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EQUIVARIANT BIFURCATIONS IN A NONLOCAL EVOLUTION MODEL

Abstract. In this work we study the bifurcations from the trivial equilibrium of the equation

$$\frac{\partial u}{\partial t}(x,t) = -u(x,t) + \tanh(\lambda(J * u)(x,t)),$$

in the space of 2τ periodic functions. This is accomplished with the help of the *equivariant branching lemma*, which allows us to take into account the simmetries present in the model. We show that the phenomenon of 'spontaneous simmetry-breaking' occurs here, that is, the bifurcating solutions are less simmetric than the trivial one.

1. Introduction

We consider here the non local evolution equation

$$\frac{\partial u(r,t)}{\partial t} = -u(r,t) + \tanh\left(\lambda J * u(r,t) + \beta h\right), \tag{1}$$

where $u(r,t)$ is a real function on $\mathbb{R} \times \mathbb{R}_+$, h, β are non negative constants and $J \in C^1(\mathbb{R})$ is a non negative even function with compact support and integral equal to 1. The $*$ above denotes convolution product, namely:

$$(J * m)(x) = \int_{\mathbb{R}} J(x-y)m(y)dy.$$

Nonlocal convolution models appear in the modelling of many phenomena, including population dynamics ([17], [13], [14]) and neuronal activity ([2], [4], [7], [10], [18], [19], [30] and [37]

In particular, equation 1 arises as a continuum limit of one-dimensional Ising spin systems with Glauber dynamics and Kac potential (see [24] and references therein); u

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represents then a magnetization density and β^{-1} the temperature of the system. Though 'local', e.g. reaction-diffusion, differential equations have also been used in this context, equation (1) seems to be the 'right' one to use, especially if one is interested in phase transitions and metastability properties (see).

It is not difficult to obtain well posedness of the problem (1) in various function spaces since the right-hand-side of (1) usually defines a global Lipschitz map. On the other hand, the investigation of qualitative properties of the associated flow seems to be harder. To begin with, the equilibria are given by the solutions of a nonlinear integral equation for which many methods used to analyze, for example, the boundary value problems that appear in the case of semilinear parabolic problems are not available. Furthermore, as it is shown in [11], nonlocality can give rise to complicated dhynamics even in the case of scalar parabolic equations.

In the last years several works dedicated to the analysis of (1) appeared in the literature. In [21] and [27], the existence and uniqueness (modulo translations) of a travelling front connecting the equilibria m_β^- and m_β^+ is proved. In the case $h = 0$ the existence of a 'standing' wave as well as its stability properties are analysed in [22] and [23]. In this case, many equilibria periodic in x also exist, as shown in [1] and [3]. The existence of a non-homogeneous stationary solution referred to as the 'bump' or 'critical droplet' in the literature, was proved in [25] for h 'sufficiently close' to 1. Another proof, which is simpler and does not require the above restriction in h is given in [28]. In the same work, the existence of a global compact attractor in $L^2(\mathbb{R})$ with a convenient weighted measure is obtained.

We considerer here the equation (1) restricted to the subspace $\mathbb{P}_{2\tau}$ of 2τ periodic functions

(with support of J contained in $[-\tau, \tau]$). The existence and continuity of global attractors with respect to parameters was proved for a slightly more general class of equations in [31] and [32].

Our aim here is to investigate the (families of) solutions bifurcating from the trivial solution of 1 (or, more precisely, its restriction to $\mathbb{P}_{2\tau}$). This equation has a natural $\mathbb{O}(2)$ equivariance. As it is well-known, after the work of many workers in the late seventies, this may have profound consequences on the bifurcation of its equilibria as explained in the now classical book of Golubstiky at all [12]. In particular, the trivial solution has full $\mathbb{O}(2)$-symmetry, but as we shall see, the bifurcating solutions are less symmetric, that is, they are symmetric under a proper subgroup of $\mathbb{O}(2)$, a phenomenon known as *spontaneous symmetry breaking* in the literature. Similar situations, for the case of semilinear elliptic equations, have been discussed by many authors, among which we cite [34], [6], [36] and [20]. The main tool used is the Equivariant Branching Lemma proved independently in [35] and [9] in a version given in [8]. This result may be seen as an equivariant version of the famous Crandall-Rabinowitz theorem on the bifurcation, giving sufficient conditions for bifurcation to occur in the presence of symmetries and information on the type of symmetry enjoyed by the bifurcating solutions.

Falar agora da bifurcao global que usa tecnicas essencialmente diferentes do s teoremas de Sturm-Liouville. The main idea is the preservation of symmetry type along each branch. VEr artigo Smoller.

This paper is organised as follows.

2. Preliminaries

2.1. **Some concepts from group representation theory.** We first recall some definitions from the representation theory of compact groups. We follow the exposition in [8] with some adaptations (see also [16]).

Definition 2.1. *Let G be a topological group. A* representation of G in a Banach space *X is a group homomorphism $\Gamma : G \to GL(X)$, where $GL(X)$ is the group (under composition) of invertible continuous linear operators in X.*

Remark 2.2. *Alternatively, the map*

$$\begin{array}{rcl} (g,x) & \to & \Gamma_g(x) \\ G\times X & \to & X \end{array}$$

is called a (linear) action of G in X and $\Gamma_g(x)$ is then denoted by $g\cdot x$ or simply gx.

We say that the representation Γ is *(strongly) continuous* if $\lim_{g\to e}\Gamma_g\xi = \xi$ for any $\xi \in X$. (Here and in the sequel, we often use the notation Γ_g for the image of g under $Gamma$).

If X is a complex (resp. real) Hilbert space the representation Γ is called *unitary* (resp. *orthogonal*) if Γ_g, is an unitary (resp. orthogonal) operator, for any $g \in G$. A closed subspace $Y \subset X$ is *invariant for* Γ if $\Gamma_g Y \subset Y$ for any $g \in G$. In this case the representation of G in Y defined by restriction is called a *subrepresentation* of Γ. The representation Γ is *irreducible* if it admits no nontrivial subrepresentation.

Suppose now that $\mathbb{F} : X \to X$ is a $\mathbb{C}^k$, $k \geq 1$ map.

Definition 2.3. *We say that $\mathbb{F}$ is* equivariant *with respecto to the action Γ if $\mathbb{F}(\Gamma_g x) = \Gamma_g mathbbF(x)$, for every g in G and every $x \in X$.*

Given a subgroup Σ of G, we may consider the points in X which are fixed by the action, restricted to Σ. Reciprocally, if x is a point in X we may consider the elements of G which fix x. More precisely, we introduce the following concepts.

