete*. This means, intuitively, that there are sequences that look like they ought to converge in norm, but which fail to have a limit in the space $PC(a,b)$. The formal definition is as follows. A sequence $\{\mathbf{a}_n\}_1^\infty$ of vectors (or functions or numbers) is called a **Cauchy sequence** if $\|\mathbf{a}_m - \mathbf{a}_n\| \to 0$ as $m, n \to \infty$, that is, if the terms in the sequence get closer and closer to each other as one goes further out in the sequence. A space $S$ of vectors (or functions or numbers) is called **complete** if every Cauchy sequence in $S$ has a limit in $S$. The real and complex number systems are complete, and it follows easily that the vector spaces $\mathbf{C}^k$ are complete for any $k$. The set $R$ of rational numbers is not: if $\{r_n\}$ is a sequence of rational numbers with an irrational limit, such as the sequence of decimal approximations to $\pi$, then $\{r_n\}$ is Cauchy but has no limit in $R$.

One can see that $PC(a,b)$ is not complete by the following simple example. Take $[a,b] = [0,1]$, and let

$$f_n(x) = x^{-1/4} \quad \text{for } x > 1/n, \qquad f_n(x) = 0 \quad \text{for } x \le 1/n.$$

If $m > n$, $f_m(x) - f_n(x)$ equals $x^{-1/4}$ when $m^{-1} < x \le n^{-1}$ and equals 0 otherwise, so

$$\|f_m - f_n\|^2 = \int_{1/m}^{1/n} x^{-1/2}\,dx = 2x^{1/2}\Big|_{1/m}^{1/n} = 2(n^{-1/2} - m^{-1/2}),$$

which tends to zero as $m, n \to \infty$. Thus the sequence $\{f_n\}$ is Cauchy; but clearly its limit, either pointwise or in norm, is the function

$$f(x) = x^{-1/4} \quad \text{for } x > 0, \qquad f(0) = 0, \tag{3.18}$$

and this function does not belong to $PC(0,1)$ because it becomes unbounded as $x \to 0$.

It is easy enough to enlarge the space $PC(a,b)$ to include functions such as (3.18) with one or more infinite singularities in the interval $[a,b]$: One simply allows improper (but absolutely convergent) integrals in the definition of the inner product and the norm. But even this is not enough. One can construct Cauchy sequences $\{f_n\}$ in which $f_n$ acquires more and more singularities as $n$ increases, in such a way that the limit function $f$ is everywhere discontinuous — and in particular, not Riemann integrable on any interval.

Fortunately, there is a more sophisticated theory of integration, the *Lebesgue integral,* which allows one to handle such highly irregular functions. The Lebesgue theory does require a very weak regularity condition called *measurability*, but this technicality need not concern us. All functions that arise in practice are measurable, and *all functions mentioned in the remainder of this book are tacitly assumed to be measurable.* For our present purposes, we do not need to know anything about the construction or detailed properties of the Lebesgue integral; all we need is a couple of definitions and a couple of facts that we shall quote without proof. Rudin [47] and Dym-McKean [19] contain brief expositions of Lebesgue integration that include most of the results we shall use; more extensive accounts of the theory can be found, for example, in Folland [25] and Wheeden-Zygmund [56].

We denote by $L^2(a,b)$ the space of **square-integrable** functions on $[a,b]$, that is, the set of all functions on $[a,b]$ whose squares are absolutely Lebesgue-integrable over $[a,b]$:

$$L^2(a,b) = \left\{ f : \int_a^b |f(x)|^2\,dx < \infty \right\}. \tag{3.19}$$

This space includes all functions for which the (possibly improper) Riemann integral $\int_a^b |f(x)|^2 dx$ converges, and one should think of it simply as the space of all functions $f$ such that the region between the graph of $|f|^2$ and the $x$-axis has finite area. Since

$$st \le \tfrac{1}{2}(s^2 + t^2)$$

(because $s^2 + t^2 - 2st = (s-t)^2 \ge 0$) for any real numbers $s$ and $t$, we have

$$|f(x)\overline{g(x)}| \le \tfrac{1}{2}\Big(|f(x)|^2 + |g(x)|^2\Big),$$

and thus if $f$ and $g$ are in $L^2(a,b)$, the integral

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

is absolutely convergent. Therefore, the definitions of the inner product and norm extend to the space $L^2(a,b)$, as do all their properties that we have discussed previously.

As in the space $PC(a,b)$, there is a slight problem with the positivity of the norm, as the condition $\int |f|^2 = 0$ does not imply that $f$ vanishes identically but only that the $f = 0$ "almost everywhere." The precise interpretation of this phrase is as follows. A subset $E$ of $\mathbf{R}$ is said to have **measure zero** if, for any $\epsilon > 0$, $E$ can be covered by a sequence of open intervals whose total length is less than $\epsilon$, that is, if there exist open intervals $I_1, I_2, \ldots$ of lengths $l_1, l_2, \ldots$ such that $E \subset \bigcup_1^\infty I_j$ and $\sum_1^\infty l_j < \epsilon$. (For example, any countable set has measure zero: If $E = \{x_1, x_2, \ldots\}$, let $I_j$ be the interval of length $\epsilon/2^j$ centered at $x_j$.) A statement about real numbers that is true for all $x$ except for those $x$ in some set of measure zero is said to be true **almost everywhere**, or for **almost every** $x$.

It can be shown that if $f \in L^2(a,b)$, the norm of $f$ is zero if and only if $f(x) = 0$ for almost every $x \in [a,b]$. Accordingly, we agree to regard two functions as equal if they are equal almost everywhere. This weakened notion of equality then validates the statement that $\|f\| = 0$ only when $f = 0$, and it turns out also to be appropriate in many other contexts. Moreover, if two *continuous* functions are equal almost everywhere then they are identically equal, so for continuous functions the ordinary notion of equality is entirely adequate.

The crucial properties of $L^2(a,b)$ that we shall need to state without proof are contained in the following theorem.

**Theorem 3.3.** (a) *$L^2(a,b)$ is complete with respect to convergence in norm.* (b) *For any $f \in L^2(a,b)$ there is a sequence $f_n$ of continuous functions on $[a,b]$ such that $f_n \to f$ in norm. In fact, the functions $f_n$ can be taken to be the restrictions to $[a,b]$ of functions on the line that possess derivatives of all orders at every point; moreover, the latter functions can be taken to be $(b-a)$-periodic or to vanish outside a bounded set.*

This theorem says that $L^2(a,b)$ is obtained by "filling in the holes" in the space $PC(a,b)$. The first assertion says that all the holes have been filled, and the second one says that nothing extra, beyond the completion of $PC(a,b)$, has been added in. For a proof, see Rudin [47], Theorems 11.38 and 11.42. We shall indicate how to prove the second assertion — that is, how to approximate arbitrary $L^2$ functions by smooth ones — in §7.1.

We are now ready to discuss the convergence of expansions with respect to orthonormal sets in $PC(a,b)$, or more generally in $L^2(a,b)$. The first step is to obtain the general form of Bessel's inequality, which is a straightforward generalization of the special case we proved in §2.1.

**Bessel's Inequality.** *If $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$, then*

$$\sum_1^\infty |\langle f, \phi_n\rangle|^2 \le \|f\|^2. \tag{3.20}$$

*Proof:* Observe that

$$\left\langle f, \langle f, \phi_n\rangle\phi_n \right\rangle = \overline{\langle f, \phi_n\rangle}\langle f, \phi_n\rangle = |\langle f, \phi_n\rangle|^2$$

and that by the Pythagorean theorem,

$$\left\|\sum_1^N \langle f, \phi_n\rangle\phi_n\right\|^2 = \sum_1^N |\langle f, \phi_n\rangle|^2.$$

Hence, for any positive integer $N$, by Lemma 3.1,

$$\begin{aligned}0 &\le \left\| f - \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 \\ &= \|f\|^2 - 2\,\mathrm{Re}\left\langle f, \sum_1^N \langle f, \phi_n\rangle \phi_n \right\rangle + \left\| \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 \\ &= \|f\|^2 - 2\sum_1^N |\langle f, \phi_n\rangle|^2 + \sum_1^N |\langle f, \phi_n\rangle|^2 \\ &= \|f\|^2 - \sum_1^N |\langle f, \phi_n\rangle|^2.\end{aligned}$$

Letting $N \to \infty$, we obtain the desired result. ∎

We are now concerned with the following problem: given an orthonormal set $\{\phi_n\}_1^\infty$ in $L^2(a,b)$, is it true that

$$f = \sum_1^\infty \langle f, \phi_n\rangle \phi_n \tag{3.21}$$

for all $f \in L^2(a,b)$? First we assure ourselves that the series on the right actually makes sense.

