Nonlinear Flux through the Boundary Versus Nonlinear Heating

Sergio Muniz Oliva*

Antonio Luiz Pereira†

1 Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^2$. In this paper we consider reaction diffusion systems with dispersion of the form

$$\begin{cases} u_t = \mathrm{Div}(a\nabla u) - \displaystyle\sum_{j=1}^{2} B_j(x)\frac{\partial u}{\partial x_j} - \lambda u + f(u), & \text{in} \quad \Omega, \\ \\ \dfrac{\partial u}{\partial n_a} = g(u), & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

where $u = (u_1, \cdots, u_N)^\top$, $N \geq 1$, $a(x) = \mathrm{diag}(a_1(x), \cdots, a_N(x))$, $a_i \in C^1(\overline{\Omega})$, $a_i(x) > m_0 > 0$, $x \in \Omega$, $1 \leq i \leq N$, $\frac{\partial u}{\partial n_a} = \langle a\nabla u, \vec{n}\rangle$, $\vec{n}$ is the outward normal and $B_j = \mathrm{diag}(b_j^1, \cdots, b_j^N)$ is continuous in $\bar{\Omega}$, $j = 1, 2$. Let $f = (f_1, \cdots, f_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$, $g = (g_1, \cdots, g_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ be smooth functions.

It has been shown by Pao [1978] that if f is a source of heat and if $g = 0$ then we have blow up in finite time. Our aim is to control the increase of heat by means of a dissipative flux through the boundary. To acomplish this goal we need to introduce some kind of "competition" between f and g. In fact one of the basic questions is: If g dissipates heat through the boundary,

*Departamento de Matemática Aplicada, Instituto de Matemática e Estatística, USP

†Departamento de Matemática, Instituto de Matemática e Estatística, USP

can we find a relation between the dissipation g and the source of heat f in such a way that we can assure the existence of global attractors?

This problem is not new, Pao [1978] introduced a relation between f and g and almost completely solved the problem for classical solutions working in the space of continous functions. Later on, Alikakos [1981], following the ideas of Friedman [1964], imposing some growth conditions on f and g, showed global existence and some asymptotic behavior of the solutions. Working in $\mathbb{R}$, Henry [1985], solved the problem completely, and assuming a nice relation between f and g (similar to Pao [1978]), showed that the system is Morse-Smale.

Another major concern is to relax most growth conditions on f and g. In this direction Carvalho and Rodriguez-Bernal [1994a] proved, for the scalar case, N=1, and $\Omega \subset \mathbb{R}^n$, $n \leq 3$, that under some growth assumptions on the nonlinearity f, the problem (**??**), with $B_j \equiv 0$, $j = 1, 2$, has a global attractor in $H^1(\Omega)$. More specifically, for $n = 2$, f and g are required to satisfy:

$$\left.\begin{array}{l} \displaystyle\liminf_{|s|\to\infty} \frac{\lambda s + f(s)}{s} \leq 0 \\ \displaystyle\liminf_{|s|\to\infty} \frac{g(s)}{s} \leq 0 \end{array}\right\} \text{one of the inequalities being strict,} \tag{2}$$

(the dissipative conditions)

and
$$\lim_{|s|\to\infty} \frac{|f'(s)|}{e^{\eta\,|s|^2}} = \lim_{|s|\to\infty} \frac{|g''(s)|}{e^{\eta\,|s|^2}} = 0,\ \forall \eta > 0, \tag{3}$$

(the growth conditions).
(see also, Hale [1988], Hale and Raugel [1992], Carvalho [1992] and Carvalho and Oliveira [1992], for the case $g = 0$ and more restrictive hypothesis on f).

These growth assumptions are used to obtain local existence of solutions for (**??**) and also play a role in obtaining energy estimates necessary to guarantee that the solution operator for (**??**) defines a global dynamical system which is bounded dissipative.

Later on Carvalho and Rodriguez-Bernal [1994b], proved, in $\mathbb{R}^2$, that if we drop (**??**) and just assume that both f and g dissipate energy, then one can actually show the existence of global attractors. The key idea is to restrict the space of initial data in such a way that no growth assumptions are needed for local existence of solutions for (**??**).