Definition 2.4. *Let Σ be a closed subgroup of G. We say that a point $x \in X$ is fixed under Σ, with respect to the representation Γ, if $(\Gamma_g x) = x$, for every g in Σ. The subspace of X consisting of all such points is denoted by $Fix_\Gamma\Sigma$. (We will often write simply $Fix\Sigma$ when no misunderstandig seems likely).*

Definition 2.5. *Let x be a point in X. The largest subgroup of G which fixes x, that is is the isotropy subgroup (or stabilizer) of x, which we denote by Stab(x).*

Definition 2.6. *If H is a subgroup of G, we define its normalizer by*

$$N(H) = \{g \in G | gHg^{-1} = H\} \ .$$

2.2. **Lyapunov-Schmidt decomposition and equivariance.** The Lyapunov-Schmidt decomposition is a procedure used to reduce an equation to as few variables as possible, after 'solving' the part that can be dealt with by the Implicit Function Theorem. The symmetries presented in the equation can be preserved int the reduced equation, called the 'bifurcation equation'. We refer to [12] or [35] for details and s.

Definition 2.7. *Let X and Y be Banach spaces. A bounded linear operator $L : X \to Y$ is called a* Fredholm operator *if its kernell $K(L)$ is finite-dimensional and its range $R(L)$ has finite codimension (and is therefore closed). The* index *of L is then the integer $ind(L) = \ker(L) - codim R(L)$.*

The following result is the main information we need about Fredholm operators.

Proposition 2.8. *If $L : X \to Y$ is a Fredholm operator, then there exist closed subspaces M and N of X and Y respectively such that $X = \ker(L) \oplus M$ and $Y = N \oplus R(L)$.*

Suppose now that Λ, X and Y are Banach spaces with $X \subset Y$ and $F : M \times X \to Y$ is a $\mathbb{C}^k, k \geq 1$ map. We write $F(\lambda, u), u \in X$ and $\lambda \in \Lambda$. We may suppose, after a change of coordinates, that $F(0,0) = 0$. We want to solve the equation $F(\lambda, u) = 0$ in a neighborhood of t $(0, 0 \in X \times \Lambda$. Suppose that $L := D_u F(0,0)$ is a Fredholm operator. We can then define continuous projections P on X onto $Ker(L)$, with $M = \ker(P)$ and Q on Y onto $R(L)$, with $N = \ker(P)$. The equation to be solved is then equivalent to the system

$$\begin{aligned} QF(v+w,\lambda) &= 0 \\ (I-Q)F(v+w,\lambda) &= 0, \end{aligned} \tag{2}$$

where we have set $v = Pu$ and $w = (I - P)u$.

Let $G : \Lambda \times \ker(L) \times M \to R(L)$ be defined by

$$G(\lambda, v, w) = Q(F(X, v + w)) \ .$$

Then $D_w G(0,0,0).w = L.w$ is an isomorphism from M into $R(L)$. By the Implicit Function Theorem, the first equation can be solved for w, that is, there exists a $\mathbb{C}^k$ function W and neighboorhoods of the origin in $\Lambda \times \ker(L)$ and M on which $QF(v + w, \lambda) = 0$ if and only if $w = W(\lambda, v)$.

The problem reduces than to solving the second equation, which now has (locally) the form

$$\varphi(\lambda, v) = (I - Q)(F(\lambda, v + W(\lambda, v)) \ , \tag{3}$$

where φ is a $\mathbb{C}^k$ map from (a neighboorhood of the origin in) $\Lambda \times \ker(L)$ into N.

It is often convenient to write equation (3) 'in coordinates'. For this, let $\{v_1, v_2, \ldots, v_m\}$ be a basis of $\ker(L)$, $\{u_1, u_2, \cdots, u_m\}$ a basis of N and $\{v_1^*, v_2^*, \ldots, v_m^*\}$ its dual basis and define

$\psi_i(\lambda, (x_1, x_2, \ldots, x_m)) = \langle v_i^*, \varphi\left(\lambda, \sum_{j=1}^m x_j v_j\right)\rangle$, for $i \in \{1, \ldots, m\}$.

The bifurcation equation becomes

$$\psi_i(\lambda, (x_1, x_2, \ldots, x_m)) = 0, \quad \psi = (\psi_1, \ldots, \psi_m). \tag{4}$$

We will need to compute derivatives of the bifurcation function. Write $\tilde{\varphi}(\lambda, x_1, x_2, \ldots, x_m) := \varphi\left(\lambda, \sum_{j=1}^m x_j v_j\right)$. Using that

$$Q\left(F\left(\lambda, \sum_{j=1}^m x_j v_j + W\left(\lambda, \sum_{j=1}^m x_j v_j\right)\right)\right) = 0 \tag{5}$$

and $W(0,0) = 0$ we obtain, after a straighforward, though somewhat lenghty computation:

- $\dfrac{\partial \tilde{\varphi}}{\partial x_i}(0,0) = 0$
- $\dfrac{\partial^2 \tilde{\varphi}}{\partial x_i \partial x_j}(0,0) = (I - Q)(D_u^2 F(0,0).(v_i, v_j));$
- $$\begin{aligned} \frac{\partial^3 \tilde{\varphi}}{\partial x_i \partial x_j \partial x_l}(0,0) &= (I-Q)(L(D_v^2 W(0,0)(v_i, v_j, v_l))) + \\ &+ (I-Q)(D_u^2 F(0,0)(v_i, D_v^2 W(0,0).(v_j, v_l))) + \\ &+ (I-Q)(D_u^2 F(0,0)(v_j, D_v^2 W(0,0).(v_i, v_l))) + \\ &+ (I-Q)(D_u^2 F(0,0))(v_l, D_v^2 W(0,0).(v_i, v_j))) + \\ &+ (I-Q)(D_u^3 F(0,0).(v_i, v_j, v_l)); \end{aligned}$$
- $\dfrac{\partial \tilde{\varphi}}{\partial \lambda}(0,0) = (I - Q)(D_\lambda F(0,0) + D_u F(0,0).(d_\lambda W(0,0)));$
- $\dfrac{\partial^2 \tilde{\varphi}}{\partial x_i \partial \lambda}(0,0) = (I - Q)(D_u^2 F(0,0).(v_i, D_\lambda W(0,0)) + D_{u\lambda}^2 F(0,0).(v_i)).$

Using again (5), $W(0,0) = 0$ and $D_v W(0,0) = 0$, we obtain

$$D_v^2 W(0,0).(v_i, v_j) = -L^{-1}(Q(D_u^2 F(0,0).(v_i v_j)))$$

and

$$D_\lambda W(0,0) = -L^{-1}(Q(D_\lambda F(0,0))) \ .$$

Therefore

- $\dfrac{\partial \psi_i}{\partial x_i}(0,0) = \left\langle v_i^*, \dfrac{\partial \tilde{\varphi}}{\partial x_i}(0,0) \right\rangle = 0 \ ;$