**Lemma 3.2.** *If $f \in L^2(a,b)$ and $\{\phi_n\}$ is any orthonormal set in $L^2(a,b)$, then the series $\sum \langle f, \phi_n\rangle \phi_n$ converges in norm, and $\left\|\sum \langle f, \phi_n\rangle \phi_n\right\| \le \|f\|$.*

*Proof:* Bessel's inequality guarantees that the series $\sum |\langle f, \phi_n\rangle|^2$ converges, so by the Pythagorean theorem,

$$\left\| \sum_m^n \langle f, \phi_n\rangle \phi_n \right\|^2 = \sum_m^n |\langle f, \phi_n\rangle|^2 \to 0 \quad \text{as } m, n \to \infty.$$

Thus the partial sums of the series $\sum \langle f, \phi_n\rangle \phi_n$ form a Cauchy sequence, and since $L^2(a,b)$ is complete, the series converges. Finally, another application of the Pythagorean theorem and Bessel's inequality gives

$$\begin{aligned}\left\| \sum_1^\infty \langle f, \phi_n\rangle \phi_n \right\|^2 = \lim_{N\to\infty} \left\| \sum_1^N \langle f, \phi_n\rangle \phi_n \right\|^2 &= \lim_{N\to\infty} \sum_1^N |\langle f, \phi_n\rangle|^2 \\ &= \sum_1^\infty |\langle f, \phi_n\rangle|^2 \le \|f\|^2.\end{aligned}$$ ∎

Now, an obvious necessary condition for (3.21) to hold for arbitrary $f$ is that the orthonormal set $\{\phi_n\}$ is as large as possible, that is, that there is no nonzero $f$ which is orthogonal to all the $\phi_n$'s. (If $\langle f, \phi_n\rangle = 0$ for all $n$, then (3.21) implies that $f = 0$.) Moreover, if (3.21) holds and the Pythagorean theorem extends to infinite sums of orthogonal vectors, Bessel's inequality (3.20) should actually be an equality. With these thoughts in mind, we arrive at the main theorem.

**Theorem 3.4.** *Let $\{\phi_n\}_1^\infty$ be an orthonormal set in $L^2(a,b)$. The following conditions are equivalent:*

*(a) If $\langle f,\phi_n\rangle = 0$ for all $n$, then $f = 0$.*

*(b) For every $f \in L^2(a,b)$ we have $f = \sum_1^\infty \langle f,\phi_n\rangle\phi_n$, where the series converges in norm.*

*(c) For every $f \in L^2(a,b)$, we have* **Parseval's equation**:

$$\|f\|^2 = \sum_1^\infty |\langle f,\phi_n\rangle|^2. \tag{3.22}$$

*Proof:* We shall show that (a) implies (b), that (b) implies (c), and that (c) implies (a).

(a) implies (b): Given $f \in L^2(a,b)$, the series $\sum\langle f,\phi_n\rangle\phi_n$ converges in norm, by Lemma 3.2. We can see that its sum is $f$ by showing that the difference $g = f - \sum\langle f,\phi_n\rangle\phi_n$ is zero. But

$$\langle g,\phi_m\rangle = \langle f,\phi_m\rangle - \sum_{n=1}^\infty \langle f,\phi_n\rangle\langle\phi_n,\phi_m\rangle = \langle f,\phi_m\rangle - \langle f,\phi_m\rangle = 0$$

for all $m$. Hence, if (a) holds, $g = 0$.

(b) implies (c): If $f = \sum\langle f,\phi_n\rangle\phi_n$, then by the Pythagorean theorem,

$$\|f\|^2 = \lim_{N\to\infty}\left\|\sum_1^N \langle f,\phi_n\rangle\phi_n\right\|^2 = \lim_{N\to\infty}\sum_1^N |\langle f,\phi_n\rangle|^2 = \sum_1^\infty |\langle f,\phi_n\rangle|^2.$$

(c) implies (a): If (c) holds and $\langle f,\phi_n\rangle = 0$ for all $n$ then $\|f\| = 0$, and therefore $f = 0$. ∎

An orthonormal set that possesses the properties (a)–(c) of Theorem 3.4 is called a **complete orthonormal set** or an **orthonormal basis** for $L^2(a,b)$. This usage of the word *complete* is different from the one discussed earlier in this section, but it is obviously appropriate in the present context. If $\{\phi_n\}$ is an orthonormal basis of $L^2(a,b)$ and $f \in L^2(a,b)$, the numbers $\langle f,\phi_n\rangle$ are called the (generalized) **Fourier coefficients** of $f$ with respect to $\{\phi_n\}$, and the series $\sum\langle f,\phi_n\rangle\phi_n$ is called the (generalized) **Fourier series** of $f$.

Often it is more convenient not to require the elements of a basis to be unit vectors. Accordingly, suppose $\{\psi_n\}$ is an orthogonal set (and recall that, according to our definition of orthogonal set, this entails $\psi_n \neq 0$ for all $n$). Let $\phi_n = \|\psi_n\|^{-1}\psi_n$; then $\{\phi_n\}$ is an orthonormal set. We say that $\{\psi_n\}$ is a **complete orthogonal set** or an **orthogonal basis** if $\{\phi_n\}$ is an orthonormal basis. In this case the expansion formula for $f \in L^2(a,b)$ and the Parseval equation take the form

$$f = \sum \frac{\langle f,\psi_n\rangle}{\|\psi_n\|^2}\psi_n \qquad \|f\|^2 = \sum \frac{|\langle f,\psi_n\rangle|^2}{\|\psi_n\|^2}. \tag{3.23}$$

Now, what about the orthonormal sets derived from Fourier series that we discussed in §3.2? We have not yet proved that they are complete, for we derived the expansion formula $f = \sum\langle f,\phi_n\rangle\phi_n$ only when $f$ was piecewise smooth, not for an arbitrary $f \in L^2(a,b)$. But there is actually very little work left to do.

**Theorem 3.5.** *The sets*

$$\left\{e^{inx}\right\}_{n=-\infty}^{\infty} \quad \textit{and} \quad \left\{\cos nx\right\}_{n=0}^{\infty} \cup \left\{\sin nx\right\}_{n=1}^{\infty}$$

*are orthogonal bases for* $L^2(-\pi, \pi)$. *The sets*

$$\left\{\cos nx\right\}_{n=0}^{\infty} \quad \textit{and} \quad \left\{\sin nx\right\}_{n=1}^{\infty}$$

*are orthogonal bases for* $L^2(0, \pi)$.

*Proof:* First consider the functions $\psi_n(x) = e^{inx}$. Suppose $f \in L^2(-\pi, \pi)$ and $\epsilon$ is a (small) positive number; we wish to show that the $N$th partial sum of the Fourier series of $f$ approximates $f$ in norm to within $\epsilon$ if $N$ is sufficiently large. By part (b) of Theorem 3.3, we can find a $2\pi$-periodic function $\widetilde{f}$, possessing derivatives of all orders, such that $\|f - \widetilde{f}\| < \epsilon/3$. Let $c_n = (2\pi)^{-1}\langle f, \psi_n\rangle$ and $\widetilde{c}_n = (2\pi)^{-1}\langle \widetilde{f}, \psi_n\rangle$ be the Fourier coefficients of $f$ and $\widetilde{f}$. By Theorem 2.5 of §2.3, we know that the Fourier series $\sum \widetilde{c}_n \psi_n$ converges uniformly to $\widetilde{f}$; hence, by Theorem 3.3, it converges to $\widetilde{f}$ in norm. Thus, if we take $N$ sufficiently large, we have

$$\left\|\widetilde{f} - \sum_{-N}^{N} \widetilde{c}_n \psi_n\right\| < \frac{\epsilon}{3}.$$

Moreover, by the Pythagorean theorem and Bessel's inequality,

$$\begin{aligned}\left\|\sum_{-N}^{N} \widetilde{c}_n \psi_n - \sum_{-N}^{N} c_n \psi_n\right\|^2 &\le \sum_{-N}^{N} |\widetilde{c}_n - c_n|^2 \\ &\le \sum_{-\infty}^{\infty} |\widetilde{c}_n - c_n|^2 \le \|\widetilde{f} - f\|^2 < \left(\frac{\epsilon}{3}\right)^2.\end{aligned}$$

Thus, if we write

$$f - \sum_{-N}^{N} c_n \psi_n = \left(f - \widetilde{f}\right) + \left(\widetilde{f} - \sum_{-N}^{N} \widetilde{c}_n \psi_n\right) + \left(\sum_{-N}^{N} \widetilde{c}_n \psi_n - \sum_{-N}^{N} c_n \psi_n\right)$$

and use the triangle inequality, we see that

$$\left\|f - \sum_{-N}^{N} c_n \psi_n\right\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves the completeness of the set $\{\psi_n\} = \{e^{inx}\}$ in $L^2(-\pi, \pi)$, and the completeness of $\{\cos nx\} \cup \{\sin nx\}$ is essentially a restatement of the same result. The completeness of $\{\cos nx\}$ and $\{\sin nx\}$ in $L^2(0, \pi)$ is an easy corollary. (Just consider the even or odd extension of $f \in L^2(0, \pi)$ to $[-\pi, \pi]$.) ∎

The normalizing constants for the functions in Theorem 3.5 are, of course, $\sqrt{1/2\pi}$ for $e^{inx}$, $\sqrt{1/\pi}$ for $\cos nx$ and $\sin nx$ on $[-\pi, \pi]$ (except for $n = 0$), and $\sqrt{2/\pi}$ for $\cos nx$ and $\sin nx$ on $[0, \pi]$ (except for $n = 0$). With this in mind, one easily sees that the Parseval equation takes the form

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = 2\pi \sum_{-\infty}^{\infty} |c_n|^2 = \frac{\pi}{2}|a_0|^2 + \pi \sum_{1}^{\infty}\left(|a_n|^2 + |b_n|^2\right), \qquad f \in L^2(-\pi, \pi),$$

where $a_n$, $b_n$, and $c_n$ are the Fourier coefficients of $f$ as defined in §2.1, and

$$\int_{0}^{\pi} |f(x)|^2\,dx = \frac{\pi}{4}|a_0|^2 + \frac{\pi}{2}\sum_{1}^{\infty} |a_n|^2 = \frac{\pi}{2}\sum_{1}^{\infty} |b_n|^2, \qquad f \in L^2(0, \pi),$$

where $a_n$ and $b_n$ are the Fourier cosine and sine coefficients of $f$ as defined in §2.4. For example, if we consider the Fourier sine series of $f(x) = x$ on $[0, \pi]$ as derived in §2.1, we find that

$$\frac{\pi}{2}\sum_{1}^{\infty} \frac{4}{n^2} = \int_0^{\pi} x^2\,dx = \frac{\pi^3}{3}, \quad \text{or} \quad \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

a result which we derived by other means in Exercise 3, §2.3.