In particular, they consider spaces of initial conditions which are embedded in $C(\overline{\Omega})$. Since all functions in this space are bounded, no growth conditions on f and g are required for local existence. They show that in such spaces the problem (**??**) has a global attractor and obtain some good estimates for the size of the attractor in the uniform norm. This problem has been studied in Carvalho and Ruas-Filho [1993], for the case $g = 0$

The goal of this paper is to combine the ideas of of competition in Pao [1978] and Henry [1985] with the ideas of Carvalho and Rodriguez-Bernal [1994b] to prove the existence of global attractors for the problem (**??**), without assuming dissipation on both f and g.

Since we are going to use many of the ideas in Carvalho and Rodriguez-Bernal [1994b], let us briefly describe them. Let $X = L^2(\Omega)$ and $A : D(A) \subset X \to X$ be the operator $A = \mathrm{diag}(A_1, \cdots A_N)$ defined by

$$D(A_i) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial n_{a_i}} = 0\},$$

$$A_i\phi = -\mathrm{Div}(a_i(x)\nabla\phi) + \textstyle\sum_{j=1}^{2} b_j^i(x)\tfrac{\partial \phi}{\partial x_j} + \lambda\phi,$$

where $\frac{\partial u}{\partial n_{a_i}} = a_i\langle \nabla u, \vec{n}\rangle$, $i = 1, \ldots, N$.

We can define the fractional powers A^α of A (see, for example, Henry [1981]) and the fractional power spaces $X^\alpha := D(A^\alpha)$ endowed with the graph norm, $\alpha \in \mathbb{R}$, where $X^\alpha = (X^{-\alpha})'$, if $\alpha < 0$. In this case we can always view A as a sectorial operator with compact resolvent from $X^{\alpha+1}$ into X^α which is positive and self adjoint.

Hale [1986] proved the existence of a local attractor for (**??**), with $g = 0$ and f satisfying (**??**), which coincides with the embedding of the attractor for $\dot{u} + \lambda u + f(u) = 0$ into the subspace of constant functions of X^α, $\alpha > \frac{3}{4}$, if the diffusion coefficient $a(x)$ is large (see also, Hale and Rocha [1987a,b] and Hale and Sakamoto [1989]). However, the techniques employed by Hale [1986] would only apply to global attractors if some a priori bound on the size of the absorbing set could be obtained and only if the diffusion coefficient is large (see Carvalho [1992] and Carvalho and Oliveira [1992]). This a priori bounds are obtained in Carvalho and Ruas-Filho [1993] for the case $g = 0$, also working in X^α, for $\alpha > \frac{3}{4}$ and $n = 3$.

Carvalho and Rodriguez-Bernal [1994b] proved the existence of a global attractor for the problem (**??**) regardless of the size of the diffusion coefficient

$a(x)$, for $g \neq 0$. They also found uniform bounds on the size of the attractor allowing the application of the results of Carvalho [1992] and Carvalho and Oliveira [1992] to the case $\alpha \neq \frac{1}{2}$ as in Hale [1986], Hale and Rocha[1987a,b] and Hale and Sakamoto [1989] (see also Fusco [1987]).

The approach followed was to show that the solution operator associated to (**??**) is globally defined, that orbits of bounded subsets of X^α, under the flow defined by (**??**), are bounded subsets of X^α and that there is a bounded set that attracts points of X^α. Since the solution operator associated to (**??**) is compact, Theorem 3.4.6 in Hale [1989] guarantees the existence of a global attractor.

Let us mention that Hale [1986], Hale and Rocha [1987a,b], Hale and Sakamoto [1989] and Carvalho and Ruas-Filho [1993], work in X^α for $\alpha > \frac{3}{4}$ and $n = 3$. The reason for this is the embedding $X^\alpha \subset C(\overline{\Omega})$. However, for $\alpha > \frac{3}{4}$, the space X^α incorporates the boundary condition $\frac{\partial u}{\partial \vec{n}_a} = 0$ and therefore, cannot be the right space to work with if $g \neq 0$. However, for $n = 2$ and $\frac{3}{4} > \alpha > \frac{1}{2}$, we have that $X^\alpha \subset C(\overline{\Omega})$ and X^α does not incorporate any boundary condition. In this case we can use a variation of constants formula for the solutions, as shown by Bernal[?] (see also Amann[1988] for a somewhat different approach). Therefore, we work in this range of α.