- $\dfrac{\partial^2\psi_i}{\partial x_i\partial x_j}(0,0) = \left\langle v_i^*, \dfrac{\partial^2\tilde{\varphi}}{\partial x_i\partial x_j}(0,0)\right\rangle = \left\langle v_i^*, D_u^2F(0,0).(v_i,v_j)\right\rangle$;
- $\dfrac{\partial^2\psi_i}{\partial x_j\partial x_l}(0,0) = \left\langle v_i^*, D_u^2F(0,0).(v_j,v_l)\right\rangle$;
- $\dfrac{\partial^2\psi_i}{\partial x_i\partial x_l}(0,0) = \langle v_i^*, D_u^2F(0,0).(v_i,v_l)\rangle$;
- $\dfrac{\partial^3\psi_i}{\partial x_i\partial x_j\partial x_l}(0,0) = \left\langle v_i^*, \dfrac{\partial^3\tilde{\varphi}}{\partial x_i\partial x_j\partial x_l}(0,0)\right\rangle = \langle v_i^*, Vangle$, where

$$\begin{aligned} V &= D_u^3F(0,0).(v_i,v_j,v_l) + D_u^2F(0,0).(v_i, D_u^2W(0,0).(v_j,v_l) \\ &\quad + D_u^2F(0,0).(v_j, D_u^2W(0,0).(v_i,v_l)) \\ &\quad + D_u^2F(0,0).(v_l, D_u^2W(0,0).(v_i,v_j)) \ ; \end{aligned}$$

- $\dfrac{\partial\psi_i}{\partial\lambda}(0,0) = \left\langle v_i^*, \dfrac{\partial\tilde{\varphi}}{\partial\lambda}(0,0)\right\rangle = \langle v_i^*, D_\lambda F(0,0)\rangle$ e
-

$$\begin{aligned} \frac{\partial\psi_i}{\partial x_j\partial\lambda}(0,0) &= \left\langle v_i^*, \frac{\partial^2\tilde{\varphi}}{\partial x_j\partial\lambda}(0,0)\right\rangle \\ &= \langle v_i^*, D^2_{u_\lambda}(0,0).v_j\rangle + \\ &+ \langle v_i^*, {D_u}^2F(0,0).(v_j, -L^{-1}(Q(D_{\lambda_s}F(0,0))))\rangle. \end{aligned}$$

Suppose we now make the additional assumption that the operator $F(\lambda,\cdot) : X \to X$ is equivariant with respect to a representation Γ in X. Then, if the projections P and Q are also chosen to be Γ-equivariant, the equivariant property propagates to the bifurcation equation. This is easily proved using the uniquennes of solutions in the first equation of (2) and equivariance of the projections (see [8]).

2.3. **The equivariant branching lemma.** Suppose the equivariant Lyapunov-Schmidt decomposition has been applied to the equation $F(\lambda,u) = 0$, where $F(\lambda,\cdot) : X \to X$ is a $\mathbb{C}^k$ map which is equivariant with respect to a representation Γ in X. As explained above, we may then suppose that the function $\varphi(\lambda,\cdot)$ appearing in the bifurcation equation

$$\varphi(\lambda,u) = 0$$

is a Γ equivariant map.

We can now state the 'Equivariant Branching Lemma' (see [8]).

Theorem 2.9. *Suppose Γ is a continuous representation of a compact group G in the Banach space X and $F : \Lambda\times X \to x$ is a $\mathbb{C}^k$ map , $k > 1$, with respect to a representation Γ in X. Suppose also that*

(i) $F(0,0) = 0$,

(ii) $L = D_uF(0,0)$. *has zero as an isolated eigenvalue with finite multiplicity. Then, for each isotropy subgroup* Σ *of* G *with respecto to* Γ*, such that* $\dim\ Fix(\Sigma) = 1$ *in* $\ker(L)$*, either one of the following situations occur (where* $g(\lambda, u) = 0$ *denotes the bifurcation equation in* $Fix(\Sigma) \cap \ker(L)$*:*

(I) *If* $\Sigma = G$ *and* $g_\lambda(0,0) \neq 0$*, there exists one branch of solutions* $u(\lambda)$*. If, in addition ,* $g''_{uu}(0,0) \neq 0$*, then* $u^2 = O(\|\lambda\|)$ *('saddle-node' bifurcation);*

(II) *If* $\Sigma < G$ *and the normalizer* $N(\Sigma)$ *of* Σ *in* G *acts trivially in* $Fix(\Sigma) \cap \ker(L)$ *then* $g(\lambda, u) = uh(\lambda, u)$*. If, in addition,* $g_{u\lambda}(0,0) \neq 0$*, then there exists a branch of solutions node* $u(\lambda)$*. If, furthermore,* $g''_{uu}(0,0) \neq 0$*, then* $u = O(\|\lambda\|)$ *('transcritical' bifurcation);*

(III) *If* $\Sigma < G$ *and* $N(\Sigma)$ *of* Σ *acts as* -1 *in* $Fix(\Sigma) \cap \ker(L)$ *then* $g(\lambda, u) = uh(\lambda, u)$*, with* h *and even function of* u*. If, in addition,* $g_{u\lambda}(0,0) \neq 0$*, then there exist two branchs of solutions* $\pm u(\lambda)$*. If, furthermore,* $g_{uuu}(0,0) \neq 0$*, then* $u = O(\|\lambda\|)$ *('pitchfork' bifurcation).*

3. Local bifurcations from the trivial solutions

We observe first that the subspace $\mathbb{P}_{2\tau}$ of of (spatially) periodic functions is invariant for the flow of (1) and u is a 2τ periodic equilibrium of (1) if and only if $v(y,t) = u(\frac{\tau}{\pi}, t)$ is a 2π periodic equilibrium of the problem

$$\frac{\partial v(y,t)}{\partial t} = -v(y,t) + \tanh\left(\lambda \tilde{J} * v(y,t)\right),$$

where $\tilde{J}(\xi) = \frac{\tau}{\pi} J(\frac{\tau}{\pi}\xi)$ is still a non negative even function supported in the interval $[-\pi, \pi]$ with integral equal to 1.

We therefore may and, in order to simplify the notation, will suppose that $\tau = \pi$.

Consider the space $\mathbb{P}_{2\pi}$ of 2π of (measurable) functions of period 2π. and let

$$\mathbb{L}^2_{per} := \{u \in \mathbb{P}_{2\pi} \mid \int_{-\pi}^{\pi} u^2\ dx < \infty, \}$$

endowed with the norm $||u|| := (\int_{-\pi}^{\pi} u^2\ dx)^{1/2}$. It is clear that $\mathbb{L}^2_{per}$ is isometric to $\mathbb{L}^2(S^1)$ but the former setting is more convenient for our purposes. It is easy to prove that the function $F : \mathbb{L}^2_{per} \to \mathbb{L}^2_{per}$ defined by $F(u) = -u + \tanh(\lambda J * u)$ is globally Lipschitzian continuous and, therefore, the Cauchy problem for (1) is well-posed in $\mathbb{L}^2_{per}$ (see [3]).

In order to prove our main result of this section, we will need some auxiliary results.