Let us sum up our theorems about the convergence of Fourier series. If $f$ is a periodic function, then the Fourier series of $f$ converges to $f$

(i) absolutely, uniformly, and in norm, if $f$ is continuous and piecewise smooth;
(ii) pointwise and in norm, if $f$ is piecewise smooth;
(iii) in norm, if $f \in L^2(a, b)$.

These results are sufficient for virtually all practical purposes. However, as we indicated in §2.6, there is more to be said on the subject. Here we shall just mention one more result that is a natural generalization of the theorems in this section. If $1 \le p < \infty$, we define $L^p(a, b)$ to be the space of Lebesgue-integrable functions $f$ on $[a, b]$ such that

$$\int_a^b |f(x)|^p\,dx < \infty.$$

If $p > 1$, the Fourier series of any $f \in L^p(-\pi, \pi)$ converges to $f$ in the "$L^p$ norm," that is, if $\{c_n\}$ are the Fourier coefficients of $f$,

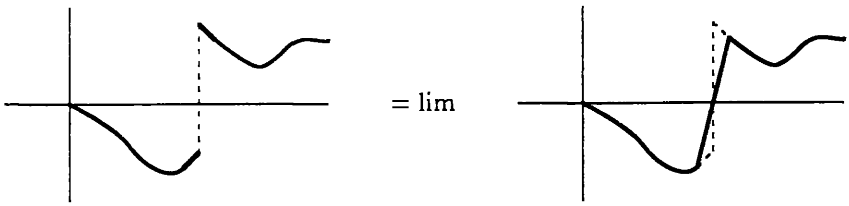
$$\int_a^b \left|\sum_{-N}^{N} c_n e^{inx} - f(x)\right|^p dx \to 0 \quad \text{as } N \to \infty.$$

However, this result is false for $p = 1$.

## *EXERCISES*

1. Show that if $f_n \in L^2(a, b)$ and $f_n \to f$ in norm, then $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(a, b)$. (Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f,\ g\rangle$.)
2. Show that $\big|\,\|f\| - \|g\|\,\big| \le \|f - g\|$. (Use the triangle inequality; consider the cases $\|f\| \ge \|g\|$ and $\|f\| \le \|g\|$ separately.) Deduce that if $f_n \to f$ in norm then $\|f_n\| \to \|f\|$.

3. Show directly that any $f \in PC(a,b)$ is the limit in norm of a sequence of continuous functions on $[a,b]$, by the argument suggested by the following picture.



4. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Suppose $c > 0$ and $d \in \mathbf{R}$, and let $\psi_n(x) = c^{1/2}\phi_n(cx+d)$. Show that $\{\psi_n\}$ is an orthonormal basis for $L^2(\frac{a-d}{c}, \frac{b-d}{c})$.
5. Finish the proof of Theorem 3.5. That is, from the completeness of $\{e^{inx}\}$ on $[-\pi,\pi]$, deduce the completeness of $\{\cos nx\} \cup \{\sin nx\}$ on $[-\pi,\pi]$ and the completeness of $\{\cos nx\}$ and $\{\sin nx\}$ on $[0,\pi]$.
6. Let $\phi_n(x) = (2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)$. In Exercise 1, §3.2, it was shown that $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(0,l)$. Prove that it is actually a basis, via the following argument.
   a. Let $\psi_k(x) = l^{-1/2}\sin(k\pi x/2l)$. Show that $\{\psi_k\}_1^\infty$ is an orthonormal basis for $L^2(0,2l)$. (This follows from Theorem 3.5 and Exercise 4.)
   b. If $f \in L^2(0,l)$, extend $f$ to $[0,2l]$ by making it symmetric about the line $x = l$, that is, define the extension $\tilde{f}$ by $\tilde{f}(x) = \tilde{f}(2l-x) = f(x)$ for $x \in [0,l]$. Show that $\langle \tilde{f}, \psi_{2n}\rangle = 0$ and $\langle \tilde{f}, \psi_{2n-1}\rangle = 2^{1/2}\langle f, \phi_n\rangle$.
   c. Conclude that if $\langle f, \phi_n\rangle = 0$ for all $n$, then $f = 0$.
7. Show that $\left\{(2/l)^{1/2}\cos(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal basis for $L^2(0,l)$. (The argument is similar to that in Exercise 6, but this time you should extend $f$ to be skew-symmetric about $x = l$, that is, $\tilde{f}(2l-x) = -\tilde{f}(x) = -f(x)$ for $x \in [0,l]$.)
8. Find the expansions of the functions $f(x) = 1$ and $g(x) = x$ on $[0,l]$ with respect to the orthonormal bases in Exercises 6 and 7.
9. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Show that for any $f, g \in L^2(a,b)$,
$$\langle f, g\rangle = \sum \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle}.$$
(Note that the case $f = g$ is Parseval's equation.)
10. Evaluate the following series by applying Parseval's equation to certain of the Fourier expansions in Table 1 of §2.1.

   a. $\displaystyle\sum_1^\infty \frac{1}{n^4}$  b. $\displaystyle\sum_1^\infty \frac{1}{(2n-1)^6}$  c. $\displaystyle\sum_1^\infty \frac{n^2}{(n^2+1)^2}$

   d. $\displaystyle\sum_1^\infty \frac{\sin^2 na}{n^4}$ $(0 < a < \pi)$

11. Suppose $f$ is of class $C^{(1)}$, $2\pi$-periodic, and real-valued. Show that $f'$ is orthogonal to $f$ in $L^2(-\pi,\pi)$ in two ways: (a) by expanding $f$ in a Fourier series and using Exercise 9 and (b) directly from the fact that $2ff' = (f^2)'$.

## 3.4 More about $L^2$ spaces; the dominated convergence theorem

In this section we continue the general discussion of $L^2$ spaces and introduce an extremely useful criterion for the integral of a limit to equal the limit of the integrals.

### *Other types of $L^2$ spaces*

The results of the previous section concerning $L^2(a,b)$ can be generalized in various ways, and we shall need some of these generalizations later on.

First, one can replace the element $dx$ of linear measure on $[a,b]$ by a weighted element of measure, $w(x)\,dx$. To be precise, suppose $w$ is a continuous function on $[a,b]$ such that $w(x) > 0$ for all $x \in [a,b]$; we call such a $w$ a **weight function** on $[a,b]$. We can then define the "weighted $L^2$ space" $L^2_w(a,b)$ to be the set of all (Lebesgue measurable) functions on $[a,b]$ such that

$$\int_a^b |f(x)|^2 w(x)\,dx < \infty,$$

and we define an inner product and norm on $L^2_w(a,b)$ by

$$\langle f,g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx, \qquad \|f\|_w = \left(\int_a^b |f(x)|^2 w(x)\,dx\right)^{1/2}.$$

This inner product and norm still satisfy the fundamental conditions (3.3)–(3.6), so the theorems of §3.1 apply in this situation. So do Theorems 3.2, 3.3, and 3.4. $w$ could also be allowed to have some singularities, as long as $\int_a^b w(x)\,dx < \infty$, or to vanish at a few points. (If $w$ vanishes on a whole subinterval of $[a,b]$, one loses the strict positivity of the norm.)

Second, one can replace the bounded interval $[a,b]$ with a half-line or the whole line, or by a region in the plane or in a higher-dimensional space. That is, let $D$ be a region in $\mathbf{R}^k$. (A "region" can be anything reasonable: an open set, or the closure of an open set, or indeed any Lebesgue measurable set. It does not have to be bounded, and indeed may be the whole space.) We define $L^2(D)$ to be the set of all functions $f$ such that

$$\int_D |f(\mathbf{x})|^2\,d\mathbf{x} < \infty,$$

and we define the inner product and norm on $L^2(D)$ by

$$\langle f,g\rangle = \int_D f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x}, \qquad \|f\| = \left(\int_D |f(\mathbf{x})|^2\,d\mathbf{x}\right)^{1/2}.$$

Here $\int_D$ is a $k$-tuple integral, and $d\mathbf{x}$ is the element of Euclidean measure in $k$-space (length when $k = 1$, area when $k = 2$, volume when $k = 3$, etc.). If one is working only with Riemann integrals, one has to worry a bit about improper integrals when $D$ is unbounded, but this problem is not serious. (The Lebesgue theory handles integrals over unbounded regions rather more smoothly.) Again, this inner product and norm satisfy (3.3)–(3.6), so the results of §3.1 are available, as is Theorem 3.4. However, the analogue of Theorem 3.2 is *false* when $D$ is unbounded (or more precisely, when $D$ has infinite measure), and a glance at its proof should show why. (See Exercise 6.) We shall state a result shortly that can be used in its place.