Observe that for $\Omega \subset \mathbb{R}$ we can take $\alpha = \frac{1}{2}$ and all the results hold true without any changes. Therefore, we restrict the presentation to the two dimensional case.

The paper will proceed as follows: in Section **??** we introduce the hypotheses that are going to be used in this paper, in Section **??** we define the spaces that are going to be used and show local existence of solutions for (**??**). In Section **??** we use the notion of sub- and super-solutions to show that the semigroup is bounded and that the solutions are defined for all time. Finally in Section **??** we show existence of global attractors and a bound for such attractors.

2 Hypotheses

In this Section we fix the hypotheses to be used throughout this paper.

(H1) *$f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $g \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ are such that, there exist $c_0, d_0 \in \mathbb{R}$ in such a way that the following condition is satisfied, for all $i = 1, \cdots, N$,*

$$\left.\begin{array}{l}\liminf_{s_i\to\pm\infty}\dfrac{f_i(s)}{s_i}\leq c_0\\ \liminf_{s_i\to\pm\infty}\dfrac{g_i(s)}{s_i}\leq d_0\end{array}\right\}\tag{4}$$

Moreover, if f, g satisfy (H1), and given the eigenvalue problem

$$\left.\begin{array}{ll}\mathrm{Div}(a\nabla v_i)-\displaystyle\sum_{j=1}^{2}B_j(x)\frac{\partial v_i}{\partial x_j}-\lambda v_i+c_0v_i=\mu_i v_i, & \text{in}\quad \Omega,\\ \dfrac{\partial v_i}{\partial n_a}=d_0 v_i \qquad \text{on}\quad \partial\Omega & \end{array}\right\}\tag{5}$$

we will assume the following,

(H2) *All eigenvalues of (**??**) are negative.*

Remark 2.1 *To avoid notational complications we will treat only the case $N = 1$, but it will be clear from the proofs that the results remain true in more dimensions and the same arguments apply if we assume (H1).*

Remark 2.2 *(H1) is the dissipation condition on the equation, however we allow either c_0 or d_0 to be positive. In other words, we allow either f or g to be a source of heat.*

Remark 2.3 *(H2) is a precise formulation of the "competition" between f and g that we mentioned in the Introduction. Notice that we cannot have both c_0 and d_0 positive. Moreover this condition states that our problem "behaves" as an intermediate case between the Dirichlet case ($d_0 = \infty$) and the Newmann case ($d_0 = 0$).*

3 Local Existence

Let us start by fixing the state spaces. First of all, it is well known that A defined by

$$D(A) = \left\{ \phi \in H^2(\Omega), \text{ such that } \frac{\partial \phi}{\partial n_a} = 0 \right\}$$
$$A\phi = -\mathrm{Div}(a\nabla\phi) + \sum_{j=1}^{2} B_j(x)\frac{\partial \phi}{\partial x_j} + \lambda\phi \tag{6}$$

where $\dfrac{\partial u}{\partial n_a} = a\langle \nabla u, \vec{n}\rangle$, generates an analytic semigroup on $X^\alpha = D(A^\alpha)$, for $0 < \alpha < 1$ which satisfies, for suitable λ

$$\left\| e^{-At}u_0 \right\|_{X^\alpha} \leq Me^{-\epsilon t}\|u_0\|_{X^\alpha}, \quad t \geq 0$$
$$\left\| e^{-At}u_0 \right\|_{X^\alpha} \leq Me^{-\epsilon t}t^{-\alpha}\|u_0\|_{X}, \quad t > 0. \tag{7}$$

for some $\epsilon > 0, M > 0$. In particular, if $B_j \equiv 0$, $j = 1, 2$, λ can be any positive number.