Consider now the operator

$$\begin{aligned} T : \mathbb{L}^2_{per} &\to \mathbb{L}^2_{per} \\ u &\to J * u \end{aligned} \tag{6}$$

We then have the following result:

Lemma 3.1. *T is a compact self-adjoint operator. Furthermore, the following assertions are equivalent.*

(a) *$\nu \in \mathbb{R}$ is an eigenvalue of T;*
(b) *There exists $n \in \mathbb{N}$ such that $\nu = \widehat{J(n)}$, where*

$$\widehat{J(n)} = \int_{\pi}^{\pi} \tilde{J}(x) \cos(nx)\, dx$$

is the n-th Fourier coefficient of the J multiplied by 2π.

In this case, if $\widehat{J(m)} \neq \widehat{J(m)}$ for for all $m \in \mathbb{N}$, $m \neq n$, then the corresponding eigenspace is generated by $u_n(x) = \cos nx$ and $v_n(x) = \sin nx$.

Proof. The compactness of T is a consequence of the Sobolev imbedding theorem and the selfadjointeness follows from the eveness of J and properties of the convolution operator.

The assertions concerning the eigenvalues and eigenfunctions of T are obtained by taking Fourier tranformation on both sides of the equality $J * u = \nu u$.

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Remark 3.2. *Suppose the eigenvalues of T are enumerated in decreasing order, that is $\lambda_0 \geq \lambda_1, \cdots, \lambda_n, \cdots$ From the properties of j it follows that $\lambda_0 = 1$ and $|\lambda_n| < \lambda_0$. Also, from the Riemmann-Lebesgue lemma, $\lambda_n \to 0$ as $n \to \infty$. Without further properties on J it is not true, in general, that $\lambda_n = J(n)$ nor that $\lambda_n > 0$. This last property, however, holds true if we suppose that $J(x)$ is decreasing for positive x (is it?), a property that will be assumed in the sequel.*

We now define a representation of the orthogonal group $G = O(2)$ in $\mathbb{L}^2_{per}$. G can be identified with the group of symmetries of $S^1 := \frac{\mathbb{R}}{2\pi\mathbb{Z}}$. generated by the translations $g_\theta(x) = x + \theta \pmod{2\pi}$ and the 'reflexion' $g_s(x) = -x \pmod{2\pi}$. We define $\Gamma : G \to GL(\mathbb{L}^2_{per})$ by giving its value at the generators.

$$\begin{aligned} \Gamma_{g_\theta} u(x) &= u(x - \theta) \\ \Gamma_{g_s} u(x) &= u(-x) \end{aligned} \tag{7}$$

Lemma 3.3. *The transformation Γ defined by 7 above can be extended to a strongly continous orthogonal representation of G in $\mathbb{L}^2_{per}$.*

The next result expresses precisely the simmetry properties of (1).

Lemma 3.4. *The function given by the right-hand-side of equation (1) that is, $F_\lambda : \mathbb{L}^2_{per} \to \mathbb{L}^2_{per}$ defined by $F_\lambda(u) = -u + \tanh(\lambda J * u)$ is equivariant with respect to the representation Γ of lemma 3.3 (for any value of the parameter λ).*

We now want to study the bifurcation of equilibria of 1 around the 'trivial' curve of equilibria $(\lambda, 0)$. The linearization of $F(\lambda, u) = -u + \tanh(\lambda J * u)$ at $(\bar{\lambda}, 0)$ is given by $D_u F(\lambda, 0) = -I + \bar{\lambda} T$. From theorem 3.1 it follows that this is a Fredholm operator of index 0 and therefore, an isomorphism if 0 is not one of its eigenvalues. Thus, the only possible bifurcation points are of the form $(\mu_n, 0)$ where $\mu_n = (\lambda_n)^{-1}$, and λ_n is one of the eigenvalues of T. By lema (3.1), $\frac{\lambda_n}{2\pi}$ must be one of the Fourier coefficients of J, and $u_n(x) = \cos nx$, $v_n(x) = \sin nx$ form a basis of $Ker(D_uF(\mu_n, 0))$.

Lemma 3.5. *Let $P : \mathbb{L}^2_{per} \to \ker(D_uF(\mu_n, 0))$ be the orthogonal projection on $Ker(D_uF(\mu_n, 0))$. Then P is equivariant with respect to the representation Γ defined by (7).*

Proof. Let $L := D_uF(\mu_n, 0)$. Since F is Γ- equivariant, by lema 3.4, so is L. Therefore
$\Gamma_g(Ker(D_uF(\mu_n, 0))) \subset Ker(D_uF(\mu_n, 0)))$, for any $g \in G$. Since Γ is orthogonal, it also follows that $\Gamma_g(Ker(D_uF(\mu_n, 0))^{\perp}) \subset Ker(D_uF(\mu_n, 0))^{\perp}$.

Therefore, for any $g \in G$ and $u \in \mathbb{L}^2_{per}$, we have

$$\begin{aligned} P(\Gamma_g u) &= P(\Gamma_g(P(u) + (u - P(u)))) \\ &= P(\Gamma_g(P(u))) + P(\Gamma_g(u - P(u))) \\ &= \Gamma_g(P(u)). \end{aligned}$$

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Remark 3.6. *Since $L := D_uF(\mu_n, 0)$ is self-adjoint, it follows that $R(L) = \ker(L)^{\perp}$ and $Q = (I - P)$ is also Γ equivariant with range $R(L)$. Thus, as observed at the end of section 2.2, in the bifurcation equation, the function*

$$\varphi : \mathbb{R} \times V \to \ker(L)$$

where V is a neighborhood of the origin, is also Γ equivariant.

In order to apply the equivariant lemma, we need some information on the representation Γ.

Lemma 3.7. *Let μ_n, u_n and v_n be the eigenfunctions of the operator T given by lemma 3.1, λ_n the corresponding eigenvalue, and $\mu_n = (\lambda_n)^{-1}$. Then, we have*

(i) $\mathrm{S}tab(u_n) = \{g_\theta \mid \theta = \frac{2k\pi}{n}\ k \in \mathbb{Z}\} \cup \{g_s \circ g_\theta \mid \theta = \frac{2k\pi}{n}\ k \in \mathbb{Z}\}$ *and*
$\mathrm{S}tab(v_n) = \{g_\theta \mid \theta = \frac{2k\pi}{n}\ k \in \mathbb{Z}\} \cup \{g_s \circ g_\theta \mid \theta = \frac{(2k+1)\pi}{n}\ k \in \mathbb{Z}\}$;

(ii) $\mathrm{N}(\mathrm{S}tab(u_n)) = \{g_\theta \mid \theta = \frac{k\pi}{n}\ k \in \mathbb{Z}\} \cup \{g_s \circ g_\theta \mid \theta = \frac{k\pi}{n}\ k \in \mathbb{Z}\}$ *and*
$\mathrm{N}(\mathrm{S}tab(v_n))\{g_\theta \mid \theta = \frac{k\pi}{n}\ k \in \mathbb{Z}\} \cup \{g_s \circ g_\theta | \theta = \frac{k\pi}{n}\ k \in \mathbb{Z}\}$;