Theorem 3.3 also needs to be reformulated; here is one good version of it.

**Theorem 3.6.** *$L^2(D)$ is complete. If $f \in L^2(D)$, there is a sequence $\{f_n\}$ that converges to $f$ in norm, such that each $f_n$ is continuous on $D$ and vanishes outside some bounded set. The $f_n$'s can be taken to be restrictions to $D$ of functions defined on all of $\mathbf{R}^k$ that have derivatives of all orders and vanish outside bounded sets.*

One can also modify $L^2(D)$ by throwing in a weight function, as before.

As a matter of fact, all one needs to develop the ideas of §3.1 are the following ingredients:

(i) a vector space $\mathscr{H}$, that is, a collection of objects that can be added to each other and multiplied by complex numbers, such that the usual laws of vector addition and scalar multiplication hold;

(ii) an inner product $\langle u, v\rangle$ on $\mathscr{H}$ and associated norm $\|u\| = \langle u, u\rangle^{1/2}$ that satisfy (3.3)–(3.6).

If, in addition, the space $\mathscr{H}$ is *complete* with respect to convergence in norm, it is called a **Hilbert space**. In this case, Bessel's inequality and Theorem 3.4 also hold. This general setup includes, but is not limited to, the spaces $\mathbf{C}^k$, $L^2(a,b)$, $L^2_w(a,b)$, and $L^2(D)$ discussed above.

Another example of a Hilbert space is the space $l^2$ of square-summable sequences. That is, the elements of $l^2$ are sequences $\{c_n\}_1^\infty$ of complex numbers such that $\sum_1^\infty |c_n|^2 < \infty$, and the inner product and norm are defined by

$$\Big\langle \{c_n\}, \{d_n\} \Big\rangle = \sum_1^\infty c_n \overline{d}_n, \qquad \Big\|\{c_n\}\Big\| = \left(\sum_1^\infty |c_n|^2\right)^{1/2}.$$

We have encountered this space before without mentioning it explicitly. Indeed, suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(a,b)$. Then the mapping that takes an $f \in L^2(a,b)$ to its sequence of coefficients $\Big\{\langle f, \phi_n\rangle\Big\}$ sets up a one-to one correspondence between $L^2(a,b)$ and $l^2$ that is linear and (by Parseval's equation) norm-preserving. Such a mapping is called a **unitary operator**.

One further comment: We suggested thinking of functions $f \in L^2(a,b)$ as vectors whose components are the values $f(x)$, $x \in [a,b]$. The reader who knows about orders of infinity may be puzzled that there are uncountably many such "components," and yet the orthonormal bases we have displayed are countable sets. The explanation is that the elements of $L^2(a,b)$ are continuous functions or limits in norm of continuous functions, and the values of a continuous function are not completely independent of each other. For example, if $f$ is continuous on $[a,b]$, then $f$ is completely determined by its values at the rational points in $[a,b]$, of which there are only countably many.

### *The dominated convergence theorem*

We now state one other result from the Lebesgue theory of integration that is of great utility even in the setting of Riemann integrable functions. It gives a general condition under which the integral of a limit is the limit of the integrals, and is an improvement on most of the theorems of this sort that one commonly encounters in calculus texts. We shall use it frequently throughout the rest of this book.

**The Dominated Convergence Theorem.** *Let $D$ be a region in $\mathbf{R}^k$ $(k = 1,2,3,\ldots)$. Suppose $g_n$ $(n = 1,2,3,\ldots)$, $g$, and $\phi$ are functions on $D$, such that*
*(a) $\phi(\mathbf{x}) \geq 0$ and $\int_D \phi(\mathbf{x})\,d\mathbf{x} < \infty$,*
*(b) $|g_n(\mathbf{x})| \leq \phi(\mathbf{x})$ for all $n$ and all $x \in D$,*
*(c) $g_n(\mathbf{x}) \to g(\mathbf{x})$ as $n \to \infty$ for all $\mathbf{x} \in D$.*
*Then $\int_D g_n(\mathbf{x})\,d\mathbf{x} \to \int_D g(\mathbf{x})\,d\mathbf{x}$.*

The proof of this theorem is beyond the scope of this book (see Rudin [47], Folland [25], or Wheeden-Zygmund [56]), but the intuition behind it can be easily explained. If $g_n \to g$ pointwise, how can the relation $\int_D g_n \to \int_D g$ fail? Consider the following two examples, in which $D$ is the real line:

$$f_n(x) = 1 \quad \text{for } n < x < n+1, \qquad f_n(x) = 0 \quad \text{otherwise.}$$
$$g_n(x) = n \quad \text{for } 0 < x < 1/n, \qquad g_n(x) = 0 \quad \text{otherwise.}$$

We have

$$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} g_n(x)\,dx = 1 \quad \text{for all } n,$$

but $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = 0$ for all $x$. The trouble is that as $n \to \infty$, the region under the graph of $f_n$ moves out to infinity to the right, and the region under the graph of $g_n$ moves out to infinity upwards, so in the limit there is nothing left. (See Figure 3.3.)

Now, the dominated convergence theorem essentially says that if this sort of bad behavior is eliminated, then the integral of the limit is the limit of the integrals. Hypothesis (a) says that the region under the graph of $\phi$ has finite area, and hypothesis (b) says that the graphs of $|g_n|$ are trapped inside this region, so they cannot leak out to infinity.

As a corollary, we obtain the following relation between pointwise convergence and convergence in norm.

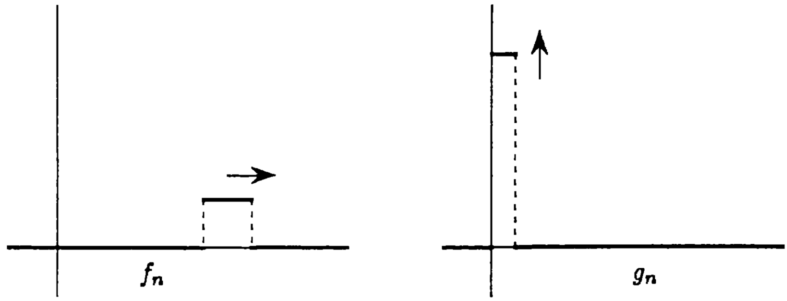



FIGURE 3.3. The examples $f_n$ and $g_n$ of sequences for which the integral of the limit is not the limit of the integral. The arrows indicate what happens as $n$ increases.

**Theorem 3.7.** *Suppose $f_n \in L^2(D)$ for all $n$ and $f_n \to f$ pointwise. If there exists $\psi \in L^2(D)$ such that $|f_n(\mathbf{x})| \le |\psi(\mathbf{x})|$ for all $n$ and all $\mathbf{x} \in D$, then $f_n \to f$ in norm.*

*Proof:* We have $|f(\mathbf{x})| = \lim |f_n(\mathbf{x})| \le |\psi(\mathbf{x})|$, and hence

$$|f_n(\mathbf{x}) - f(\mathbf{x})|^2 \le \Big(|f_n(\mathbf{x})| + |f(\mathbf{x})|\Big)^2 \le |2\psi(\mathbf{x})|^2.$$

Therefore, we can apply the dominated convergence theorem, with $g_n = |f_n - f|^2$, $g = 0$, and $\phi = |2\psi|^2$, to conclude that

$$\|f_n - f\|^2 = \int_D |f_n(\mathbf{x}) - f(\mathbf{x})|^2 \, d\mathbf{x} \to 0. \qquad \blacksquare$$

***Best approximations in $L^2$***

If $\{\phi_n\}$ is an orthonormal basis for $L^2(D)$, where $D$ is any interval in $\mathbf{R}$ or region in $\mathbf{R}^n$, we have $\sum \langle f, \phi_n \rangle \phi_n = f$ for all $f \in L^2(D)$. On the other hand, suppose $\{\phi_n\}$ is an orthonormal set in $L^2(D)$ that is not complete. If $f \in L^2(D)$, what significance can we attach to the series $\sum \langle f, \phi_n \rangle \phi_n$? We know that it converges by Lemma 3.2. In general its sum will not be $f$, but it is the unique *best approximation* to $f$ in norm among all functions of the form $\sum c_n \phi_n$. (The latter sum converges in norm precisely when $\sum |c_n|^2 < \infty$, as the argument used to prove Lemma 3.2 shows.) We state this result as a theorem.

**Theorem 3.8.** *If $\{\phi_n\}$ is an orthonormal set in $L^2(D)$ and $f \in L^2(D)$, then*

$$\Big\| f - \sum \langle f, \phi_n \rangle \phi_n \Big\| \le \Big\| f - \sum c_n \phi_n \Big\|$$

*for all choices of $c_n$ with $\sum |c_n|^2 < \infty$. Equality holds only when $c_n = \langle f, \phi_n \rangle$ for all $n$.*

*Proof:* We have

$$f - \sum c_n\phi_n = \Big(f - \sum\langle f, \phi_n\rangle\phi_n\Big) + \sum\Big(\langle f, \phi_n\rangle - c_n\Big)\phi_n.$$

Now, $f - \sum\langle f, \phi_n\rangle\phi_n$ is easily seen to be orthogonal to all $\phi_n$; see the first part of the proof of Theorem 3.4. Hence, by the Pythagorean theorem (and a simple limiting argument, if there are infinitely many $\phi_n$),

$$\Big\|f - \sum c_n\phi_n\Big\|^2 = \Big\|f - \sum\langle f, \phi_n\rangle\phi_n\Big\|^2 + \sum\Big|\langle f, \phi_n\rangle - c_n\Big|^2.$$

The last sum on the right is clearly nonnegative, and it is zero precisely when $c_n = \langle f, \phi_n\rangle$ for all $n$; this establishes the theorem. ∎

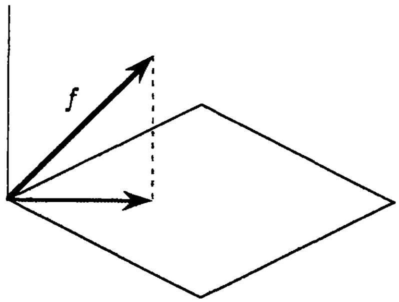



FIGURE 3.4. A vector $f$ and its orthogonal projection onto a plane.