Let $\tilde{A}$ be the extension of A to the space $X^\beta = H^{2\beta}$ with $\beta < 0$. $\tilde{A}$ is also a sectorial operator and generates an analytic semigroup.

Suppose $\frac{1}{2} < \alpha < \frac{3}{4}$. Following Rodriguez-Bernal [1993] we consider, for $u \in X^\alpha$, the functional $h(u)$ which acts on test functions in the following way

$$\langle h(u), \phi\rangle = \langle f_\Omega(u), \phi\rangle + \langle g_\Gamma(u), \gamma(\phi)\rangle,$$

where γ denotes the trace operator, and $f_\Omega : X^\alpha \to L^2(\Omega)$ and $g_\Gamma : X^\alpha \to H^{\frac{1}{2}}(\Omega)$ are the maps defined by

$$\begin{gathered} f_\Omega(u)(x) = f(u(x)), \text{ and} \\ g_\Gamma(u) = \gamma(g_\Omega(u), \text{ with} \\ g_\Omega(u)(x) = g(u(x)). \end{gathered}$$

With this, we can state the following

Theorem 3.1 *Assume that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are C^1 and C^2 functions respectively,*

$$\frac{1}{2} < \alpha < \frac{3}{4}, \quad -\frac{1}{2} < \beta < -\frac{1}{4} \textit{ and } \alpha < \beta + 1.$$

Then the abstract parabolic problem

$$\begin{cases} \dfrac{du}{dt} = \tilde{A}u + h(u) \\ \\ u(0) = u_0 \in X^\alpha \end{cases} \tag{8}$$

has an unique solution for any $u_0 \in X^\alpha$, which is given by the variation of constants formula

$$T(t)u_0 = e^{\tilde{A}t}u_0 + \int_0^t e^{\tilde{A}(t-s)} h(T(s)u_0) ds. \tag{9}$$

Moreover, if the maximal interval of existence of the solution $T(t)u_0$ is $[0, t_{max}[$ then either $t_{max} = +\infty$ or $\|T(t)u_0\|_{X^\alpha} \to \infty$ as $t \to t_{max}$.

Proof:

The result follows from Henry's result [1981], once we prove that $h : X^\alpha \to X^\beta$ is Lipschitz continuous in bounded sets of X^α. This can be done using the continuity of the imbedding $X^\alpha \hookrightarrow C^0(\Omega)$. Details can be found in Carvalho and Rodriguez-Bernal [1994b].

Remark 3.1 *With the same hypotheses on f and g it can be proved (Rodriguez-Bernal [1993]) that any solution u of (**??**) satisfies*

$$t \to u \in H^2(\Omega), \text{ and } \frac{\partial u}{\partial n_a} = g \text{ in } H^{\frac{1}{2}}(\Gamma).$$

*So u can be considered a solution of our original problem (**??**).*

4 Boundedness of the Semigroup

In this section we prove that solutions of (**??**) with initial data in X^α, are globally defined and orbits of bounded subsets of X^α, under the flow determined by (**??**), are also bounded in X^α.

To acomplish this goal, we use comparison results. We start by defining the concepts of sub- and super-solutions.

Definition 4.1 *A C^2 function $\bar{u}: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ ($\underline{u}$ respectively) is a super-solution (sub-solution) of the problem*

$$\begin{cases} u_t = Div(a\nabla u) - \sum_{j=1}^{2} B_j(x)\dfrac{\partial u}{\partial x_j} - \lambda u + f(u), \qquad in \quad \Omega, \\ \\ \dfrac{\partial u}{\partial n_a} = g(u), \qquad on \quad \partial\Omega \\ \\ u(0) = u_0. \end{cases} \tag{10}$$

if it satisfies

$$\begin{aligned} &\bar{u}_t \geq Div(a\nabla \bar{u}) - \textstyle\sum_{j=1}^{2} B_j(x)\frac{\partial \bar{u}}{\partial x_j} - \lambda \bar{u} + f(\bar{u}), \qquad in \quad \Omega, \\ &\frac{\partial \bar{u}}{\partial n_a} \geq g(\bar{u}), \qquad on \quad \partial\Omega \\ &\bar{u}(0) \geq u_0. \end{aligned} \tag{11}$$

(and respectively with the $\geq$ sign replaced by the $\leq$ sign).