(iii) $\mathrm{F}ix(\mathrm{S}tab(u_n)) \cap \ker(D_uf((\mu_n, 0)) = [u_n]$ *and*
$\mathrm{F}ix(\mathrm{S}tab(v_n)) \cap \ker(D_uf(\mu_n, 0)) = [v_n]$;

(iv) *The normalizers $N(\mathrm{S}tab(u_n))$ and $N(\mathrm{S}tab(v_n))$ act as $-\mathbf{1}$ in $\mathrm{F}ix(\mathrm{S}tab(u_n)) \cap \ker(D_uf((\mu_n, 0))$ and $\mathrm{F}ix(\mathrm{S}tab(v_n)) \cap \ker(D_uf(\mu_n, 0))$ respectively.*

Proof. We prove only the assertions about u_n, since the arguments for v_n are very similar. Let $g \in \mathrm{S}tab(u_n)$. If $g = g_\theta$ for some θ, then $\cos(n(x + \theta)) = \cos(nx)$ for any $x \in \mathbb{R}$ which happens if, and only if $\theta = \frac{2k\pi}{n}$. If $g = g_s \circ g_\theta$, then $\cos(n(x + \theta)) = \cos(-n(x + \theta) = \cos(nx)$ and the same conclusion above holds for θ. This proves (i).

To prove (ii), we first observe that, since $H = Stab(u_n)$ is finite

$$N(H) = \{g \in G \mid gHg^{-1} = H\} = \{g \in G \mid gHg^{-1} \subset H\}.$$

Also, for any θ, τ in S^1, $g_\tau \circ g_\theta = g_\theta \circ g_\tau$ and $g_s \circ g_\theta = g_{-\theta} \circ g_s$.

We analyze two cases for $g \in G$: $g = g_\theta$ and $g = g_s g_\theta$. If $g_\theta H g_{-\theta} \subset H$, then for any $g_s g_{-\tau} \in H$, we must have $g_\theta g_s g_\tau g_{-\theta} = g_s g_{\tau-2\theta} \in H$, which happens if and only if $\theta = \frac{k\pi}{n}$ for some $k \in \mathbb{Z}$. Reciprocally, if $\theta = \frac{k\pi}{n}$ then, for any $g_s g_\tau \in H$, we have $g_\theta g_s g_\tau g_{-\theta} = g_s g_{\tau-2\theta}$. which belongs to H, since $\tau - 2\theta = \frac{2k\pi}{n}$ for some $k \in \mathbb{Z}$. If $g_\tau \in H$, then $g_\theta g_\tau g_{-\theta} = g_\tau \in H$.

Therefore $g_\theta \in N(H)$ if and only if $\theta = \frac{k\pi}{n}$, for some $k \in \mathbb{Z}$.

If $g_s g_\theta H g_{-\theta} g_s \subset H$, then for any $g_s g_{-\tau} \in H$, we have $g_s g_\theta g_s g_\tau g_{-\theta} g_s = g_s g_{2\theta-\tau} \in H$, which happens if and only if $\theta = \frac{k\pi}{n}$ for some $k \in \mathbb{Z}$. Reciprocally, if $\theta = \frac{k\pi}{n}$ then, for any $g_s g_\tau \in H$, we have $g_s g_\theta g_s g_\tau g_{-\theta} g_s = g_s g_{2\theta-\tau}$. which belongs to H, since $2\theta - \tau = \frac{2k\pi}{n}$ for some $k \in \mathbb{Z}$. For $g_\tau \in H$ $g_s g_\theta g_\tau g_{-\theta} g_s = g_{-\tau}$, which clearly belongs to H.

Therefore $g_s g_\theta \in N(H)$ if and only if $\theta = \frac{k\pi}{n}$, for some $k \in \mathbb{Z}$, proving (ii).

By lemma 3.1, $\ker(D_u f(\mu_n, 0)) = [u_n, v_n]$. Furthermore, $g_s \in Stab(u_n)$ by (i) and $\Gamma_{g_s} v_n(x) = v_n(-x) = -v_n(x)$. Thus $v_n \notin Fix(Stab(u_n))$. This proves (iii).

Finally, observe that if $g = g_\theta$, with $\theta = \frac{k\pi}{n}$, we have

$$\Gamma_g u_n(x) = u_n(x - \theta) = \begin{cases} u_n(x) & \text{if } 2|k, \\ -u_n(x) & \text{if } 2 \not| \; k \; ; \end{cases}$$

If $g = g_s g_\theta$, with $\theta = \frac{k\pi}{n}$, we have

$$\Gamma_g u_n(x) = u_n(-x + \theta) = \begin{cases} u_n(x) & \text{if } 2|k, \\ -u_n(x) & \text{if } 2 \not| \; k \; ; \end{cases}$$

This proves (iv). ∎

Consider the equation

$$F(\lambda, u) = -u + \tanh(\lambda J * u) = 0, \tag{8}$$

for the equilibria of 1 in a neighboorhod of $(\mu_n, 0)$.

Let $L = D_u F(\mu_n, 0)$, $P : \mathbb{L}^2_{per} \to \ker(D_u F(\mu_n, 0)$ be the orthogonal projection on $\ker(D_u F(\mu_n, 0)$

We set $v = Pu$ and $w = (I - P)u$.

As observed in (3.6), the corresponding bifurcation equation

$$\varphi(\lambda, v) = P(F(\lambda, v + W(\lambda, v)) \;, \tag{9}$$

defined for v in a neighboorhod of $\ker(L)$, is also Γ equivariant.

We can now state the main result of this section.

Theorem 3.8. *The bifurcation points of the equation (8) with respect to the 'trivial curve' of solutions $(\lambda, 0)$, are the points $(\mu_n, 0)$, where $\mu_n = (\lambda_n)^{-1}$ and λ^{-1} is one of the eigenvalues of the operator T defined in (6). More precisely, if $\Sigma = \mathrm{Stab}(u_n)$ and $g(\lambda, x)$ is the bifurcation equation (9) derived from (8) restricted to $\mathrm{Fix}(\Sigma)$, then $g(\lambda, x) = xh(\lambda, x)$ in a neighborhood of $(\mu_n, 0)$, where h is an even function of x, and there exist two branches of solutions $\pm u(\lambda)$, such that $x^2 = O(\|\lambda - \mu_n\|)$ ('pitchfork bifurcation').*

Proof. As already observed after lemma 3.4, $D_uF(\lambda, 0)$ is an isomorphism if $\mu_n \neq (\lambda_n)^{-1}$, so the necessity conclusion follows immediately.