The pictorial intuition behind Theorem 3.8 is shown in Figure 3.4. The horizontal plane represents the space of functions (or vectors) of the form $\sum c_n\phi_n$; the sum $\sum\langle f, \phi_n\rangle\phi_n$ is the closest point to $f$ in this plane, namely, the orthogonal projection of $f$ onto the plane.

One situation in which Theorem 3.8 is particularly useful is when $\{\phi_n\}$ is simply a finite subset of an orthonormal basis.

**Corollary 3.1.** *Suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(D)$. If $f \in L^2(D)$, the partial sum $\sum_1^N\langle f, \phi_n\rangle\phi_n$ of the series $\sum_1^\infty\langle f, \phi_n\rangle\phi_n$ is the best approximation in norm to $f$ among all linear combinations of $\phi_1, \ldots, \phi_N$.*

***EXERCISES***

1. Show that $\left\{e^{2\pi i(mx+ny)}\right\}_{m,n=-\infty}^{\infty}$ is an orthonormal set in $L^2(D)$ where $D$ is any square whose sides have length one and are parallel to the coordinate axes.
2. Find constants $a, b, A, B, C$ such that $f_0(x) = 1$, $f_1(x) = ax + b$, and $f_2(x) = Ax^2 + Bx + C$ are an orthonormal set in $L^2_w(0, \infty)$ where $w(x) = e^{-x}$. (Hint: $\int_0^\infty x^n e^{-x}dx = n!$.)

3. Let $D$ be the unit disc $\{x^2+y^2 \leq 1\}$, and let $f_n(x,y) = (x+iy)^n$. Show that $\{f_n\}_0^\infty$ is an orthogonal set in $L^2(D)$, and compute $\|f_n\|$ for all $n$. (Hint: In polar coordinates, $x+iy = re^{i\theta}$ and $dx\,dy = r\,dr\,d\theta$.)
4. Suppose $\{\phi_n\}$ is an orthonormal set in $L^2_w(D)$. Show that $\{w^{1/2}\phi_n\}$ is an orthonormal set in $L^2(D)$ (with respect to the weight function 1).
5. Suppose $f : [a,b] \to [c,d]$ and $f'(x) > 0$ for $x \in [a,b]$. Show that if $\{\phi_n\}$ is an orthonormal basis for $L^2(c,d)$, then $\{\phi_n \circ f\}$ is an orthonormal basis for $L^2_w(a,b)$ where $w = f'$.
6. Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n \to 0$ uniformly but $f_n \not\to 0$ in norm.
7. What is the best approximation in norm to the function $f(x) = x$ on the interval $[0,\pi]$ among all functions of the form (a) $a_0 + a_1 \cos x + a_2 \cos 2x$, (b) $b_1 \sin x + b_2 \sin 2x$, (c) $a \cos x + b \sin x$?

## 3.5 Regular Sturm-Liouville problems

In §1.3 we arrived at the orthogonal bases $\{\cos nx\}_0^\infty$ and $\{\sin nx\}_1^\infty$ for $L^2(0,\pi)$ by solving the boundary value problems

$$u''(x) + \lambda^2 u(x) = 0, \qquad u'(0) = u'(\pi) = 0$$

and

$$u''(x) + \lambda^2 u(x) = 0, \qquad u(0) = u(\pi) = 0.$$

We derived the orthogonal basis $\{e^{inx}\}_{-\infty}^\infty$ for $L^2(-\pi,\pi)$ by considering periodic functions, but we could also have found it by solving the boundary value problem

$$u''(x) + \lambda^2 u(x) = 0, \qquad u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi).$$

In fact, there is a large class of boundary value problems on an interval $[a,b]$ that lead to orthogonal bases for $L^2(a,b)$. These problems are the subject of the present section.

First, a bit of conceptual background from finite-dimensional linear algebra. We recall that a linear transformation $T : \mathbf{C}^k \to \mathbf{C}^k$ is called *self-adjoint* or *Hermitian* if

$$\langle T\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, T\mathbf{b}\rangle \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbf{C}^k.$$

(When $T$ is described by a matrix $(T_{ij})$, this means that $T_{ji} = \overline{T_{ij}}$.) It is one of the basic results of linear algebra, known as the *spectral theorem* or the *principal axis theorem,* that whenever $T$ is self-adjoint there is an orthonormal basis of $\mathbf{C}^k$ consisting of eigenvectors for $T$. What we are aiming for is an analogue of this theorem for differential operators acting on the space $L^2(a,b)$.

Suppose then that $S$ and $T$ are linear operators that are defined on certain subspaces $\mathscr{D}_S$ and $\mathscr{D}_T$ of $L^2(a,b)$ and map them into $L^2(a,b)$. We say that $S$ and $T$ are **adjoint** to each other (or that $T$ is the adjoint of $S$, or vice versa) if

$$\langle S(f), g\rangle = \langle f, T(g)\rangle \quad \text{for all } f \in \mathscr{D}_S \text{ and } g \in \mathscr{D}_T.$$

$S$ is called **self-adjoint** or **Hermitian** if

$$\langle S(f), g \rangle = \langle f, S(g) \rangle \quad \text{for all } f, g \in \mathscr{D}_S.$$

(These definitions will suffice for our purposes; in more advanced work one needs to be more careful about specifying the domains $\mathscr{D}_S$ and $\mathscr{D}_T$.)

Now suppose $L$ is a second-order linear differential operator,

$$L(f) = rf'' + qf' + pf,$$

where $r$, $q$, and $p$ are real functions of class $C^{(2)}$ on $[a,b]$. We shall assume that the leading coefficient $r$ is nonvanishing on $[a,b]$, as the existence of "singular points" where $r = 0$ complicates the theory considerably. (Later we shall sometimes allow $r$ to vanish at one or both endpoints.) For the time being, we take the domain of $L$ to be the space of all twice continuously differentiable functions on $[a,b]$.

What is the adjoint of $L$? If we write out the integral defining $\langle L(f), g \rangle$, we can move the derivatives from $f$ onto $g$ by integration by parts, thus:

$$\int_a^b (rf'')\overline{g}\,dx = -\int_a^b f'(r\overline{g})'dx + rf'\overline{g}\Big|_a^b = \int_a^b f(r\overline{g})''dx + \Big[rf'\overline{g} - f(r\overline{g})'\Big]_a^b,$$
$$\int_a^b (qf')\overline{g}\,dx = -\int_a^b f(q\overline{g})'dx + qf\overline{g}\Big|_a^b.$$

We therefore have

$$\begin{aligned}\langle L(f), g \rangle &= \int_a^b (rf'' + qf' + pf)\overline{g}\,dx \\ &= \int_a^b f\Big[(r\overline{g})'' - (q\overline{g})' + p\overline{g}\Big]dx + \Big[rf'\overline{g} - f(r\overline{g})' + qf\overline{g}\Big]_a^b \qquad (3.24) \\ &= \langle f, L^*(g) \rangle + \Big[r(f'\overline{g} - f\overline{g}') + (q - r')f\overline{g}\Big]_a^b,\end{aligned}$$

where $L^*$ is the **formal adjoint** of $L$ defined by

$$L^*(g) = (rg)'' - (qg)' + pg = rg'' + (2r' - q)g' + (r'' - q' + p)g. \qquad (3.25)$$

(Here we have used the assumption that $r$, $q$, and $p$ are real.) We say that $L$ is **formally self-adjoint** if $L^* = L$. On comparing the coefficients of $L^*$ with $L$, we see that this happens precisely when $2r' - q = q$ and $r'' - q' = 0$, that is, when $q = r'$. In this case, $L$ has the form

$$L(f) = rf'' + r'f' + pf = (rf')' + pf, \qquad (3.26)$$

and moreover, the second boundary term at the end of (3.24) vanishes. We have therefore proved the following.