A basic result for our arguments is the following

Theorem 4.1 *(Pao [1978])*
*If f is locally Lipschitz and $\bar{u}$ and $\underline{u}$ are respectively a super- and sub-solution of the problem (**??**), satisfying*

$$\underline{u} \leq \bar{u}, \;\; in \; \Omega \times (0, T),$$

*then, there exists a solution u of (**??**) such that*

$$\underline{u} \leq u \leq \bar{u}, \;\; in \; \Omega \times (0, T).$$

Let φ be the first positive normalized eigenfunction of (**??**) and $m = \min_{x \in \bar{\Omega}} \varphi(x)$. We know that $m > 0$. For each $\theta \in \mathbb{R}_+$, define

$$\Sigma_\theta = \left\{ u \in X^\alpha : |u(x)| \leq \theta\varphi(x), \text{ for all } x \in \bar{\Omega} \right\}.$$

From the dissipative hypothesis (H1) on f and g, we know that there exists

$\xi \in \mathbb{R}$, such that

$$\frac{f(s)}{s} \leq c_0 \text{ and } \frac{g(s)}{s} \leq d_0,$$

for all s with $|s| \geq \xi$.

Lemma 4.1 *If $\theta m \geq \xi$ then Σ_θ is a positively invariant set for the local solution of (**??**).*

Proof:

Let

$$\Sigma_\theta^1 = \{u \in X^\alpha : u(x) \leq \theta\varphi(x), \text{ for all } x \in \bar{\Omega}\}$$

$$\Sigma_\theta^2 = \{u \in X^\alpha : u(x) \geq -\theta\varphi(x), \text{ for all } x \in \bar{\Omega}\}$$

Since $\Sigma_\theta = \Sigma_\theta^1 \cap \Sigma_\theta^2$ it is enough to show that Σ_θ^1 and Σ_θ^2 are positively invariant.

Let $u_0 \in \Sigma_\theta^1$, and suppose, for contradiction, that there exists $t_0 \in [0, t_{max}[$ and $x_0 \in \bar{\Omega}$ such that

$$T(t_0)u_0(x_0) \geq \theta\varphi(x_0).$$

Consider $\bar{v}(t) = e^{\mu(t-t_0)}\theta\varphi$, where μ is the eigenvalue associated with φ. We have that

$$\begin{aligned}
\frac{\partial \bar{v}}{\partial t} = & \quad \operatorname{Div}(a\nabla\bar{v}) - \textstyle\sum_{j=1}^{2} B_j(x)\frac{\partial \bar{v}}{\partial x_j} - \lambda\bar{v} + c_0\bar{v} \geq \\
& \quad \operatorname{Div}(a\nabla\bar{v}) - \textstyle\sum_{j=1}^{2} B_j(x)\frac{\partial \bar{v}}{\partial x_j} - \lambda\bar{v} + f(\bar{v}) \\
\frac{\partial \bar{v}}{\partial n_a} = & \quad d_0\bar{v} \geq g(\bar{v}),
\end{aligned}$$

for all $t \in]0, t_0]$.

Thus $\bar{v}$ is a super-solution for the problem (**??**). It follows from Theorem **??** that

$$T(t)u_0 \leq \bar{v}(t), \text{ in } \bar{\Omega} \text{ for all } t \in [0, t_0[.$$

In particular, $T(t_0)u_0(x_0) \leq \theta\varphi(x_0)$ and we reach a contradiction.

To prove that Σ_θ^2 is positively invariant we proceed in a similar way, using now that $\underline{v} = -\bar{v}$ is a sub-solution for the problem (**??**).