To prove the sufficiency, we have to check that the hypotheses of Theorem (2.9) are met . Now, $F(\mu_n, 0) - 0$ so (i) is satisfied (the fact that $\mu_n \neq 0$ is, of course, irrelevant). Also, $L = D_uF(\mu_n, 0)$ has 0 as a (double) eigenvalue by Theorem (3.1). Let $\Sigma := \mathrm{S}tab(u_n)$. Then $\dim\ Fix(\Sigma) \cap \ker(L) = 1$, by lemma 3.7. We then need to show that the conditions of alternative III of Theorem (2.9) hold. Clearly $\Sigma < G$ and $N(\Sigma)$ acts as $-\mathbf{1}$ in $\mathrm{F}ix(\Sigma)) \ \cap\ \ker(L)$ by lemma 3.7. It remains only to prove the nonvanishhting conditions on the derivatives of the bifurcation function $g(\lambda, x)$, where

$$g(\lambda, x) = \langle u_n\,, P(F(\lambda, xu_n + W(\lambda, xu_n)))\rangle.$$

Firstly, observe that

$$\begin{aligned}
D_uF(\lambda, u).\dot{u} &= -\dot{u} + \lambda \mathrm{s}ech^2(\lambda J * u) J * \dot{u}\ ;\\
D^2_uF(\lambda, u).(\dot{u}, \dot{v}) &= -2(\lambda)^2(J * \dot{u})(J * \dot{v})\mathrm{s}ech^2(\lambda J * u).\tanh(\lambda J * u);\\
D^3_uF(\lambda, u).(\dot{u}, \dot{v}, \dot{w}) &= -2(\lambda)^3(J * \dot{u})(J * \dot{v})(J * \dot{w})\mathrm{s}ech^4(\lambda J * u)\\
&\quad +4(\lambda)^3(J * \dot{u})(J * \dot{v})(J * \dot{w})\mathrm{s}ech^2(\lambda J * u)\tanh^2(\lambda J * u)\ ;\\
D_\lambda F(\lambda, u).\dot{\lambda} &= \mathrm{s}ech^2(\lambda J * u) J * u\ ;\\
D^2_{u\lambda}F(\lambda, u).u_n &= \mathrm{s}ech^2(\lambda J * u) J * u_n\\
&\quad -8\tau^2\lambda(J * u)(J * u_n)\mathrm{s}ech^2(\lambda J * u)\tanh(\lambda J * u).
\end{aligned}$$

With $(\lambda, u) = (\mu_n, 0)$, we obtain

$$D_uF(\mu_n, 0).\dot{u} = -\dot{u} + \lambda_n^{-1} J * \dot{u}\ ; \tag{1}$$

$$D^2_2F(\mu_n, 0).(\dot{u}, \dot{v}) = 0\ ; \tag{2}$$

$$D^3_uF(\mu_n, 0).(\dot{u}, \dot{v}, \dot{w}) = -2\lambda_n^{-3}(J * \dot{u})(J * \dot{v})(J * \dot{w})\ ; \tag{3}$$

$$D_\lambda F(\mu_n, 0).\dot{\lambda} = 0\ ; \tag{4}$$

$$D^2_{u\lambda}F(\mu_n, 0).u_0 = 2\tau J * u_0\ . \tag{5}$$

Using the computations at the end of section 2.2, we obtain

$$D_uW(\mu_n,0).u_n = D_uW(\mu_n,0).v_n = 0 \ ; \tag{6}$$
$$D_v^2W(\mu_n,0).(u_n,u_n) = -L^{-1}(Q(D_u^2F(\mu_n,0).(0,u_n))) \ ; \tag{7}$$
$$D_v^2W(\mu_n,0).(v_n,v_n) = -L^{-1}(Q(D_u^2F(\mu_n,0).(v_n,v_n))) \ . \tag{8}$$

From (6), (7) and (8) it follows that

$$D_v^2W(\mu_n,0).(u_n,u_n) = D_v^2W(\mu_n,0).(v_n,v_n) = 0 \ . \tag{9}$$

From (1),(2),(3),(4),(5),(6), (9) and the computations at the end of section 2.2, we obtain

$$\begin{aligned}
\frac{\partial^2 g}{\partial x \partial \lambda}(\mu_n,0) &= \langle u_n, 2\tau\lambda_n u_n\rangle \\
&= 2\tau\lambda_n\|u_n\|^2 \\
&= 2\tau^2\lambda_n \ ; \\
\frac{\partial^3 g}{\partial x^3}(\mu_n,0) &= \langle u_n, -2P(u_n^3)\rangle \\
&= -2\langle u_n, P(u_n^3)\rangle \\
&= -\frac{2}{\tau}\langle u_n, \langle u_n^3, u_n\rangle u_n\rangle \\
&= -2\langle u_n, u_n^3\rangle \\
&= -\frac{3}{2}\tau \ ;
\end{aligned}$$

4. Global bifurcations

In this section we prove that, under certain conditions, the curve of equilibria bifurcating the trivial curve $(\lambda,0)$ can be globally continued. To achieve this we introduce some subspaces of $\mathbb{L}^2_{per}$, related to the fixed point spaces $\mathrm{F}ix(u_n)$ and $\mathrm{F}ix(v_n)$ of the previous section.

4.1. **Invariant subspaces.**

Proposition 4.1. *For any $n \in \mathbb{N}$, let*

$$\begin{aligned} X_n &= \{u \in \mathbb{L}^2_{per} \,|\, u(x+\frac{2\pi}{n}) = -u(x) \text{ and } u(-x) = u(x)\}\ , \\ Y_n &= \{u \in \mathbb{L}^2_{per} \,|\, u(x+\frac{2\pi}{n}) = -u(x) \text{ and } u(-x) = -u(x)\}\ , \\ Z_n &= \{u \in \mathbb{L}^2_{per} \,|\, u(x+\frac{2\pi}{n}) = u(x) \text{ and } u(-x) = u(x)\}\ , \\ W_n &= \{u \in \mathbb{L}^2_{per} \,|\, u(x+\frac{2\pi}{n}) = u(x) \text{ and } u(-x) = -u(x)\}\ ,. \end{aligned}$$

Then X_n, Y_n, Z_n and W_n are closed subspaces of $\mathbb{L}^2_{per}$, which are invariant for the flow of (1).

Proof. We only sketch the simple proof.

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_n converging to u_0 em $\mathbb{L}^2_{per}$. Then, there exists a subsequence $(u_{k_s})_{s\in\mathbb{N}}$, $(u_{k_s})_{s\in\mathbb{N}}$ converging to u_0 uniformly, except in a set of null measure in $-[\pi,\pi]$. The closedness property follows then easily from uniqueness of the limit. The invariance property is a consequence of uniqueness of solutions of (1).

The subspaces X_n, Y_n, Z_n and W_n are related to the Fix spaces which played an important role in the equivariant bifurcation result of section 3 as described in the next result.

Proposition 4.2. *Let u_n and v_n be the eigenfunction of the operator T of lemma 3.1 and X_n, Y_n e Z_n as in lemma 4.1. Then, we have*

(i) $u_n \in X_n$ *and* $v_n \in Y_n$*;*
(ii) $\mathrm{F}ix(\mathrm{S}tab(u_n)) = X_n \oplus Z_n$*;*
(iii) $\mathrm{F}ix(\mathrm{S}tab(v_n)) = Y_n \oplus Z_n$.