**Lagrange's Indentity.** *If $L$ is formally self-adjoint,*

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \Big[r(f'\overline{g} - f\overline{g}')\Big]_a^b. \tag{3.27}$$

Evidently the discrepancy between formal and actual self-adjointness lies in the endpoint terms in (3.27). They can be eliminated by restricting $L$ to a smaller domain, consisting of functions that satisfy suitable boundary conditions. More precisely, for a second-order operator $L$ it is usually appropriate to impose two independent boundary conditions of the form

$$\begin{aligned} B_1(f) &= \alpha_1 f(a) + \alpha_1' f'(a) + \beta_1 f(b) + \beta_1' f'(b) = 0, \\ B_2(f) &= \alpha_2 f(a) + \alpha_2' f'(a) + \beta_2 f(b) + \beta_2' f'(b) = 0, \end{aligned} \tag{3.28}$$

where the $\alpha$'s and $\beta$'s are constants. We say that the boundary conditions (3.28) are **self-adjoint** (relative to the operator $L$) if

$$\Big[r(f'\overline{g} - f\overline{g}')\Big]_a^b = 0 \quad \text{for all } f, g \text{ satisfying (3.28).}$$

Almost all the boundary conditions that arise in practice are of the form

$$\begin{aligned} &\alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0 \\ &\Big(\alpha, \alpha', \beta, \beta' \in \mathbf{R}; \quad (\alpha, \alpha') \neq (0,0); \quad (\beta, \beta') \neq (0,0)\Big). \end{aligned} \tag{3.29}$$

Boundary conditions of the form (3.29) are called **separated**, since each one involves a condition at only one endpoint. Separated boundary conditions are always self-adjoint (relative to any operator $L$). In fact, if $f$ and $g$ both satisfy the boundary condition at $a$,

$$\alpha f(a) + \alpha' f'(a) = 0, \qquad \alpha g(a) + \alpha' g'(a) = 0, \tag{3.30}$$

then the expression $r(f'\overline{g} - f\overline{g}')$ vanishes at $x = a$; likewise at $b$. This is obvious when $\alpha' = 0$, in which case (3.30) becomes $f(a) = g(a) = 0$; on the other hand, if $\alpha' \neq 0$, we can rewrite (3.30) as

$$f'(a) = cf(a), \qquad g'(a) = cg(a) \qquad (c = -\alpha/\alpha'),$$

so that

$$r(a)[f'(a)\overline{g(a)} - f(a)\overline{g'(a)}] = cr(a)[f(a)\overline{g(a)} - f(a)\overline{g(a)}] = 0.$$

There is also one set of nonseparated boundary conditions that is commonly used, namely, the **periodic** boundary conditions

$$f(a) = f(b), \qquad f'(a) = f'(b). \tag{3.31}$$

These are self-adjoint relative to $L$ provided that $r(a) = r(b)$, for then the endpoint evaluations at $a$ and $b$ in (3.27) cancel each other out.

Now we are ready to formulate the boundary value problems that lead to orthogonal bases for $L^2(a, b)$.

*Definition.* A **regular Sturm-Liouville problem** on the interval $[a, b]$ is specified by the following data:

(i) a formally self-adjoint differential operator $L$ defined by $L(f) = (rf')' + pf$, where $r$, $r'$, and $p$ are real and continuous on $[a, b]$ and $r > 0$ on $[a, b]$;

(ii) a set of self-adjoint boundary conditions, $B_1(f) = 0$ and $B_2(f) = 0$, for the operator $L$;

(iii) a positive, continuous function $w$ on $[a, b]$.

The object is *to find all solutions $f$ of the boundary value problem*

$$\begin{aligned} &L(f) + \lambda w f = 0, \quad \text{i.e.,} \ \left[r(x)f'(x)\right]' + p(x)f(x) + \lambda w(x)f(x) = 0, \\ &B_1(f) = B_2(f) = 0, \end{aligned} \tag{3.32}$$

*where $\lambda$ is an arbitrary constant.*

(A comment on condition (i): We have assumed from the outset that $r$ does not vanish on $[a, b]$, so either $r > 0$ or $r < 0$. If $r < 0$, we simply replace $r$, $p$, and $\lambda$ by $-r$, $-p$, and $-\lambda$, which leaves (3.32) unchanged.)

For most values of $\lambda$, the only solution of (3.32) is the trivial one, $f(x) \equiv 0$. If (3.32) has nontrivial solutions, $\lambda$ is called an **eigenvalue** for the Sturm-Liouville problem, and the corresponding nontrivial solutions are called **eigenfunctions**. (This usage of the term *eigenvalue* is somewhat specialized. $\lambda$ is an eigenvalue in the usual sense of the word, not of the operator $L$ but rather of the operator $M$ defined by $M(f) = -w^{-1}L(f)$.) If $f$ and $g$ satisfy (3.32), then so does any linear combination $c_1 f + c_2 g$ (this is just the superposition principle at work), so the set of all eigenfunctions for a given eigenvalue $\lambda$, together with the zero function, is a linear space called the **eigenspace** for $\lambda$.

We summarize the elementary properties of eigenvalues and eigenfunctions in the following theorem, which displays the importance of eigenfunctions from the point of view of orthogonal sets. We recall that if $w > 0$ is a weight function on $[a, b]$, the weighted inner product $\langle f, g\rangle_w$ is given by

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx = \langle wf, g\rangle = \langle f, wg\rangle. \tag{3.33}$$

**Theorem 3.9.** *Let a regular Sturm-Liouville problem (3.32) be given.*

*(a) All eigenvalues are real.*

*(b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $w$; that is, if $f$ and $g$ are eigenfunctions with eigenvalues $\lambda$ and $\mu$, $\lambda \neq \mu$, then*

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\,dx = 0.$$

*(c) The eigenspace for any eigenvalue $\lambda$ is at most 2-dimensional. If the boundary conditions are separated, it is always 1-dimensional.*

*Proof:* (a) If $\lambda$ is an eigenvalue, with eigenfunction $f$, then

$$\lambda\|f\|_w^2 = \langle\lambda wf, f\rangle = -\langle L(f), f\rangle = -\langle f, L(f)\rangle = \langle f, \lambda wf\rangle = \bar{\lambda}\langle f, wf\rangle = \bar{\lambda}\|f\|_w^2.$$

Here we have used (3.27) and (3.33) and the fact that $f$ satisfies self-adjoint boundary conditions. Since $\|f\|_w^2 > 0$, we conclude that $\bar{\lambda} = \lambda$, that is, $\lambda$ is real.

(b) Suppose $L(f) + \lambda wf = 0$ and $L(g) + \mu wg = 0$, where $f$ and $g$ are nonzero. We have just shown that $\lambda$ and $\mu$ must be real, and by the same sort of argument,

$$\lambda\langle f, g\rangle_w = \langle\lambda wf, g\rangle = -\langle L(f), g\rangle = -\langle f, L(g)\rangle = \langle f, \mu wg\rangle = \mu\langle f, g\rangle_w.$$

Thus, if $\lambda \neq \mu$ we must have $\langle f, g\rangle_w = 0$.

(c) The fundamental existence theorem for ordinary differential equations (see Appendix 5) says that for any constants $c_1$ and $c_2$ there is a unique solution of $L(f) + \lambda wf = 0$ satisfying the initial conditions $f(a) = c_1$, $f'(a) = c_2$. That is, a solution is specified by two arbitrary constants $c_1$ and $c_2$, so the space of *all* solutions of $L(f) + \lambda wf = 0$ is 2-dimensional. Hence the space of solutions satisfying the given boundary conditions is *at most* 2-dimensional. Moreover, if the boundary conditions are separated, one of them has the form $\alpha f(a) + \alpha' f'(a) = 0$. This imposes the linear relation $\alpha c_1 + \alpha' c_2 = 0$ on the constants $c_1$ and $c_2$ and hence reduces the dimension of the solution space to one. (Of course the other boundary condition will usually reduce the dimension to zero; this is why there are nontrivial solutions only for certain special values of $\lambda$.) ∎

At this point it is not evident that a given Sturm-Liouville problem has any eigenfunctions at all. But, in fact, there are as many as anyone could wish for.

**Theorem 3.10.** *For every regular Sturm-Liouville problem*

$$(rf')' + pf + \lambda wf = 0, \qquad B_1(f) = B_2(f) = 0$$

*on* $[a, b]$, *there is an an orthonormal basis* $\{\phi_n\}_1^\infty$ *of* $L_w^2(a, b)$ *consisting of eigenfunctions. If* $\lambda_n$ *is the eigenvalue for* $\phi_n$, *then* $\lim_{n\to\infty} \lambda_n = +\infty$. *Moreover, if* $f$ *is of class* $C^{(2)}$ *on* $[a, b]$ *and satisfies the boundary conditions* $B_1(f) = B_2(f) = 0$, *then the series* $\sum\langle f, \phi_n\rangle\phi_n$ *converges uniformly to* $f$.

In more detail, the content of Theorem 3.10 is as follows. By Theorem 3.9(c), for each eigenvalue $\lambda$ there are either one or two independent eigenfunctions. In the latter case we can choose the two eigenfunctions to be orthogonal to each other with respect to the weight $w$. (If $\langle f_1, f_2\rangle_w \neq 0$, we can replace $f_2$ by $\tilde{f}_2 = f_2 - cf_1$ where $c$ is chosen to make $\langle f_1, \tilde{f}_2\rangle = 0$.) If we put all these eigenfunctions together, by Theorem 3.9(b) we obtain an orthogonal set; and Theorem 3.10 says that this set is actually a basis. This implies, in particular, that the set of eigenvalues is countably infinite.

We shall take Theorem 3.10 on faith for the present, but we shall prove it in the case of separated boundary conditions in §10.3. A proof of the general case, as well as its generalization to higher-order differential equations, can be found in Naimark [40], Chapter II.

*Example.* Consider the problem

$$f'' + \lambda f = 0, \qquad f'(0) = \alpha f(0), \quad f'(l) = \beta f(l). \tag{3.34}$$

First let us dispose of the case $\lambda = 0$. The general solution of $f'' = 0$ is $f(x) = c_1 + c_2 x$. The boundary condition at 0 says that $c_2 = \alpha c_1$, and the boundary condition at $l$ says that $c_2 = \beta(c_1 + c_2 l)$. The only solution of this pair of equations is $c_1 = c_2 = 0$ unless $\beta = \alpha/(1 + l\alpha)$, in which case we may take $c_1 = 1$ and $c_2 = \alpha$.