Lemma 4.2 *If B is a bounded subset of X^α, with $\frac{1}{2} < \alpha < \frac{3}{4}$, then $\cup_{t\geq 0} T(t)B$ is also a bounded subset of X^α.*

Proof:

Since the inclusion map $i : X^\alpha \hookrightarrow C^0(\bar{\Omega})$ is continuous, there exists $\theta \in \mathbb{R}$ such that $B \subset \Sigma_\theta$. We can, of course assume that $\theta m \geq \xi$. Lemma **??** implies that $T(t)u_0 \in \Sigma_\theta$, for all $t \in [0, t_{max}[$ so

$$\|T(t)u_0\|_\infty \leq \theta\|\varphi\|_\infty.$$

Applying the variation of constants formula, we obtain

$$\|T(t)u_0\|_{X^\alpha} \leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\int_0^t (t-s)^{-\alpha+\beta}e^{-\epsilon(t-s)}\|h(T(s)u_0)\|_{X^\beta}ds,$$

where $M, \epsilon > 0$ are constants depending only on the semigroup e^{At} and $-\frac{1}{2} < \beta < -\frac{1}{4}$, $\alpha < \beta + 1$.

To compute the norm $\|h(T(s)u_0)\|_{X^\beta}$, let $\phi \in X^{-\beta} = H^{-2\beta}(\Omega)$. We have

$$\langle h(T(s)u_0), \phi\rangle_{-2\beta,-2\beta} = \int_\Omega f_\Omega(T(s)u_0)\phi(x)dx + \int_\Gamma g_\gamma(T(s)u_0)\gamma(\phi(x))dx$$

$$\leq \|f_\Omega(T(s)u_0)\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} + \|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma)}\|\gamma(\phi)\|_{L^2(\Gamma)}$$

$$\leq \left(\|f_\Omega(T(s)u_0)\|_{L^2(\Omega)} + K\|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma)}\right)\|\phi\|_{H^{-2\beta}},$$

where K is a bound for the continuous linear map $\gamma : H^{-2\beta}(\Omega) \to L^2(\Gamma)$.

Thus,

$$\|T(t)u_0\|_{X^\alpha} \leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\int_0^{t_{max}} \Big(K\|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma)}$$

$$+\|f_\Omega(T(s)u_0)\|_{L^2(\Omega)}\Big)(t-s)^{-\alpha+\beta}e^{-\epsilon(t-s)}ds$$

$$\leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\int_0^{t_{max}} \Big[K\|g_\Gamma(T(s)u_0)\|_\infty|\Gamma|^{\frac{1}{2}}$$

$$+\|f_\Omega(T(s)u_0)\|_\infty|\Omega|^{\frac{1}{2}}\Big](t-s)^{-\alpha+\beta}e^{-\epsilon(t-s)}ds$$

$$\leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\Bigg[\sup_{|\tau|\leq\theta\|\varphi\|_\infty}|g(\tau)||\Gamma|^{\frac{1}{2}}$$

$$+\sup_{|\tau|\leq\theta\|\varphi\|_\infty}|f(\tau)||\Omega|^{\frac{1}{2}}\Bigg]\int_0^\infty (t-s)^{-\alpha+\beta}e^{-\epsilon(t-s)}ds,$$

for all $t \in [0, t_{max}[$. Therefore, $\|T(t)u_0\|_{X^\alpha}$ is bounded by a constant depending only on B. In particular, $t_{max} = \infty$.

5 Existence of Global Attractors

The first step to show the existence of global attractors will be to show a "contraction property" of the sets Σ_θ, similar to the property for rectangles, considered by Carvalho [1993]. It is interesting to notice that we cannot use rectangles here since they are not invariant (unless f and g are both negative). In fact if f is positive at some point $x_0 \in \Omega$ and a constant function u_0 is chosen as an initial condition at time t_0 then $\frac{\partial u}{\partial t}(u_0, t_0, x_0) > 0$, so $T(t)u_0$ grows at the point x_0 for some time. A similar argument can be used for a point on $\partial\Omega$ if $g > 0$. We show in fig.1 below the result of a simulation (for N=1)...