Proof. The assertion (i) is immediate. If $u \in \mathrm{F}ix(\mathrm{S}tab(u_n))$, define v_1, v_2 by $v_1(x) = \frac{u(x)-u(x+\pi/n}{2}$ and $v_2(z) = \frac{u(x)+u(x-\pi/n}{2}$. One easily checks that $v_1 \in X_n$ and $v_2 \in Z_n$. If $u \in X_n \cap Z_n$, then $u(x) = -u(x)$ for any $x \in \mathbb{R}$ and thus $u = 0$. Therefore $\mathrm{F}ix(\mathrm{S}tab(u_n)) \subset X_n \oplus Z_n$ The converse inclusion is immediate, so the assertion (ii) follows. The proof o (iii) is completely analogous.

Now, consider the operator T defined in 6. From the properties of the convolution, it follows that the subspaces X_n, Y_n, Z_n and W_n are left invariant by T. Concerning the spectral properties of $L := T\big|_{X_n}$, we have the following result whose proof can be given along the same lines of (3.1).

Lemma 4.3. *L is a compact self-adjoint operator. Furthermore, the following assertions are equivalent.*

(a) $\nu \in \mathbb{R}$ *is an eigenvalue of T;*
(b) *There exists $k \in \mathbb{N}$ such that $\nu = \widehat{J(k)}$, where $k \in \mathbb{N}$ is of the form $k = (2l+1)n$ and $\widehat{J(k)}$ is as in lemma 3.1.*

In this case, if $\widehat{J(k)} \neq \widehat{J(s)}$ for for all $k, s \in \mathbb{N}$, $k \neq s$, then μ is a simple eingevalue with corresponding eigenfunction by $u_n(x) = \cos nx$.

The linearization of $F(\lambda, u) = -u + \tanh(\lambda J * u)$ restricted to X_n at $(\bar{\lambda}, 0)$ is given by $D_u F(\lambda, 0) = -I + \bar{\lambda} L$. It follows that the only possible bifurcation points of $F(\lambda, u) = -u + \tanh(\lambda J * u) = 0$ restricted to X_n, along the trivial curve of equilibria $(\lambda, 0)$ are the points $(\mu_k, 0)$ where $\mu_k = (\lambda_k)^{-1}$ and λ_k is one of the eigenvalues of L. By lemma (4.3) $\frac{\lambda_k}{2\pi}$ is one of the Fourier coefficients of J, $k \in \mathbb{N}$ is of the form $k = (2l+1)n$ and $Ker(D_u F(\mu_k, 0)) = [u_k(x)]$.

In fact, one can prove the converse assertion, using the Crandall-Rabinowitz theorem. More precisely:

Theorem 4.4. *The bifurcation points of the equation (8) restricted to the invariant subspace X_n with respect to the 'trivial curve' of solutions $(\lambda, 0)$, are the points $(\mu_k, 0)$, where $\mu_k = (\lambda_k)^{-1}$ and λ_k^{-1} is one of the eigenvalues of the operator L. Furthermore, the nontrivial solutions of (8) in a neighborhood of $(\mu_k, 0)$ are C^1 close to the corresponding eigenfunction u_k of L.*

Remark 4.5. *Similar results are obtained substituting Y_n, Z_n or W_n for X_n.*

4.2. **Global bifurcation in X_n.** Our global continuation result is based on the following result.

Theorem 4.6. *Let X be a Banach space, $U \subset \mathbb{R} \times X$ an open set and f a differentiable map from U into X. Suppose that $f(\lambda, 0) \equiv 0$ and $(\lambda, 0)$ is a bifurcation point of $f(\lambda, u) = 0$ with respect to the trivial curve $(\lambda, 0)$. Let S denote the closure of the set of nontrivial zeros of f. Then there exists a connected component C of S containing $(\lambda_0, 0)$ satisfying one and only one of the following assertions:*

(i) *C is not compact (if $U = \mathbb{R} \times X$, then C is unbounded);*
(ii) *C meets the line $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ at $(\mu, 0)$, where $\mu \neq \lambda_0$ is another bifurcation point.*

Proof. See [29] e [34]. ∎

The next two lemmas will be needed in the proof of the main result of this section.

Lemma 4.7. *Let J be a C^1, nonnegative even function in $\mathbb{R}$ with support contained in $[-2\pi/n, 2\pi/n]$ and is stricly decreasing in $[0, \pi/n[\cap supp J$. Suppose $u \in Y_n$, $u \geq 0$ in $[0, \pi/2n[$ and there exist $x_0 \in [0, \pi/2n[$ and $\delta > 0$ such that $u > 0$ in $[x_0 - \delta, x_0[$. Then $(J * u)(x_0) > 0$.*

Proof. We have

$$
\begin{aligned}
(J*u)(x_0) &= \int J(x-x_0)u(x)dx = \int_{-\pi/n+x_0}^{\pi/n+x0} J(x-x_0)u(x)dx \\
&= \int_0^{x_0} J(x-x_0)u(x)dx + \int_{\pi/n-x_0}^{\pi/n} J(x-x_0)u(x)dx \\
&+ \int_{x_0}^{\pi/2n} J(x-x_0)u(x)dx + \int_{\pi/2n}^{\pi/n-x_0} J(x-x_0)u(x)dx \\
&+ \int_{-x_0}^{0} J(x-x_0)u(x)dx + \int_{\pi/n}^{\pi/n+x_0} J(x-x_0)u(x)dx \\
&+ \int_{-\pi/2n}^{-x_0} J(x-x_0)u(x)dx + \int_{-\pi/n+x_0}^{-\pi/2n} J(x-x_0)u(x)dx\,. \qquad (10)
\end{aligned}
$$

We prove that

$$
\int_0^{x_0} J(x-x_0)u(x)dx + \int_{\pi/n-x_0}^{\pi/n} J(x-x_0)u(x)dx > 0. \qquad (11)
$$

Making the change of variables $y = x - \pi/n$ and $z = -y$ and using that $u(y+\pi/n) = -u(y)$, $u(z) = u(-z)$ we obtain, for the second integral in (11)

$$
\begin{aligned}
\int_{\pi/n-x_0}^{\pi/n} J(x-x_0)u(x)dx &= \int_{-x_0}^{\pi/n} J(y+\pi/n-x_0)u(y+\pi/n)dy \\
&= -\int_{-x_0}^{0} J(y+\pi/n-x_0)u(y)dy \\
&= \int_{x_0}^{0} J(-z+\pi/n-x_0)u(z)dz \\
&= -\int_0^{x_0} J(\pi/n-z-x_0)u(z)dz
\end{aligned}
$$

Since $0 \le x_0 - x < \pi/n - x - x_0$ for all $x \in [0, x_0]$, we have $J(x-x_0) = J(x_0-x) > J(\pi/n - x - x_0) > 0$. Furthermore $J(x-x_0)u(x) > 0$ for all $x \in [x_0-\delta, x_0[$. Thus

$$
\begin{aligned}
\int_0^{x_0} J(x-x_0)u(x)dx &> \int_0^{x_0} J(\pi/n-x-x_0)u(z)dx \\
&= -\int_{\pi/n-x_0}^{\pi/n} J(x-x_0)u(x)dx.
\end{aligned}
$$

This proves (11).