Now for $\lambda \neq 0$, let us set $\lambda = \nu^2$, where $\nu$ is positive real or positive imaginary according as $\lambda > 0$ or $\lambda < 0$. (By Theorem 3.9(a), we need only consider real $\lambda$.) The general solution of the differential equation $f'' + \lambda f = 0$ is

$$f(x) = c_1 \cos \nu x + c_2 \sin \nu x \qquad (\lambda = \nu^2).$$

Since $f(0) = c_1$ and $f'(0) = \nu c_2$, the boundary condition at 0 says that $c_2 = (\alpha/\nu)c_1$. Since a constant multiple of a solution is a solution, we may choose $c_1 = \nu$, $c_2 = \alpha$, so that

$$f(x) = \nu \cos \nu x + \alpha \sin \nu x. \tag{3.35}$$

Now the boundary condition at $l$ says that

$$-\nu^2 \sin \nu l + \alpha\nu \cos \nu l = \beta(\nu \cos \nu l + \alpha \sin \nu l),$$

or

$$(\alpha - \beta)\nu \cos \nu l = (\alpha\beta + \nu^2) \sin \nu l,$$

or finally

$$\tan \nu l = \frac{(\alpha - \beta)\nu}{\alpha\beta + \nu^2}. \tag{3.36}$$

For the case of imaginary $\nu$ (i.e., $\lambda < 0$) we set $\nu = i\mu$ and use the fact that $\tan ix = i \tanh x$ to rewrite (3.36) as

$$\tanh \mu l = \frac{(\alpha - \beta)\mu}{\alpha\beta - \mu^2}. \tag{3.37}$$

In both cases we need only consider positive values of $\nu$ and $\mu$, since the actual eigenvalue is $\nu^2$ or $-\mu^2$.

If $\nu$ satisfies (3.36), then the function $f$ defined by (3.35) is an eigenfunction for the problem (3.34). In general it is not normalized, but finding the normalization is a simple matter of calculus, and the equation (3.36) can often be used to simplify the result. As an illustration, let us work out the case $\beta = -\alpha$. (Other cases are considered in Exercises 5 and 6.) If $f$ is given by (3.35), then

$$\begin{aligned}
\|f\|^2 &= \int_0^l (\nu^2 \cos^2 \nu x + 2\alpha\nu \sin \nu x \cos \nu x + \alpha^2 \sin^2 \nu x)\, dx \\
&= \left[ \tfrac{1}{2}\nu^2(x + \nu^{-1} \cos \nu x \sin \nu x) + \alpha \sin^2 \nu x + \tfrac{1}{2}\alpha^2(x - \nu^{-1} \cos \nu x \sin \nu x) \right]_0^l \\
&= \tfrac{1}{2}(\nu^2 + \alpha^2)l + \frac{(\nu^2 - \alpha^2)}{2\nu} \cos \nu l \sin \nu l + \alpha \sin^2 \nu l.
\end{aligned}$$

But if $\beta = -\alpha$, (3.36) gives

$$\frac{(\nu^2 - \alpha^2)}{2\nu} = \frac{\alpha}{\tan \nu l} = \frac{\alpha \cos \nu l}{\sin \nu l},$$

so

$$\|f\|^2 = \tfrac{1}{2}(\nu^2 + \alpha^2)l + \alpha(\cos^2 \nu l + \sin^2 \nu l) = \tfrac{1}{2}(\nu^2 + \alpha^2)l + \alpha. \tag{3.38}$$

There is no way to describe the values of $\nu$ and $\mu$ that solve the transcendental equations (3.36) and (3.37) in closed form (except when $\alpha = \beta$), but it is easy to find them graphically. Namely, they are the values at which the curves $y = \tan \nu l$ and $y = (\alpha - \beta)\nu/(\alpha\beta + \nu^2)$ in the $\nu y$-plane, or $y = \tanh \mu l$ and $y = (\alpha - \beta)\mu/(\alpha\beta - \mu^2)$ in the $\mu y$-plane, intersect. The relative configuration of these curves depends on $\alpha$ and $\beta$; we shall display a couple of representative cases here and let the reader work out some others as exercises.

*Case I.* $\alpha = 1$, $\beta = -1$, $l = \pi$. Here the situation is as depicted in Figure 3.5. There is an infinite sequence of positive solutions to (3.36), say $\nu_1 < \nu_2 < \cdots$, and $\nu_n$ is approximately $n - 1$ when $n$ is large. There are no positive solutions to (3.37). Hence, there is an infinite sequence of positive eigenvalues $\lambda_n = \nu_n^2$ for (3.34), with $\lambda_n \approx (n-1)^2$ for $n$ large, and no negative eigenvalues. (Zero is not an eigenvalue since $-1 \neq 1/(1+\pi)$.) The (unnormalized) eigenfunctions are given by (3.35):

$$f_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x.$$

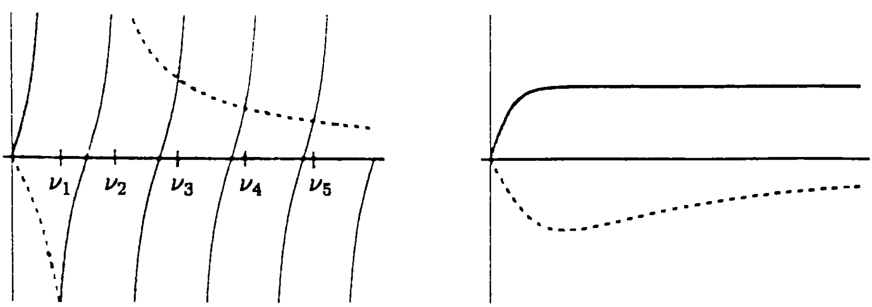



FIGURE 3.5. Left: the graphs of $\tan \pi\nu$ (solid) and $2\nu/(\nu^2 - 1)$ (dashed); the numbers $\nu_n$ are the values of $\nu$ at which the graphs intersect. Right: the graphs of $\tanh \pi\mu$ (solid) and $-2\mu/(\mu^2 + 1)$ (dashed).

*Case II.* $\alpha = 1$, $\beta = 4$, $l = \pi$. Here the situation is as depicted in Figure 3.6. Again there is an infinite sequence $\{\nu_n\}_1^\infty$ of positive solutions to (3.36), this time with $\nu_n \approx n$ for large $n$; and zero is not an eigenvalue of (3.34) since $4 \neq 1/(1+\pi)$. But now there is also one positive solution $\mu_0$ to (3.37). Hence, there is an infinite sequence of positive eigenvalues $\lambda_n = \nu_n^2$ for (3.34) and one negative eigenvalue $\lambda_0 = -\mu_0^2$. The (unnormalized) eigenfunction for $\lambda_n = \nu_n^2$ is

$$f_n(x) = \nu_n \cos \nu_n x + \sin \nu_n x,$$

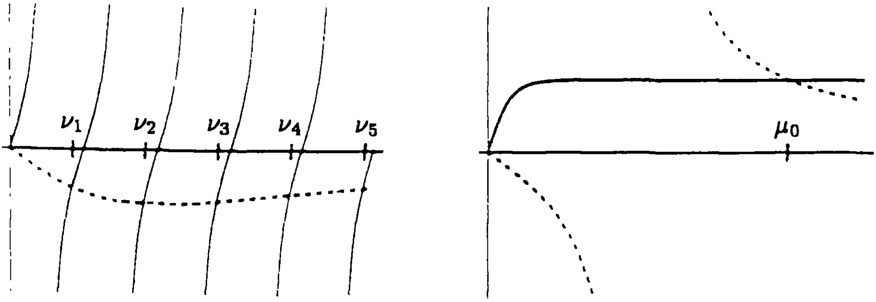



FIGURE 3.6. Left: the graphs of $\tan \pi\nu$ (solid) and $-3\nu/(\nu^2+4)$ (dashed). Right: the graphs of $\tanh \pi\mu$ (solid) and $3\mu/(\mu^2-4)$ (dashed). The numbers $\nu_n$ and $\mu_0$ are the values of $\nu$ and $\mu$ at which the graphs intersect.