Lemma 5.1 *Suppose $\bar{\theta} \in \mathbb{R}$ satisfy $\bar{\theta}m > \xi$. Then, for any θ there exists a $\bar{t}$ such that*

$$T(t)\Sigma_\theta \subset \Sigma_{\bar{\theta}},$$

for all $t \geq \bar{t}$.

Proof:

Let $u \in \Sigma_\theta$. We can suppose without loss of generality that $\theta \geq \bar{\theta}$. Let $\bar{v} = e^{t\mu}\theta\varphi$, $\underline{v} = -\bar{v}$. As in Lemma **??**, we can prove that $\bar{v}$ and $\underline{v}$ are super- and sub-solutions respectively. Thus, using Theorem **??** and the uniqueness of solution, we have that

$$\underline{v} \leq T(t)u \leq \bar{v},$$

as long as $e^{\mu t}\theta \geq \bar{\theta}$.

So, $T(t)u$ enters $\Sigma_{\bar{\theta}}$ eventually. Since $\Sigma_{\bar{\theta}}$ is positively invariant, the result follows.

Theorem 5.1 *The problem (**??**) has a global attractor $\mathcal{A}$ in X^α. Furthermore $u(x) \in \Sigma_\theta$, for all $x \in \bar{\Omega}$, $u \in \mathcal{A}$ if $\theta m \geq \xi$.*

Proof:

Since, by Lemma **??**, $T(t)$ takes bounded sets of X^α into bounded sets of X^α for any $t \geq 0$, and the semigroup regularizes the solutions, only point dissipativeness remains to be proved (see Hale [1988]).

Let $\bar{\theta} \in \mathbb{R}$ be such that $\bar{\theta}m \geq \xi$. If u is any element of X^α, it follows from the continuity of the imbedding $X^\alpha \hookrightarrow C^0(\bar{\Omega})$ that $u \in \Sigma_\theta$, for some θ and then, applying Lemma **??**, we conclude that $T(t)u \in \Sigma_{\bar{\theta}}$, for t big enough. Let $v = T(t_0)u \in \Sigma_{\bar{\theta}}$.

Applying the variation of constants formula, as in Lemma **??**, we obtain

$$\begin{aligned} \|T(t)v\|_{X^\alpha} \leq \quad & Me^{-\epsilon t}\|v\|_{X^\alpha} + M\int_0^\infty \Big[K\|g_\Gamma(T(s)v)\|_\infty|\Gamma|^{\frac{1}{2}} \\ & + \|f_\Omega(T(s)v)\|_\infty|\Omega|^{\frac{1}{2}}\Big](t-s)^{-\alpha+\beta}e^{-\epsilon(t-s)}ds, \end{aligned}$$

where M and K are independent of v.

Observing that $T(s)v \in \Sigma_{\bar{\theta}}$, for any $s \geq 0$, we conclude that, for t sufficiently large,

$$\|T(t_0+t)u\|_{X^\alpha} = \|T(t)v\|_{X^\alpha} \leq M+$$

$$M\left[\sup_{|\tau|\leq\bar{\theta}\|\varphi\|_\infty}|g(\tau)||\Gamma|^{\frac{1}{2}} + \sup_{|\tau|\leq\bar{\theta}\|\varphi\|_\infty}|f(\tau)||\Omega|^{\frac{1}{2}}\right]\int_0^\infty(t-s)^{-\alpha+\beta}e^{-\epsilon(t-s)}ds,$$

for t large.

Thus, the set in X^α bounded by the right-hand side above, attracts points. This proves point dissipativeness. Furthermore, since $\mathcal{A} \subset \Sigma_\theta$, for some θ, it follows from Lemma **??**, taking t large, that

$$\mathcal{A} = T(t)\mathcal{A} \subset \Sigma_{\tilde{\theta}}, \qquad \text{if } \tilde{\theta} > \xi$$

so

$$\mathcal{A} \subset \Sigma_{\bar{\theta}} = \bigcap_{\tilde{\theta}m > \xi} \Sigma_{\tilde{\theta}},$$

which proves the second part of the thesis.

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Eletronic Mail: smo@ime.usp.br, alpereir@ime.usp.br