The positivity of the other integrals in (10) is similarly proved.

■

Lemma 4.8. *Suppose J is as in lemma 4.7 and let $u \in Y_n$, $u \geq 0$ em $[0, \pi/2n[$ and u not identically zero in $[\pi/2n - \delta, \pi/2n[$ for $\delta > 0$. Then $(J' * u)(\pi/2n) < 0$.*

Proof. Since J' is supported in $]-\pi/n\,, \pi/n[$, J' is odd (??) and $u(x - \pi/2n) = -u(x + \pi/2n)$, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
(J' * u)(\pi/2n) &= \int J'(x)u(x - \pi/2n)dx = -\int_{-\pi/n}^{\pi/n} J'(x)u(x + \pi/2n)dx \\
&= -\int_{-\pi/n}^{0} J'(x)u(x + \pi/2n)dx - \int_{0}^{\pi/n} J'(x)u(x + \pi/2n)dx \\
&= -\int_{-\pi/n}^{0} J'(x)u(x + \pi/2n)dx + \int_{0}^{\pi/n} J'(x)u(x - \pi/2n)dx.
\end{aligned}$$

Using that $J' \geq 0$ in $]-\pi/n, 0]$, *supp*$J' \subset [-\pi/n\,, \pi/n]$, $u \geq 0$ in $[-\pi/2n, \pi/2n]$ and $u \not\equiv 0$ $[\pi/2n - \delta, \pi/2n[$, we obtain

$$\int_{-\pi/n}^{0} J'(x)u(x + \pi/2n)dx \;\leq\; 0. \tag{12}$$

Since $J' \leq 0$ in $[0, \pi/n[$, *supp*$J' \subset [-\pi/n\,, \pi/n]$ and $u \geq 0$ em $]-\pi/2n, \pi/2n[$, it follows that

$$\int_{0}^{\pi/n} J'(x)u(x - \pi/2n)dx) \;<\; 0. \tag{13}$$

From (12) and (13) $(J' * u)(\pi/2n) \;<\; 0$, as claimed (acho que os sinais estao trocados!).
■

We are now in a position to prove our main result of this section.

Theorem 4.9. *Let J be as in , X_n as in proposition 4.1. and $(\mu_k, 0)$ one of the bifurcation points of (8) restricted to X_n, given by theorem 4.4. Then $(\mu_k, 0)$ is a global bifurcation point of (8). More precisely, there exists a connected component C of the closure of the set o nontrivial zeroes of (8) containing $(\mu_k, 0)$ and contained in X_n, which intersects $\{\lambda\} \times X_n$, for any $\lambda \geq \mu_k$.*

Proof. By theorem 4.6, there is a connected component C_k of the closure of the set o nontrivial zeroes of (8) containing $(\mu_k, 0)$, which is either noncompact or meets the line $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ at $(\mu_s, 0)$, where $\mu_s \neq \mu_0$ is another bifurcation point in X_n. We show that the second alternative is not possible.

Let

$$F_k = \{(\lambda, u) \in C_k \,|\, u > 0 \text{ in } [0, \pi/2k[\,\text{and}\, u'(\pi/2k) < 0\} \cup \{(\mu_k, 0)\}.$$

We prove that F_k is both open and closed in C_k (in the C^1 topology).

We first show closedness.

Let $(\lambda_j\,, u_j)_{j\in\mathbb{N}}$ be a sequence in F_k with $\lambda_j \to \lambda$ and $u_j \to u$. Since $u_j > 0$ in $[0, \pi/2k[$, for all $j \in \mathbb{N}$, then $u \geq 0$ in $[0, \pi/2k[)$. We prove that u cannot be identically zero, unless e $\lambda = \mu_k$.

In fact, suppose $\lambda \neq \mu_k$ and $u \equiv 0$. By theorem 4.6, λ must then be one of the bifurcation points μ_s given by Theorem 4.4. We may suppose that $s > k$. But then, it follows fom theorem 4.4 that $u < 0$ in an open set of $[0, \pi/2k]$ which is a contradiction.

We show that $(\lambda, u) \in F_k$. If there exists $x_0 \in]0, \pi/2k[$ such that $u(x_0) = 0$, then $u > 0$ in $[x_0 - \delta, x_0[$ for some $\delta > 0$ since $u \not\equiv 0$. Thus, by lemma 4.7, $(J * u)(x_0) > 0$ which cannot happen, since u is an equilibrium. By lemma 4.8, $u'(\pi/2k) > 0$. Thus, $(\lambda, u) \in F_k$ and F_k is closed in C_k .

We now show that F_k is open in C_k .

Let $(\lambda_0\,, u_0) \in F_k$. If $(\lambda_0\,, u_0) = (\mu_k, 0)$, there exists a neighborhood of it in $\mathbb{R}\times Y_n \cap C_k$ contained in F_k by Theorem 4.4. Suppose $\lambda_0 \neq \mu_k$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $\delta < \inf_{x\,\in\,[0,\pi/2k-\varepsilon]} u_0(x)$ and$u_0' < -\delta$ em $]\pi/2k - \varepsilon, \pi/2k]$. Let $u \in C^1([-\pi/k\,, \pi/k])$ such that $||u - u_0||_{C^1([-\pi/k\,,\pi/k])} < \delta$. Then

$$|u(x) - u_0(x)| < \delta, \text{ for all } x \in [0, \pi/2k - \varepsilon] \tag{14}$$

and

$$|u'(x) - u_0'(x)| < \delta, \text{ for all } x \in]\pi/2k - \varepsilon, \pi/2k]. \tag{15}$$

From (14, $u(x) > 0$, for all $x \in [0, \pi/2k - \varepsilon]$ and, by (15) $u'(x) < 0$, for all $x \in]\pi/2k - \varepsilon, \pi/2k]$. Thus $u'(\pi/2k) < 0$ and u is strictly decreasing in $]\pi/2k - \varepsilon, \pi/2k]$. Since $u > 0$ in $[0, \pi/2k - \varepsilon]$ and strictly decreasing in $]\pi/2k - \varepsilon, \pi/2k[$, it follows that $u > 0$ in $[0, \pi/2k[$, showing that F_k is open in C_k . Finally F_k is not empty by Theorem 4.4. Thus $F_k = C_k$ does not intersect the λ axis and so, by Theorem 4.6 cannot be compact. From results in [31], the second component of C_k must be in a compact set. Also, there are no nontrivial equilibria if $\lambda \leq 1$. From this, the claimed result follows immediately. ■

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