and the eigenfunction for $\lambda_0 = -\mu_0^2$ is

$$f_0(x) = \mu_0 \cosh \mu_0 x + \sinh \mu_0 x.$$

## *EXERCISES*

1. Under what condition on the constants $c$ and $c'$ are the boundary conditions $f(b) = cf(a)$ and $f'(b) = c'f'(a)$ self-adjoint for the operator $L(f) = (rf')' + pf$ on $[a, b]$? (Assume as usual that $r$ and $p$ are real.)
2. Show that the problem (3.34) has no negative eigenvalues if $\alpha > 0 > \beta$ and exactly one negative eigenvalue if $\beta > \alpha > 0$ or $0 > \beta > \alpha$.
3. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f(0) = 0$, $f'(l) = 0$ on $[0, l]$. (Cf. Exercise 6, §3.3.)
4. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f'(0) = 0$, $f(l) = 0$ on $[0, l]$. (Cf. Exercise 7, §3.3.)
5. Find the normalized eigenfunctions for the problem (3.34) in the case $\alpha = 0$. (The answer is a bit different in the cases $\beta > 0$, $\beta = 0$, and $\beta < 0$.)
6. Find the normalized eigenfunctions for the problem (3.34) in the case $\beta = 0$. (Hint: The change of variable $x \to l - x$ essentially reduces this to Exercise 5.)
7. Find the eigenvalues and normalized eigenfunctions for the problem $f'' + \lambda f = 0$, $f(0) = 0$, $f'(1) = -f(1)$.
8. The Sturm-Liouville theory can be generalized to higher-order equations. As an example, consider the operator $L(f) = f^{(4)}$ on the interval $[0, l]$.
   a. Prove the analogue of Lagrange's identity for $L$:

$$\int_0^l \left[f^{(4)}(x)\overline{g}(x) - f(x)\overline{g}^{(4)}(x)\right]dx = \left[f'''\overline{g} - f\overline{g}''' + f'\overline{g}'' - f''\overline{g}'\right]_0^l. \quad (*)$$

   b. For the fourth-order equation $L(f) - \lambda f = 0$ one needs four boundary conditions involving $f$, $f'$, $f''$, and $f'''$. Such a set of boundary conditions is called self-adjoint for $L$ if the right side of $(*)$ vanishes whenever $f$ and $g$ both satisfy the conditions. Show that one obtains a self-adjoint set of boundary conditions by imposing any of the following pairs of conditions at $x = 0$ and any one of them at $x = l$:

$$f = f' = 0, \qquad f = f'' = 0, \qquad f' = f''' = 0, \qquad f'' = f''' = 0.$$

c. Show that the eigenvalues for the equation $L(f) - \lambda f = 0$, subject to any self-adjoint set of boundary conditions, are all real, and that eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(0, l)$.

d. One can show that the analogue of Theorem 3.10 holds here, i.e., there is an orthonormal basis of eigenfunctions. For example, consider the boundary conditions $f(0) = f''(0) = 0$, $f(l) = f''(l) = 0$. Show that $f_n(x) = \sin(n\pi x/l)$ is an eigenfunction. What is its eigenvalue? Why can you guarantee immediately that there are no other independent eigenfunctions?

9. Suppose $p$, $q$, and $r$ are real functions of class $C^{(2)}$ and that $r > 0$. The differential equation $rf'' + qf' + pf + \lambda f = 0$ can be written in the form $L(f) + \lambda w f = 0$ where $w$ is an arbitrary positive function and $L(f) = wrf'' + wqf' + wpf$. Show that $w$ can be always be chosen so that $L$ is formally self-adjoint.

The following two problems use the fact that the general solution of the Euler equation

$$x^2 f''(x) + axf'(x) + bf(x) = 0 \qquad (x > 0)$$

is $c_1 x^{r_1} + c_2 x^{r_2}$ where $r_1$ and $r_2$ are the zeros of the polynomial $r(r-1) + ar + b$. (If the two zeros coincide, the general solution is $c_1 x^{r_1} + c_2 x^{r_1} \log x$.) In case $r_1$ and $r_2$ are complex, it is useful to recall that $x^{is} = e^{is \log x}$.

10. Find the eigenvalues and normalized eigenfunctions for the problem

$$(xf')' + \lambda x^{-1} f = 0, \qquad f(1) = f(b) = 0 \qquad (b > 1).$$

Expand the function $g(x) = 1$ in terms of these eigenfunctions. (Hint: in computing integrals, make the substitution $y = \log x$. Orthonormality here is with respect to the weight $w(x) = x^{-1}$.)

11. Find the eigenvalues and normalized eigenfunctions for the problem

$$(x^2 f')' + \lambda f = 0, \qquad f(1) = f(b) = 0 \qquad (b > 1).$$

12. Consider the Sturm-Liouville problem

$$(rf')' + pf + \lambda f = 0, \qquad f(a) = f(b) = 0. \qquad (**)$$

a. Show that if $f$ satisfies $(**)$, then

$$\lambda \int_a^b |f|^2\,dx = \int_a^b r|f'|^2\,dx - \int_a^b p|f|^2\,dx.$$

(Hint: Use the fact that $\lambda f = -(rf')' - pf$ and integrate by parts.)

b. Deduce that if $p(x) \leq C$ for all $x$, then all the eigenvalues $\lambda$ of (**) satisfy $\lambda \geq -C$.

c. Show that the conclusion of (b) still holds if the boundary conditions $f(a) = f(b) = 0$ are replaced by $f'(a) - \alpha f(a) = f'(b) - \beta f(b) = 0$ where $\alpha \leq 0$ and $\beta \geq 0$. (Hint: The analogue of part (a) in this situation is

$$\lambda \int_a^b |f|^2\,dx = \int_a^b r|f'|^2\,dx - \int_a^b p|f|^2\,dx + \beta r(b)|f(b)|^2 - \alpha r(a)|f(a)|^2.)$$

## 3.6 Singular Sturm-Liouville problems

In §3.5 we considered the differential equation

$$rf'' + r'f' + pf + \lambda wf = 0 \tag{3.39}$$

on a closed, bounded interval $[a,b]$, in which $r$, $r'$, $p$, and $w$ were assumed continuous on $[a,b]$ and $r$ and $w$ were assumed strictly positive on $[a,b]$. However, it often turns out in practice that one or more of these assumptions must be weakened, leading to the so-called **singular Sturm-Liouville problems.** Specifically, we allow the following modifications of the basic setup:

(i) The leading coefficient $r$ may vanish at one or both endpoints of $[a,b]$. In addition, the weight $w$ may vanish or tend to infinity at one or both endpoints, and the function $|p|$ may tend to infinity at one or both endpoints.

(ii) The interval $[a,b]$ may be unbounded, that is, $a = -\infty$ and/or $b = \infty$.

There is an extensive theory of these more general boundary value problems, but it is beyond the scope of this book. (Complete treatments can be found in Dunford-Schwartz [18] and Naimark [40]; see also Titchmarsh [52].) We shall merely sketch a few of the main features here, and we shall discuss specific examples in Chapters 5 and 6 and Sections 7.4 and 10.4.

The first problem is to decide what sort of boundary conditions to impose. Since we wish to use the machinery of inner products and orthogonality, we wish to use only solutions of (3.39) that are square-integrable. Now, in the regular case, all solutions of (3.39) are continuous on $[a,b]$ and hence belong to $L^2_w(a,b)$. However, under condition (i), the solutions to (3.39) may fail to be square-integrable because they blow up at one or both endpoints; whereas under condition (ii), solutions may fail to be square-integrable because they do not decay at infinity. Thus, we distinguish two cases concerning the behavior of solutions at each endpoint; to be definite, we consider the endpoint $a$.

*Case I.* All solutions of (3.39) belong to $L^2_w(a,c)$ for $a < c < b$. (It turns out that if this condition is satisfied for one value of $\lambda$, then it is satisfied for all values of $\lambda$.) In this case, we impose a boundary condition at $a$. In some cases it may be of the form $\alpha f(a) + \alpha' f'(a) = 0$, as before, but it may also be a condition on the limiting behavior of $f$ and $f'$ at $a$ — for example, the condition that $f(x)$ should remain bounded as $x \to a$.

*Case II.* Not all solutions of (3.39) belong to $L^2_w(a,c)$. In this case we impose no boundary condition at $a$ beyond the one that automatically comes with the problem, namely, that the solution should belong to $L^2_w(a,b)$.

In any event, we require the boundary conditions to be self-adjoint, i.e., if $f$ and $g$ satisfy the boundary conditions then the boundary term in Lagrange's identity should vanish. Precisely, since $f$ and $g$ may have singularities at $a$ and $b$, or $a$ and/or $b$ may be infinite, this requirement should be formulated as

$$\lim_{\delta,\epsilon\to 0}\Big[r(f'\overline{g} - f\overline{g}')\Big]_{a+\delta}^{b-\epsilon} = 0. \tag{3.40}$$

(3.40) implies that

$$\langle L(f), g\rangle = \langle f, L(g)\rangle \quad \text{where} \quad L(f) = (rf')' + pf,$$

for any smooth functions $f$ and $g$ that satisfy the boundary conditions, and once this equation is established, the proof of Theorem 3.9 goes through without change. Therefore, the eigenvalues are all real and the eigenfunctions with distinct eigenvalues are orthogonal to each other.

However, the situation with Theorem 3.10 is different: in general, *there is no guarantee that there will be enough eigenfunctions to make an orthonormal basis.* Sometimes there are, sometimes there aren't. In the latter case, it is still possible to expand arbitrary functions in $L^2_w(a,b)$ in terms of solutions of the differential equation (3.39) that satisfy the given boundary conditions, but the expansion will involve an integral rather than (or in addition to) an infinite series.

For example, consider the differential equation

$$f'' + \lambda f = 0 \quad \text{on} \quad (-\infty,\infty).$$

The general solution is

$$c_1\cos\nu x + c_2\sin\nu x \quad \text{or} \quad c_1e^{i\nu x} + c_2e^{-i\nu x} \quad (\lambda = \nu^2).$$

None of these functions, for any value of $\lambda$, belongs to $L^2(-\infty,\infty)$, except for the trivial case $c_1 = c_2 = 0$. However, any $f \in L^2(-\infty,\infty)$ can be written as a "continuous superposition" (i.e., integral) of the functions $e^{i\nu x}$ as $\nu$ ranges over all real numbers, by means of the Fourier transform. This is the subject of Chapter 7.